

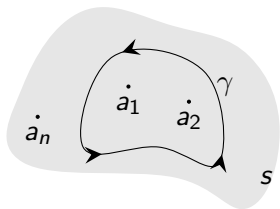
A Formal Proof of Cauchy's Residue Theorem

Wenda Li and Lawrence C. Paulson

University of Cambridge

{wl302,lp15}@cam.ac.uk

August 21, 2016



Informally, suppose f is holomorphic (i.e. complex differentiable) on an open set s except for a finite number of points $\{a_1, a_2, \dots, a_n\}$ and γ is some closed path, Cauchy's residue theorem states

$$\oint_{\gamma} f = 2\pi i \sum_{k=1}^n n(\gamma, a_k) \text{Res}(f, a_k)$$

where

- ▶ $n(\gamma, a_k)$ is the winding number of γ about a_k
- ▶ $\text{Res}(f, a_k)$ is the residue of f at a_k

Overview

1. Background: the multivariate analysis library in Isabelle/HOL
2. Main proof of the residue theorem
3. Application to improper integrals
4. Corollaries: the argument principle and Rouché's theorem
5. Related work
6. Conclusion

The multivariate analysis library in Isabelle/HOL

- ▶ about 70000 LOC (for now) on topology, analysis and linear algebra
- ▶ originates from John Harrison's work in HOL Light
 - ▶ \mathbb{R}^n vs. type classes
 - ▶ scripted proofs vs. structured proofs

Background: contour integration

Contour integration is mathematically defined as

$$\oint_{\gamma} f = \int_0^1 f(\gamma(t))\gamma'(t)dt.$$

In Isabelle/HOL, we have

definition *has_contour_integral* ::

```
"(complex  $\Rightarrow$  complex)  $\Rightarrow$  complex  $\Rightarrow$  (real  $\Rightarrow$  complex)
 $\Rightarrow$  bool"
```

```
where "(f has_contour_integral i)  $\gamma \equiv$ 
(( $\lambda x$ . f( $\gamma$  x) * vector_derivative  $\gamma$  (at x within {0..1}))
has_integral i) {0..1}"
```

and also *contour_integral* of type

```
(complex  $\Rightarrow$  complex)  $\Rightarrow$  (real  $\Rightarrow$  complex)  $\Rightarrow$  complex
```

Background: valid paths

A valid path is a piecewise continuously differentiable function on $[0..1]$.

lemma *valid_path_def:*

fixes $\gamma :: "real \Rightarrow 'a :: real_normed_vector"$

shows "*valid_path* $\gamma \longleftrightarrow \text{continuous_on } \{0..1\} \gamma \wedge$

$(\exists s. \text{finite } s \wedge \gamma \text{ C1_differentiable_on } (\{0..1\} - s))"$

Background: winding numbers

The winding number $n(\gamma, z)$ is the number of times the path γ travels counterclockwise around the point z :

$$n(\gamma, z) = \frac{1}{2\pi i} \oint_{\gamma} \frac{dw}{w - z}$$

A lemma to illustrate this definition is as follows:

lemma *winding_number_valid_path*:

fixes $\gamma :: \text{"real"} \Rightarrow \text{"complex"}$ **and** $z :: \text{complex}$

assumes *"valid_path γ "* **and** *" $z \notin \text{path_image } \gamma$ "*

shows *"winding_number γ z*

= $1/(2\text{pi}*i) * \text{contour_integral } \gamma (\lambda w. 1/(w - z))$ "*

Background: Cauchy's integral theorem

Cauchy's integral theorem states: given f is a holomorphic function on an open set s , which contains a closed path γ and its interior, then

$$\oint_{\gamma} f = 0$$

theorem *Cauchy_theorem_global*:

fixes $s::\text{"complex set"}$ **and** $f::\text{"complex} \Rightarrow \text{complex"}$

and $\gamma::\text{"real} \Rightarrow \text{complex"}$

assumes $\text{"open } s\text{"}$ **and** $\text{"}f \text{ holomorphic_on } s\text{"}$

and $\text{"valid_path } \gamma\text{"}$ **and** $\text{"pathfinish } \gamma = \text{pathstart } \gamma\text{"}$

and $\text{"path_image } \gamma \subseteq s\text{"}$

and $\text{"}\bigwedge w. w \notin s \implies \text{winding_number } \gamma \ w = 0\text{"}$

shows $\text{"}(f \text{ has_contour_integral } 0) \ \gamma\text{"}$

Main proof: singularities

Cauchy's integral theorem does not apply when there are singularities.

For example, consider $f(w) = \frac{1}{w}$ so that f has a pole at $w = 0$, and γ is the circular path $\gamma(t) = e^{2\pi it}$:

$$\oint_{\gamma} \frac{dw}{w} = \int_0^1 \frac{1}{e^{2\pi it}} \left(\frac{d}{dt} e^{2\pi it} \right) dt = \int_0^1 2\pi i dt = 2\pi i \neq 0$$

Main proof: residue

The residue of f at z can be defined as

$$\text{Res}(f, z) = \frac{1}{2\pi i} \oint_{\text{circ}(z)} f$$

where $\text{circ}(z)$ is a small counterclockwise circular path around z .



definition *residue*

`:: "(complex \Rightarrow complex) \Rightarrow complex \Rightarrow complex"`

where `"residue f z = (SOME int. $\exists \delta > 0. \forall \epsilon > 0. \epsilon < \delta \longrightarrow$
(f has_contour_integral 2*pi*i*int) (circlepath z epsilon))"`

Main proof: Cauchy's residue theorem

lemma *Residue_theorem*:

fixes s $pts::\text{"complex set"}$ **and** $f::\text{"complex} \Rightarrow \text{complex"}$

and $\gamma::\text{"real} \Rightarrow \text{complex"}$

assumes *"open s"* **and** *"connected s"*

and *"finite pts"* **and** *"f holomorphic_on s-pts"*

and *"valid_path γ "* **and** *"pathfinish γ = pathstart γ "*

and *"path_image $\gamma \subseteq s$ -pts"*

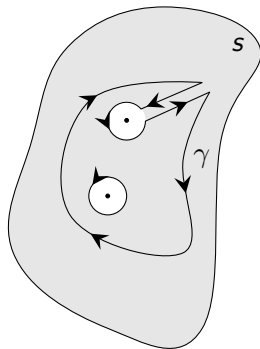
and *" $\forall z. (z \notin s) \longrightarrow \text{winding_number } \gamma z = 0$ "*

shows *"contour_integral γ f*

*= 2*pi*i*($\sum_{p \in pts. \text{winding_number } \gamma p * \text{residue } f p}$)"*

Main proof: remarks

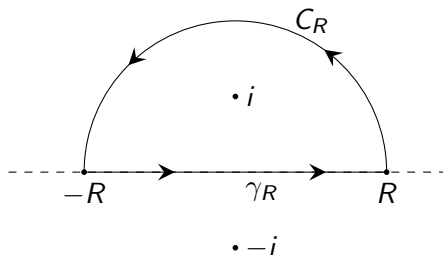
- ▶ the core idea to induct on the number of singularity points
- ▶ gap in informal proofs



Application to improper integrals

The residue theorem can be used to solve improper integrals:

$$\int_{-\infty}^{\infty} \frac{dx}{x^2 + 1} = \pi$$



Proof: let

$$f(z) = \frac{1}{z^2 + 1}.$$

since $\lim_{R \rightarrow \infty} \oint_{C_R} f = 0$ we have

$$\int_{-\infty}^{\infty} \frac{dx}{x^2 + 1} = \lim_{R \rightarrow \infty} \oint_{\gamma_R + C_R} f$$

$$\int_{-\infty}^{\infty} \frac{dx}{x^2 + 1} = \pi$$

$$\Leftrightarrow \lim_{R \rightarrow \infty} \oint_{\gamma_R} f = \pi$$

$$\Leftrightarrow \lim_{R \rightarrow \infty} \left(\oint_{\gamma_R} f + \oint_{C_R} f \right) = \pi$$

$$\Leftrightarrow \lim_{R \rightarrow \infty} \oint_{\gamma_R + C_R} f$$

$$= 2\pi i \operatorname{Res}(\gamma_R + C_R, i) = \pi$$

```
def f≡"λx::real. 1/(x^2+1)"
def f'≡"λx::complex. 1/(x^2+1)"
```

```
have "((λR. integral {- R..R} f) → pi) at_top
      = ((λR. contour_integral (γ_R R) f')
          → pi) at_top"
```

```
also have "... = ((λR. contour_integral (C_R R) f'
      + contour_integral (γ_R R) f') → pi) at_top"
```

```
also have "... = ((λR. contour_integral
      (C_R R +++ γ_R R) f') → pi) at_top"
```

```
also have "..."
```

```
proof -
```

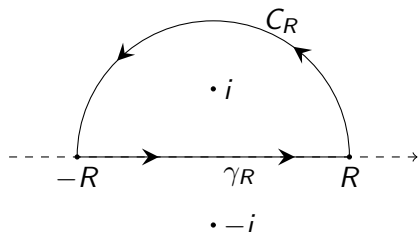
```
  have "contour_integral (C_R R +++ γ_R R) f' = pi"
    when "R>1" for R
```

```
  then show ?thesis
```

```
qed
```

```
finally have "((λR. integral {- R..R}
      (λx. 1 / (x^2 + 1))) → pi) at_top"
```

Application to improper integrals (3)



lemma *improper_Ex*:

" $\text{Lim at_top } (\lambda R. \text{integral } \{-R..R\} (\lambda x. 1/(x^2+1))) = \text{pi}$ "

Remarks

- ▶ about 300 LOC
- ▶ the main difficulty is to show $n(\gamma_R + C_R, i) = 1$ and $n(\gamma_R + C_R, -i) = 0$

Corollaries: the argument principle

The argument principle states: suppose f is holomorphic on a connected open set s except for a finite number of poles and γ is a valid closed path, then

$$\oint_{\gamma} \frac{f'}{f} = 2\pi i \left(\sum_{z \in \text{zeros}} n(\gamma, z) \text{zorder}(f, z) - \sum_{z \in \text{poles}} n(\gamma, z) \text{porder}(f, z) \right)$$

where

- ▶ f' is the first derivative of f ;
- ▶ $\text{zorder}(f, z)$ and $\text{porder}(f, z)$ are the order of a zero and a pole respectively;
- ▶ zeros and poles are the zeros and poles respectively of f .

Corollaries: Rouché's theorem

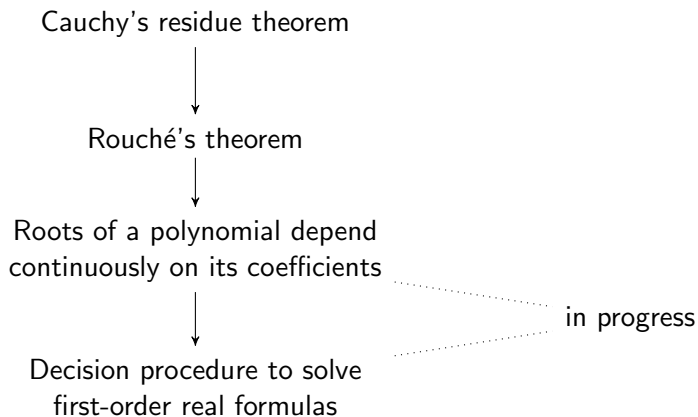
Given two functions f and g holomorphic on a connected open set containing a valid path γ , if

$$|f(w)| > |g(w)| \quad \forall w \in \text{image of } \gamma$$

then Rouché's Theorem states

$$\sum_{z \in \text{zeros}(f)} n(\gamma, z) \text{zorder}(f, z) = \sum_{z \in \text{zeros}(f+g)} n(\gamma, z) \text{zorder}(f + g, z)$$

Big picture



To the best of our knowledge, our formalization of Cauchy's residue theorem is novel among major proof assistants.

- ▶ HOL Light: comprehensive library for complex analysis, to which it should be not hard to port our result;
- ▶ Coq: Coquelicot and C-Corn, but they are mainly about real analysis and some fundamental theorems (e.g. Cauchy's integral theorem) are not yet available.

Conclusion

To conclude, I have mainly covered

- ▶ the multivariate analysis library in Isabelle/HOL
- ▶ the residue theorem
- ▶ application to improper integrals
- ▶ corollaries: the argument principle and Rouché's theorem

Thank you for your attention!