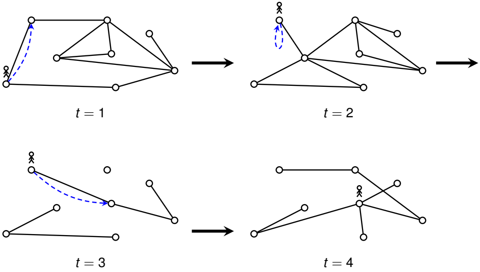


Random walks on dynamic graphs: Mixing times, hitting times, and return probabilities

Thomas Sauerwald and Luca Zanetti

to appear in ICALP'19, full version arXiv:1903.01342

7 May 2019

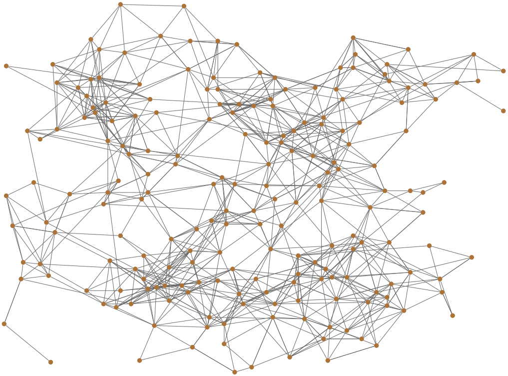
Outline

Random Walks and Markov Chains

Random Walks on Graphs

A class of Markov chains where a particle is moving on the vertices of a graph:

- start from some specified vertex
- at each step, jump to a randomly chosen neighbor



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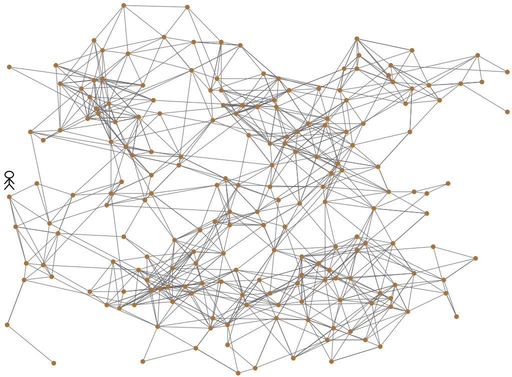

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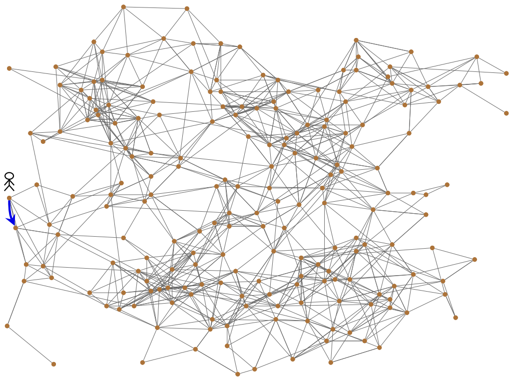

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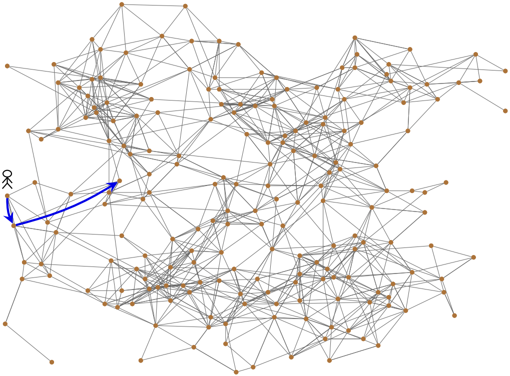

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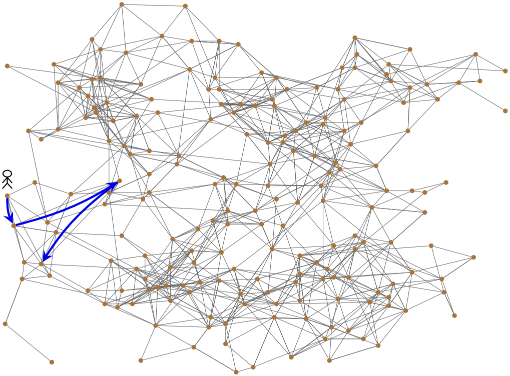

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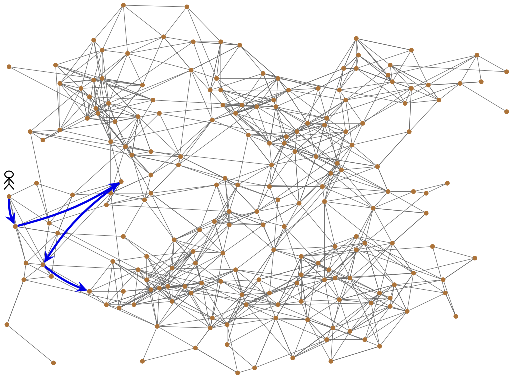

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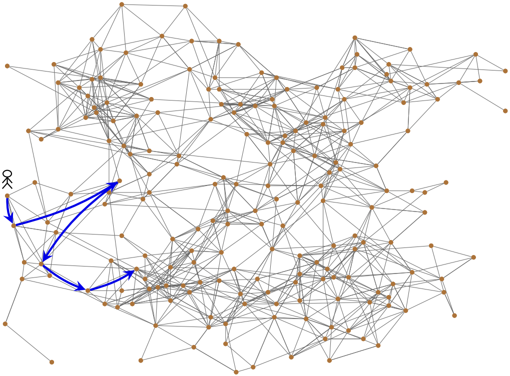

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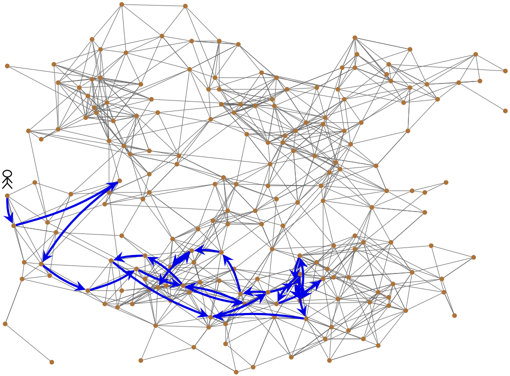

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Hitting and Cover Times

- Let $t_{hit}(u, v)$ be the expected time for a random walk to go from u to v
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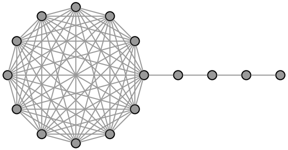

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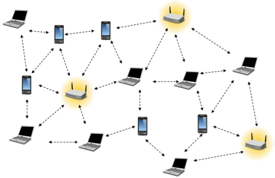

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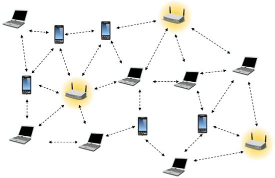

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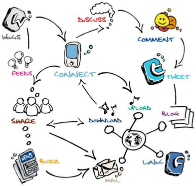

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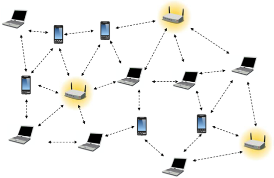

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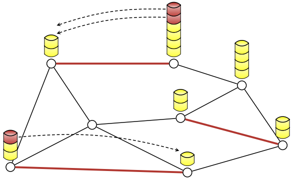
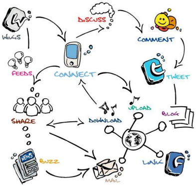


(Distributed) Algorithms



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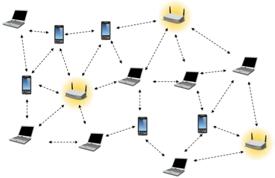

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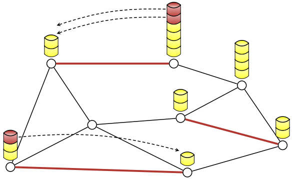

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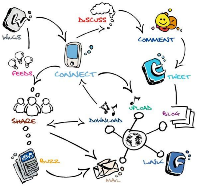

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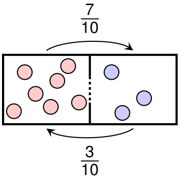

Particle Processes

Random Walk on a Dynamic Graph Sequence

Lazy Random Walks

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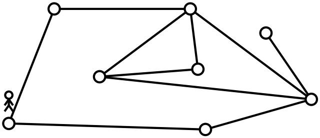

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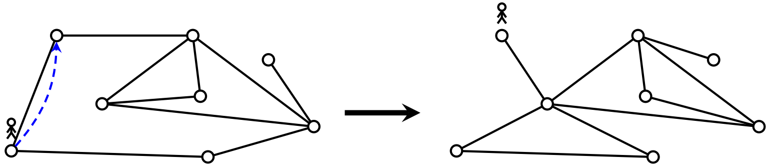

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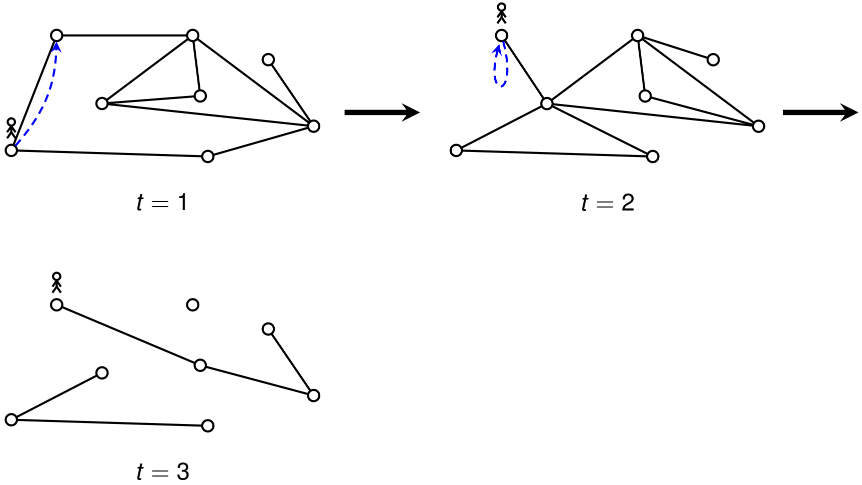

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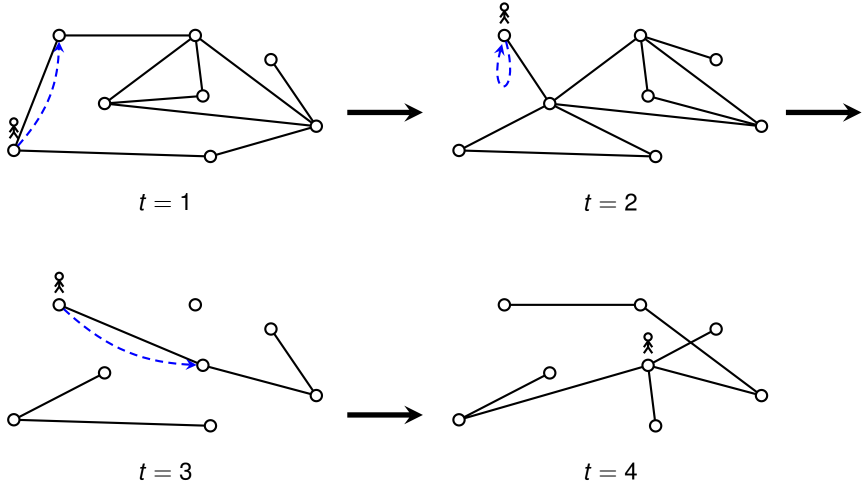

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For dynamic connected graphs:

- If $\pi^{(t)}$ changes over time, in general, we don't have mixing
- Can we at least say something about hitting times?

Related Work: A Dichotomy for dynamic graphs

Avin, Koucky, and Lotker (ICALP'08, RSA'18)

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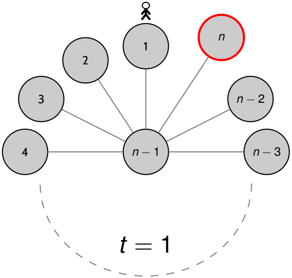

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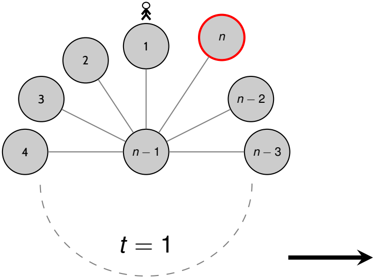

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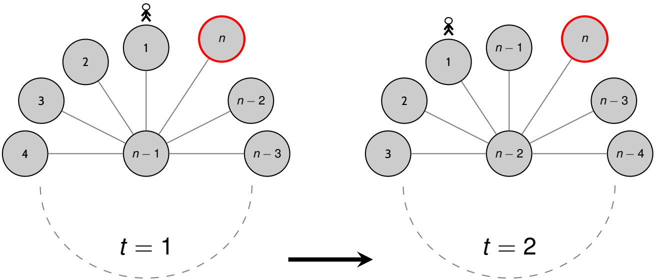

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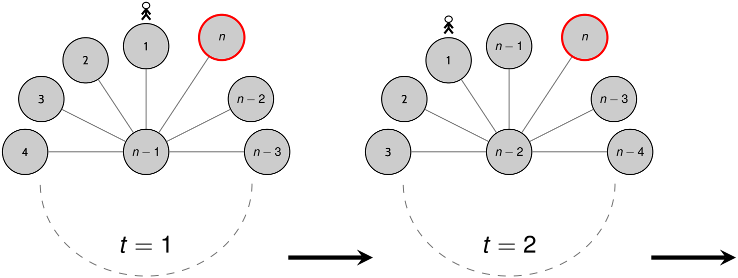

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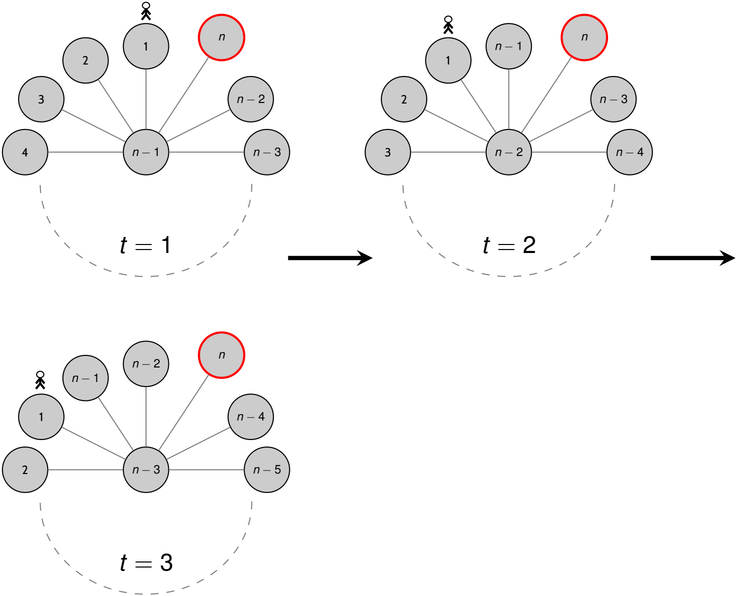

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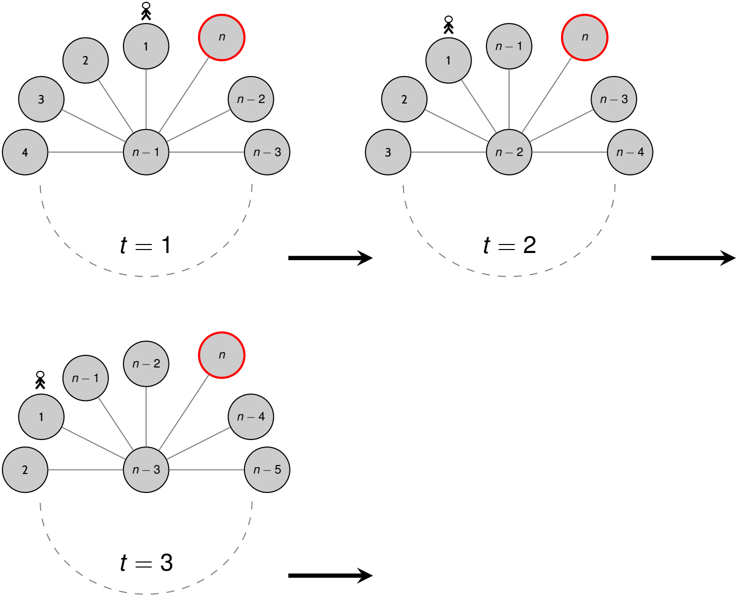

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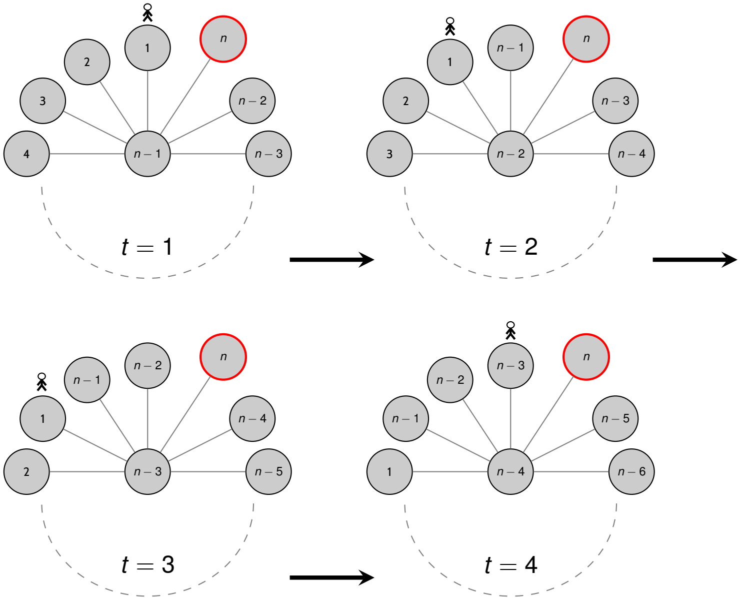

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   - hitting (and covering) can take exponential time
   - this holds even if $\pi^{(t)}$ changes *slowly*
2. If all graphs are connected and regular ($\Rightarrow$ $\pi^{(t)}$ is always uniform),
   - mixing in $O(n^2 \log(n))$ steps
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How can we derive these results?

Classical Proof (Spanning Tree Approach)

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For any static graph G, $t_{cov}(G) \leq 2(n-1)|E|)$.

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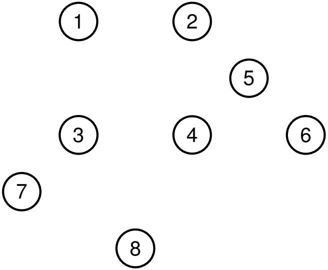

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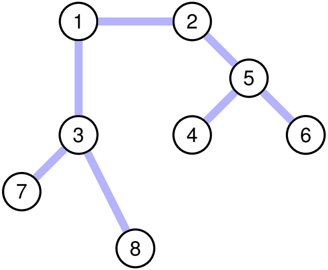

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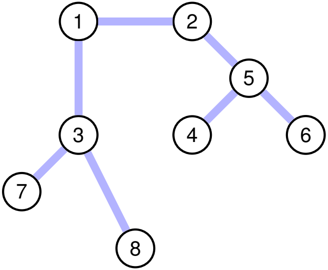

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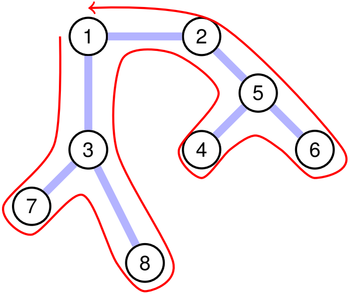

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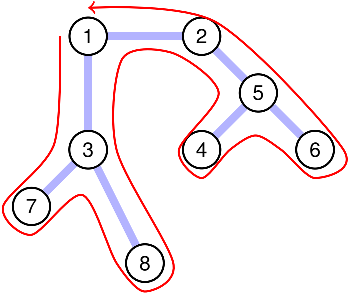

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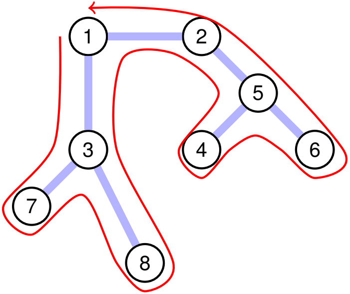

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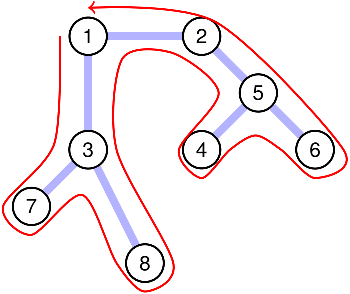

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Both proofs crucially rely on a static spanning tree or static shortest path!

Return Times on Dynamic Graphs

A fundamental fact of the return times is that:

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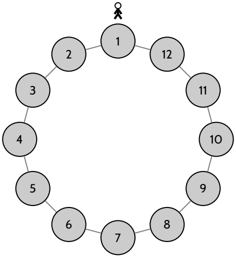

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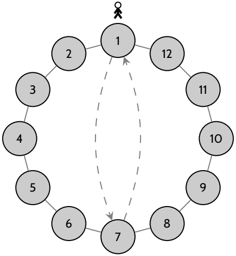

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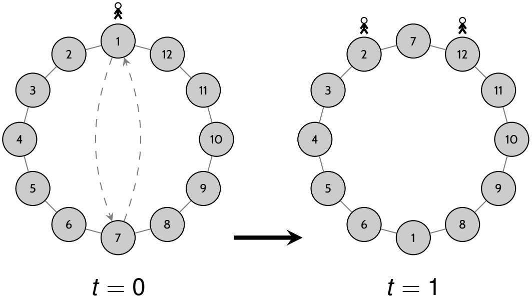

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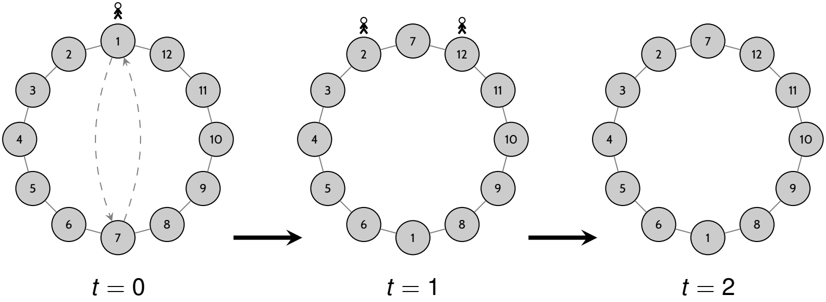

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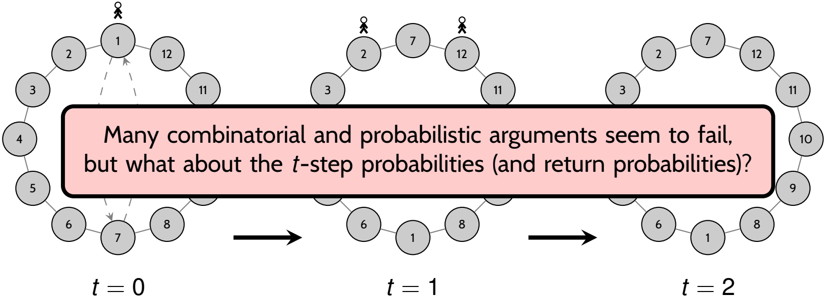
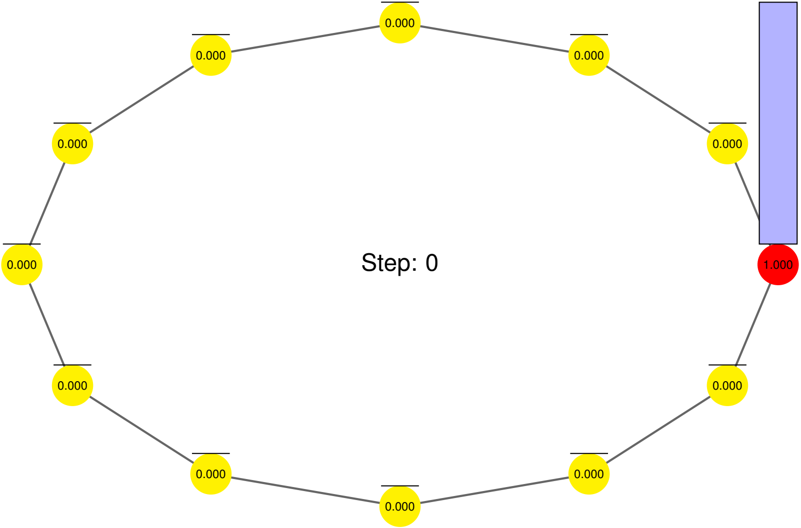


Many combinatorial and probabilistic arguments seem to fail, but what about the t-step probabilities (and return probabilities)?

Diffusion of a Random Walk on a Static Cycle



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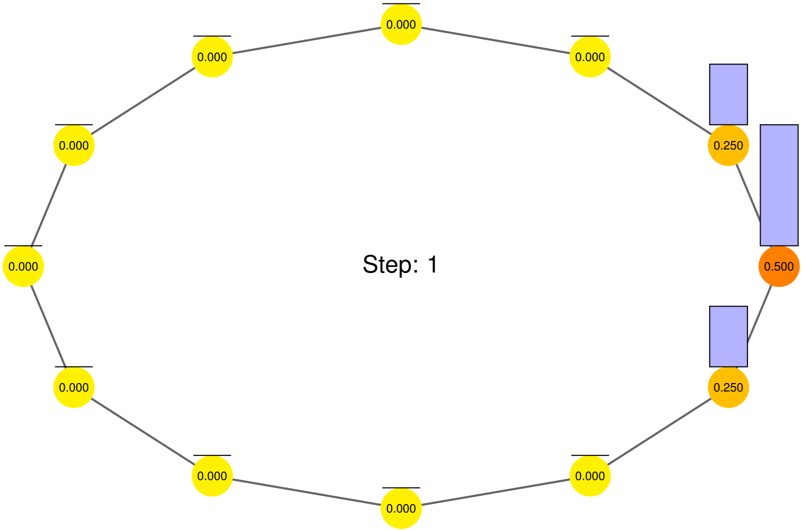

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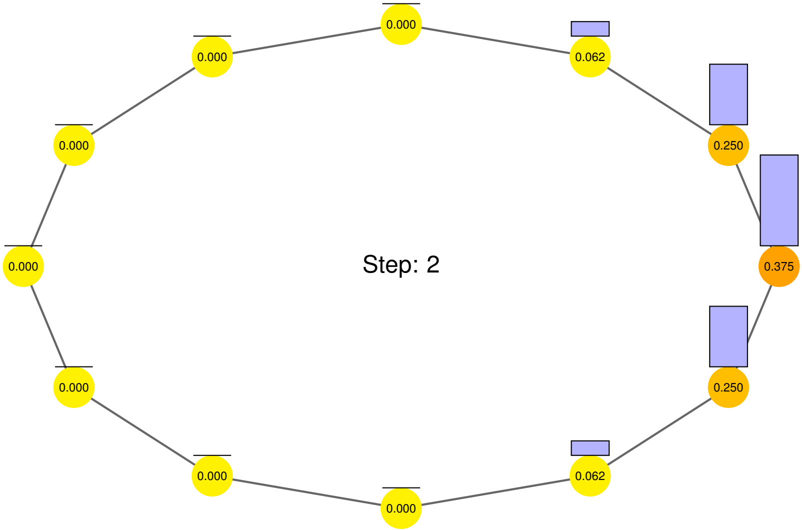

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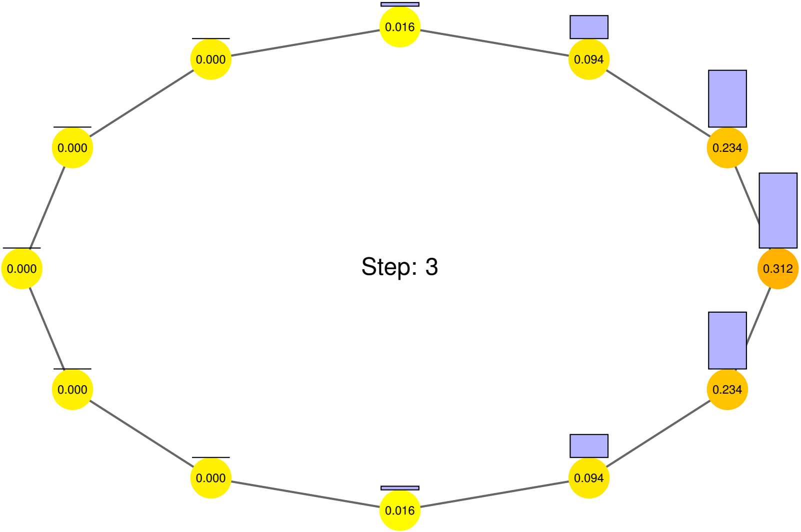

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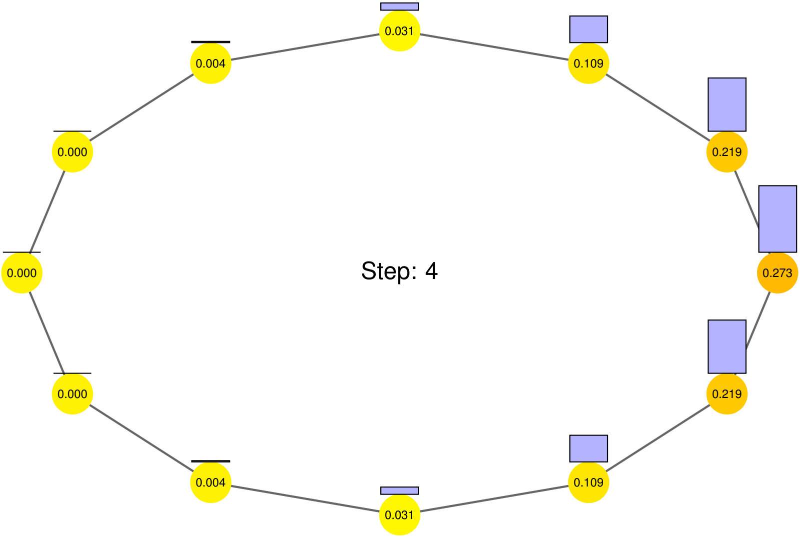

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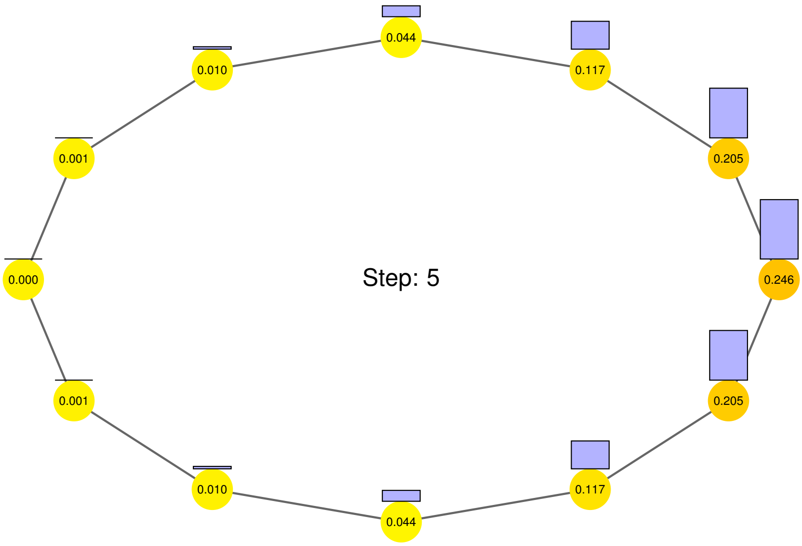

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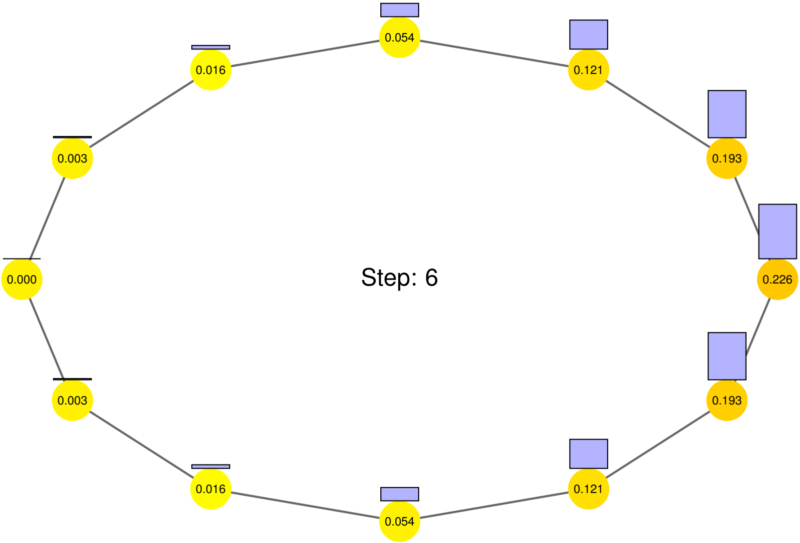

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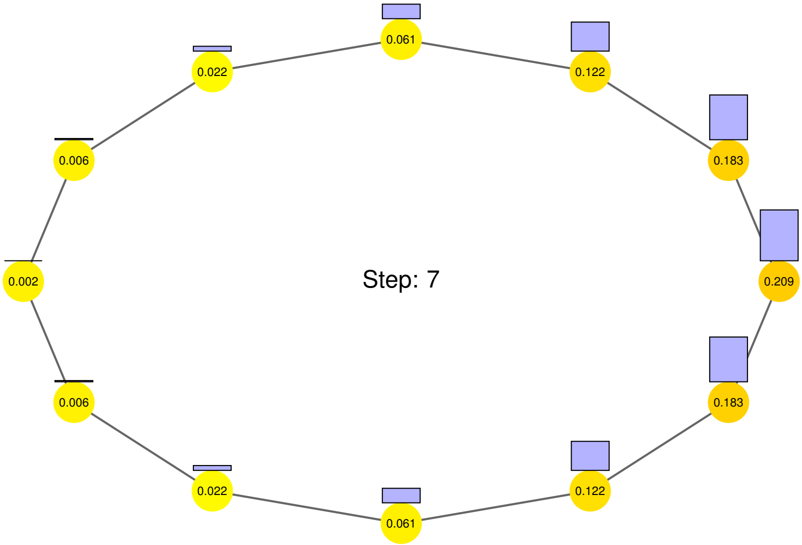

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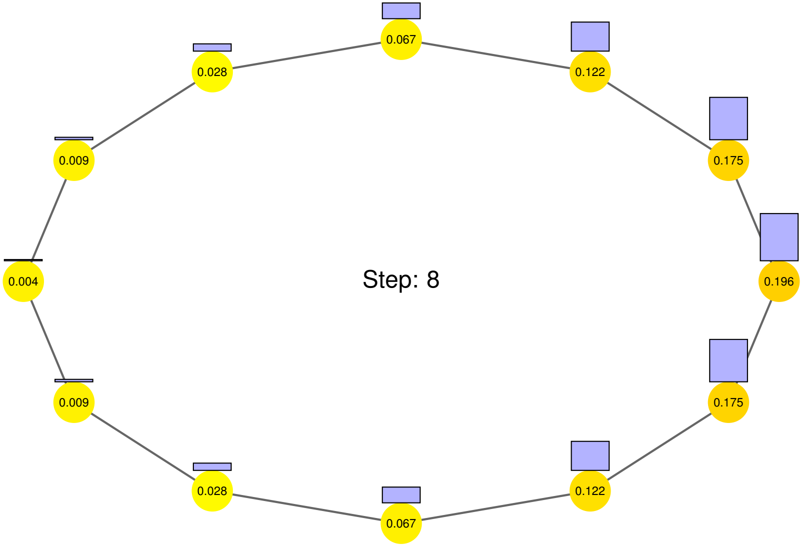

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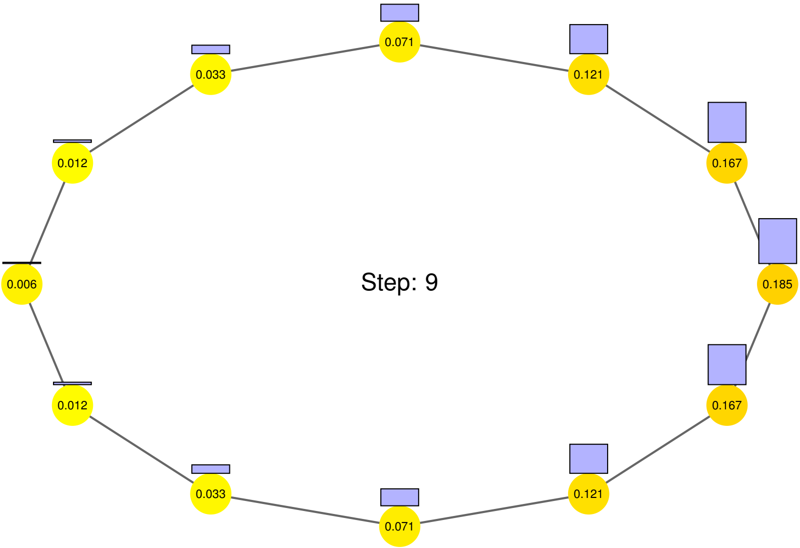

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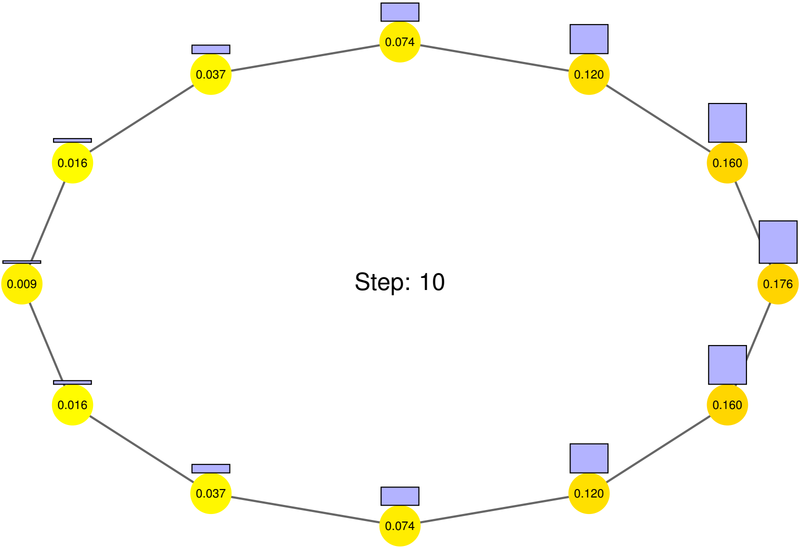

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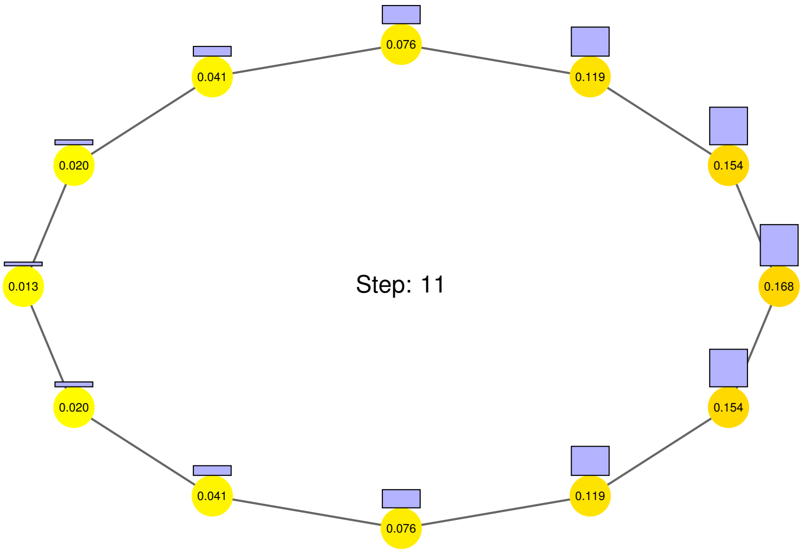

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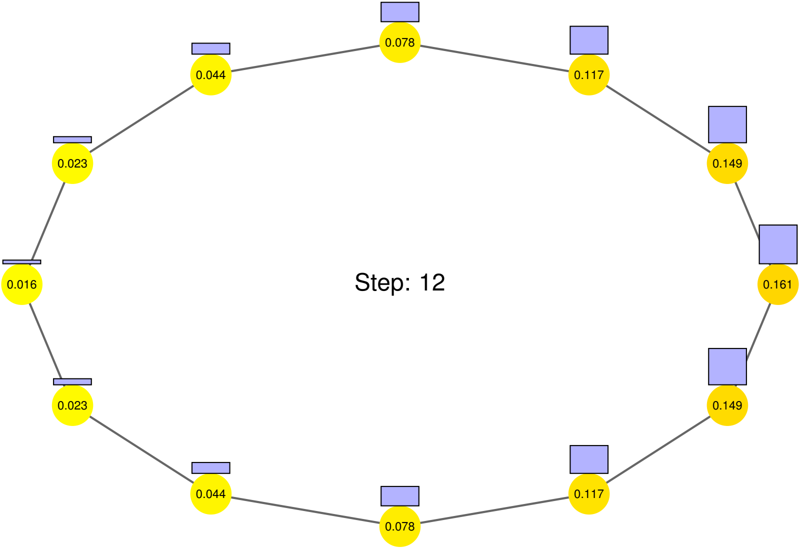

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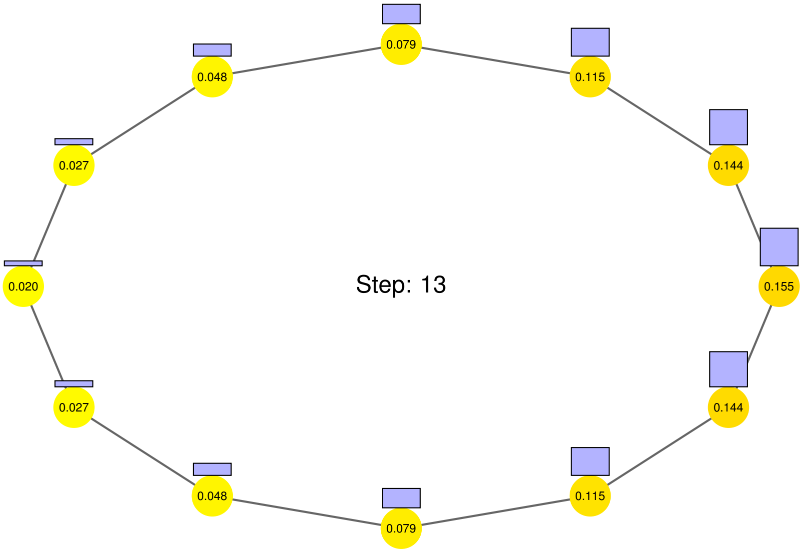

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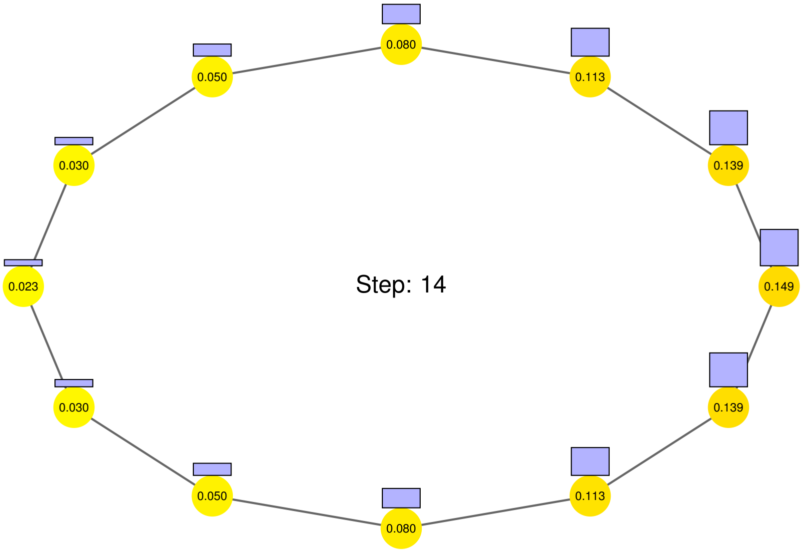

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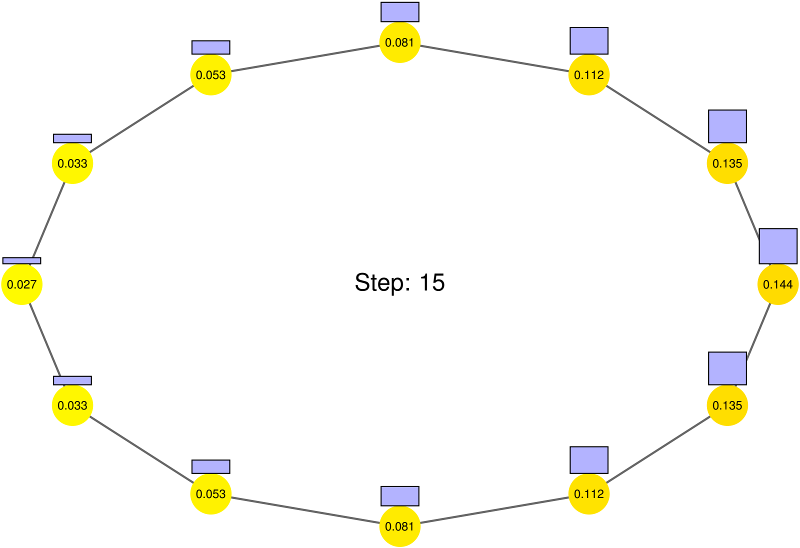

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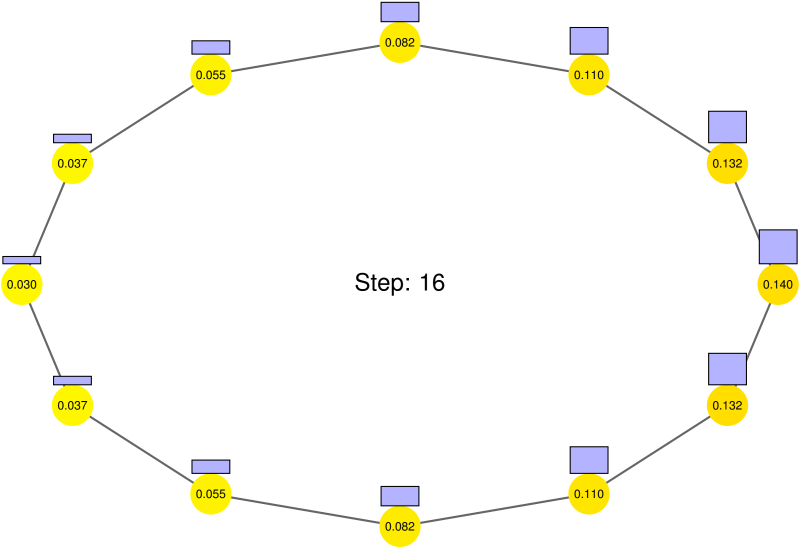

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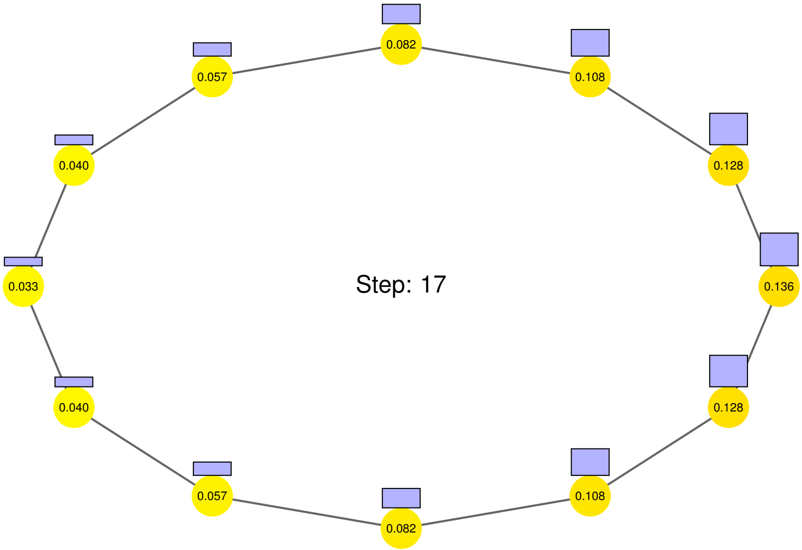

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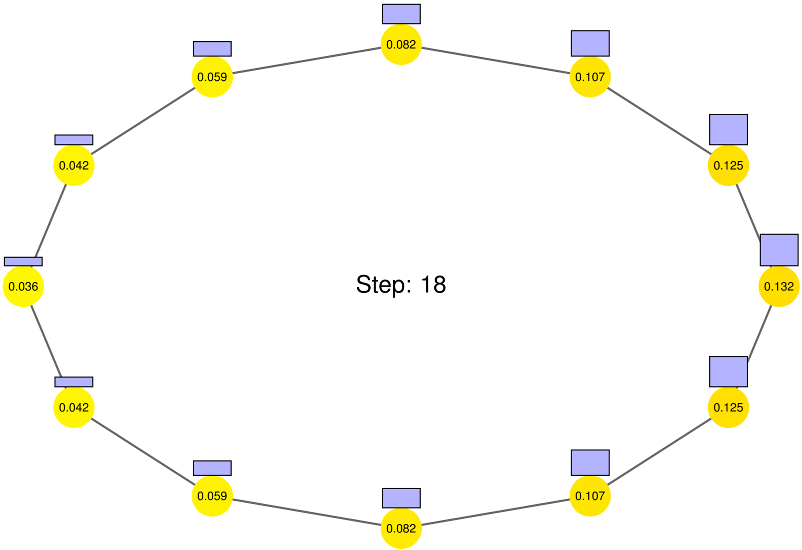

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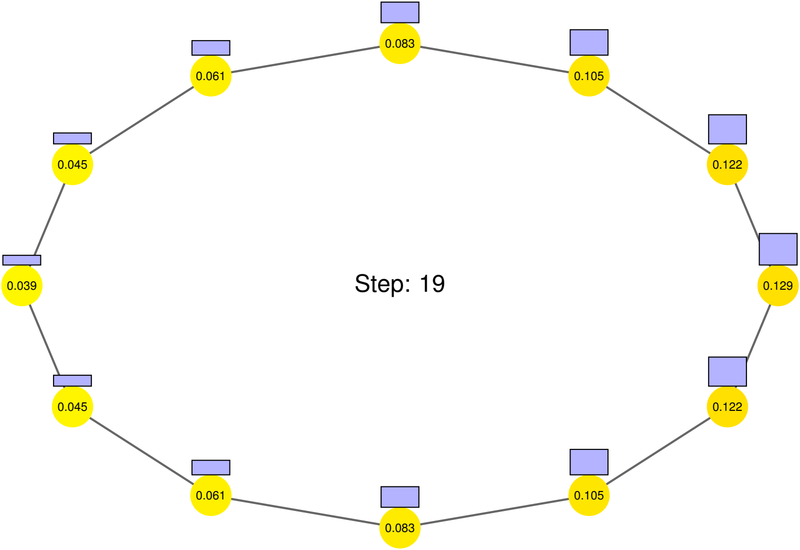

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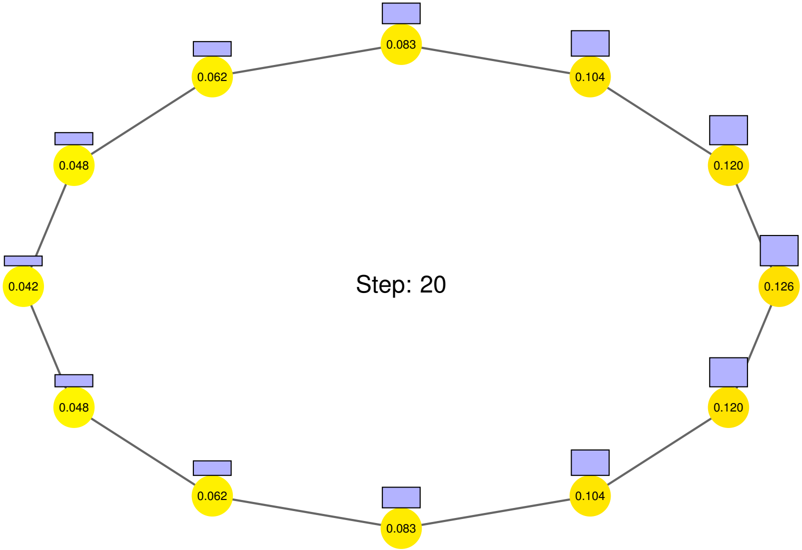

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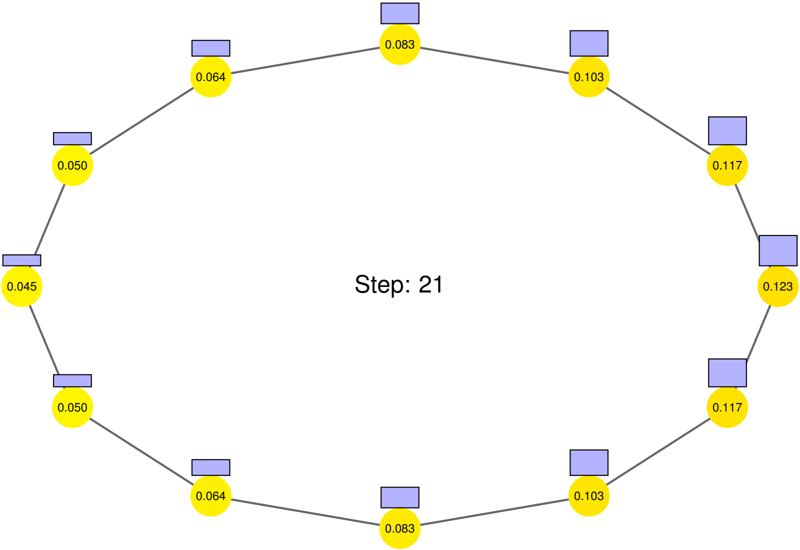

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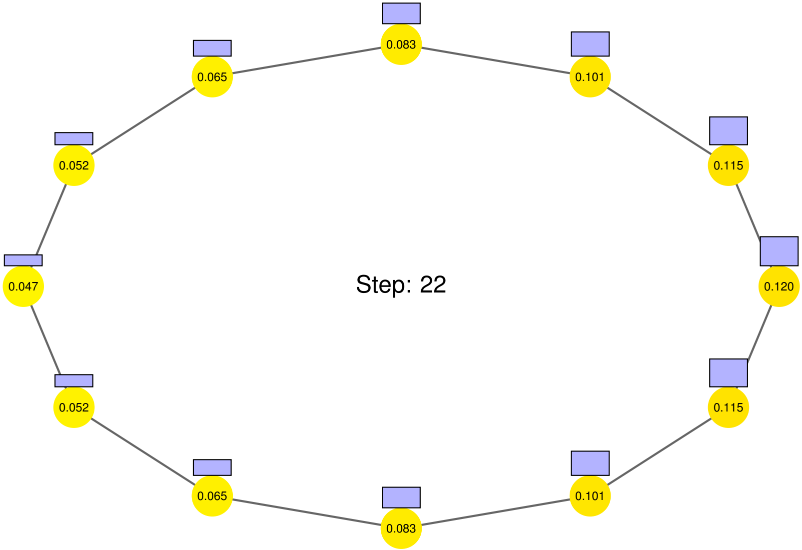

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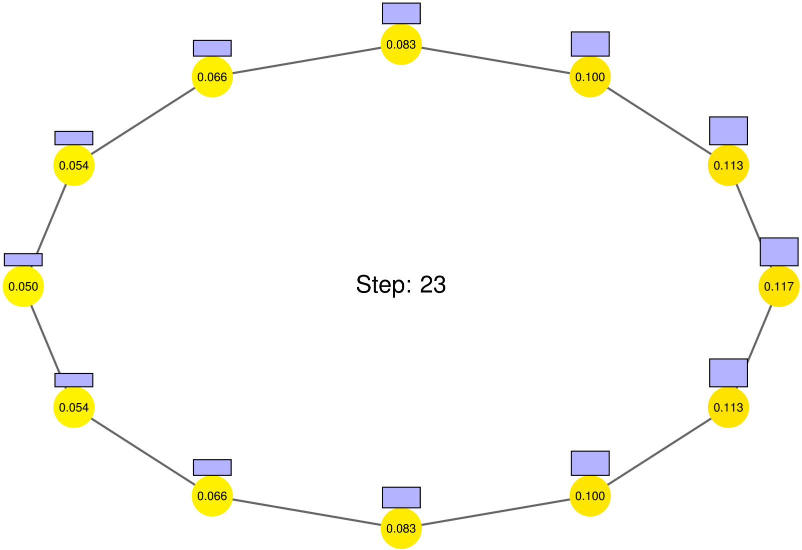

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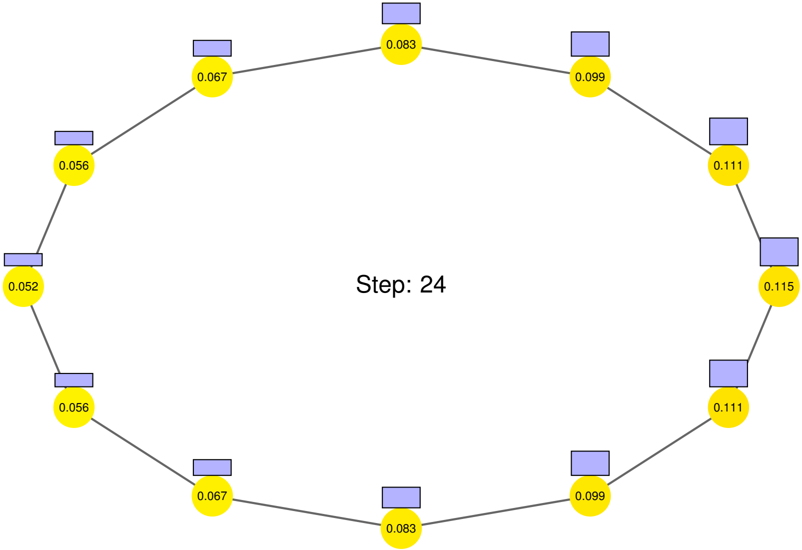

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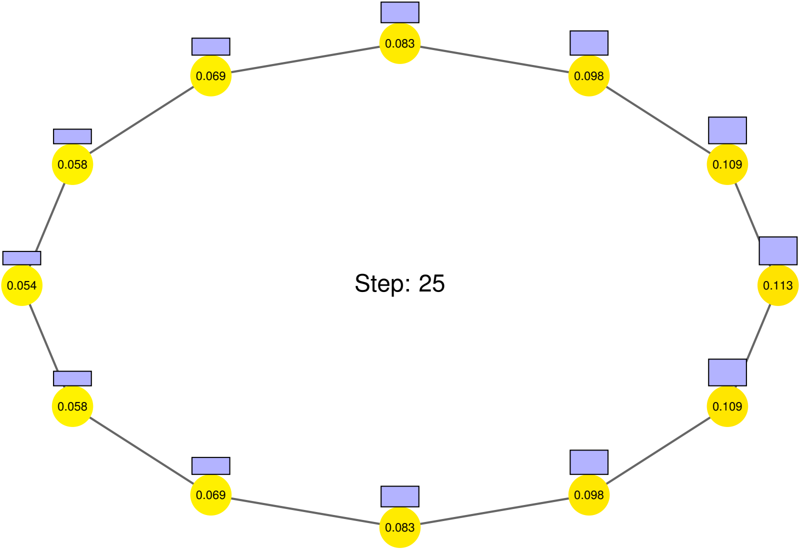

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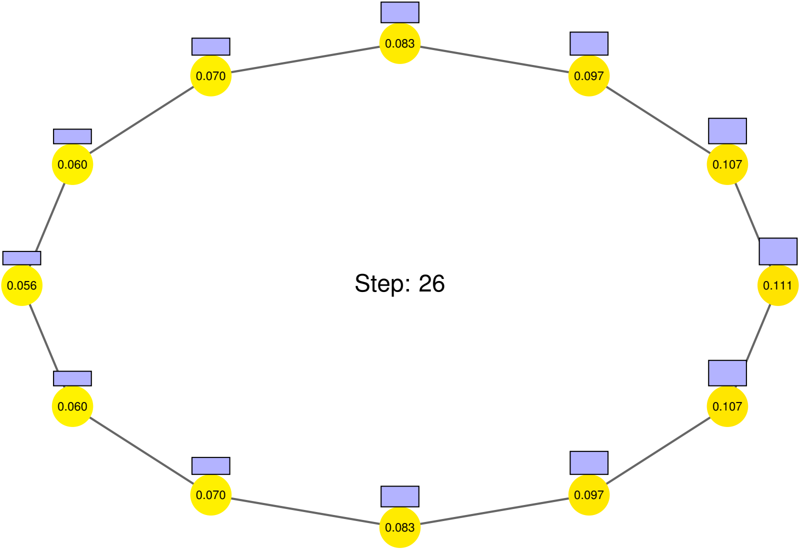

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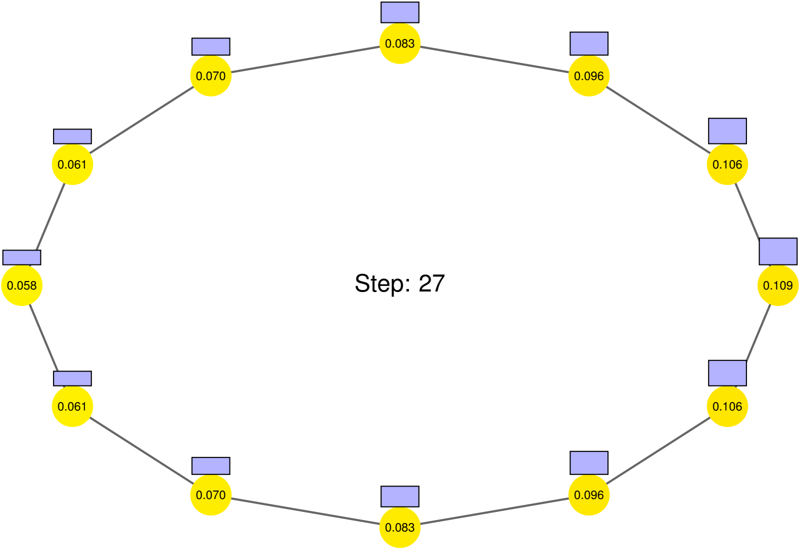

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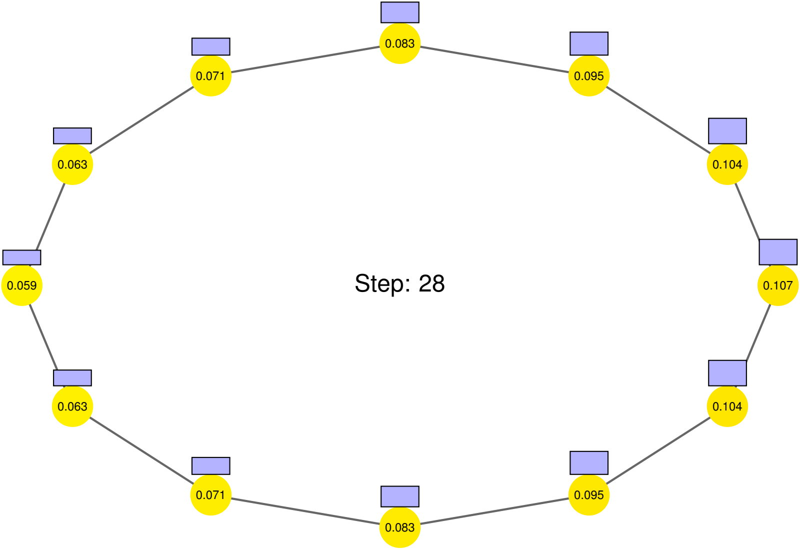

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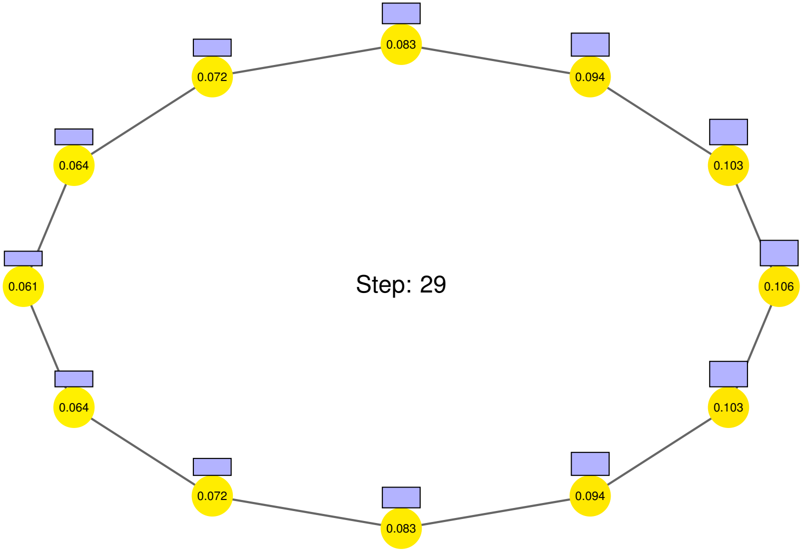

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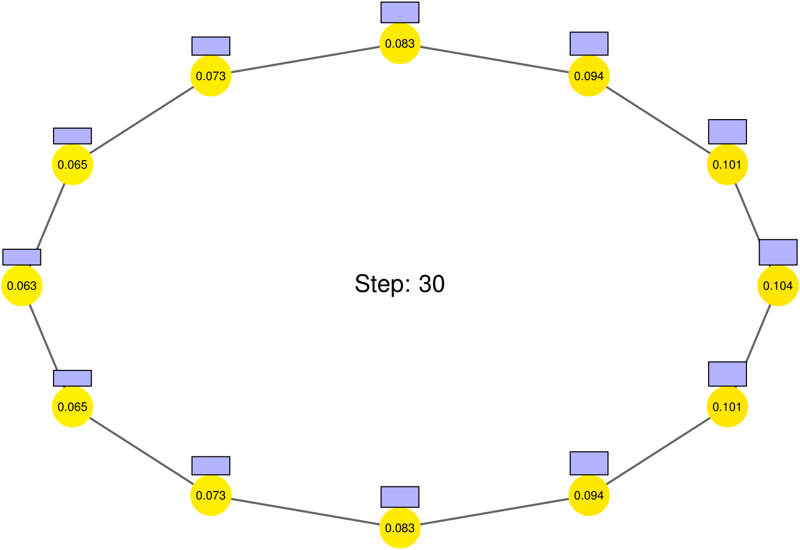

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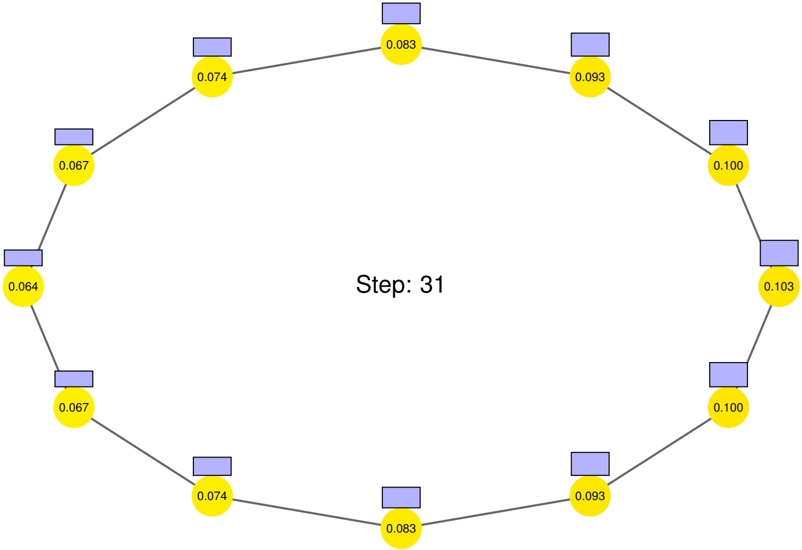

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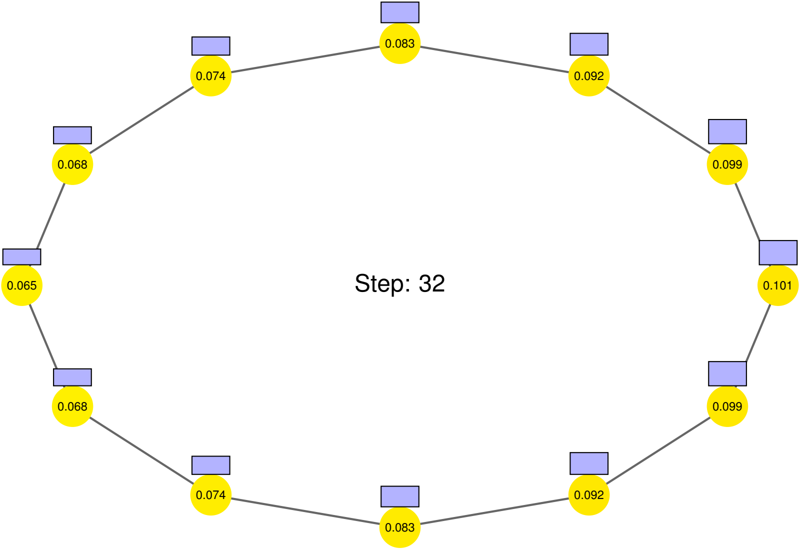

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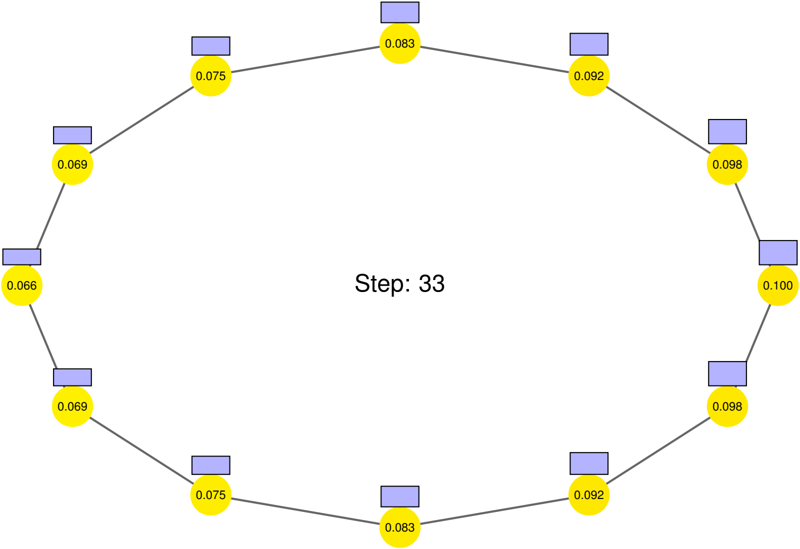

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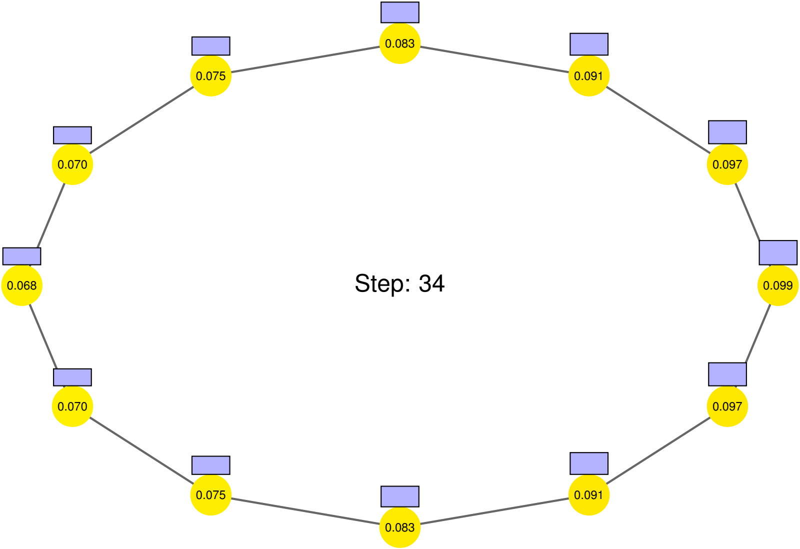

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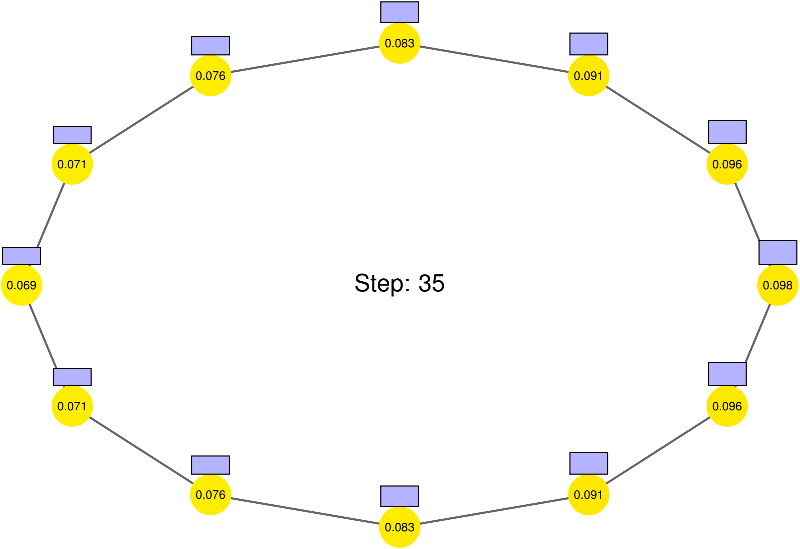

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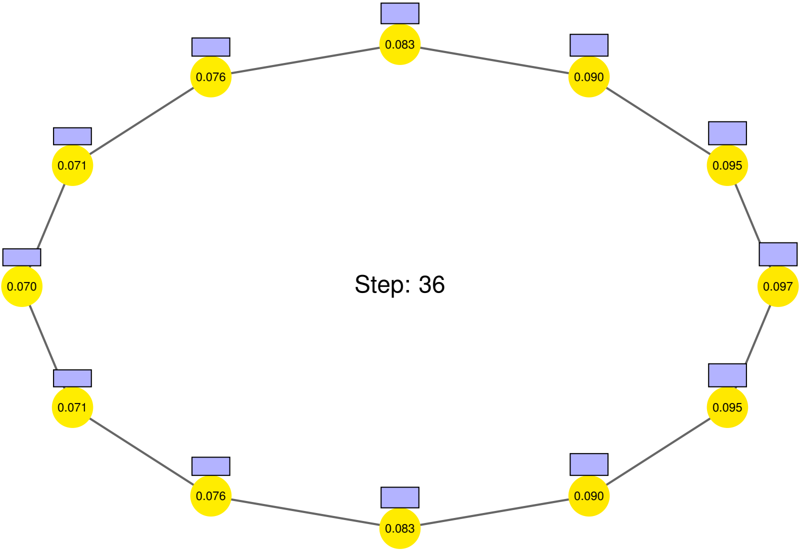

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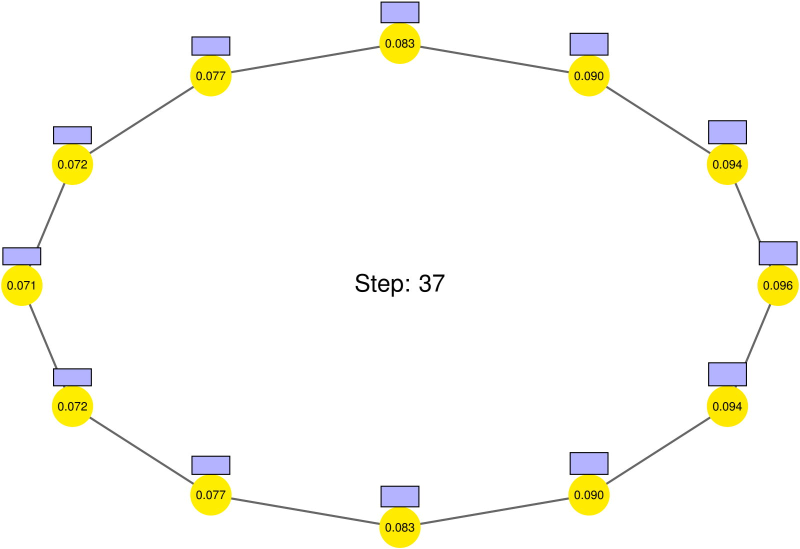

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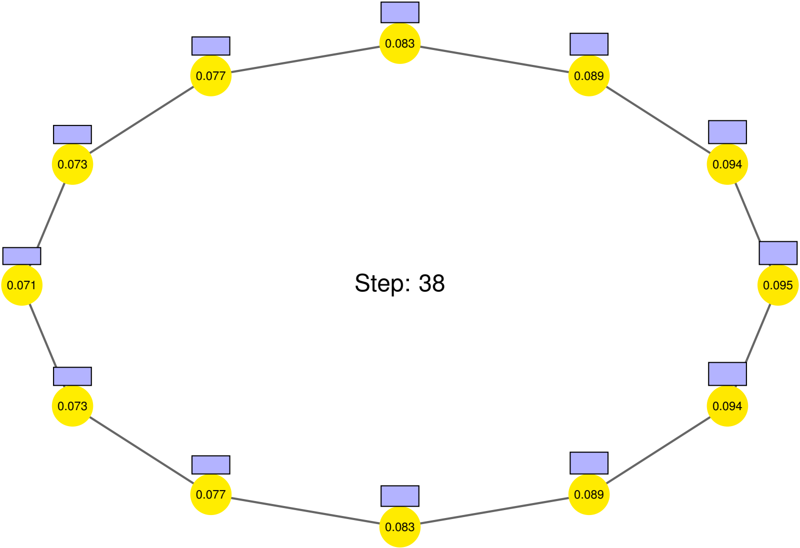

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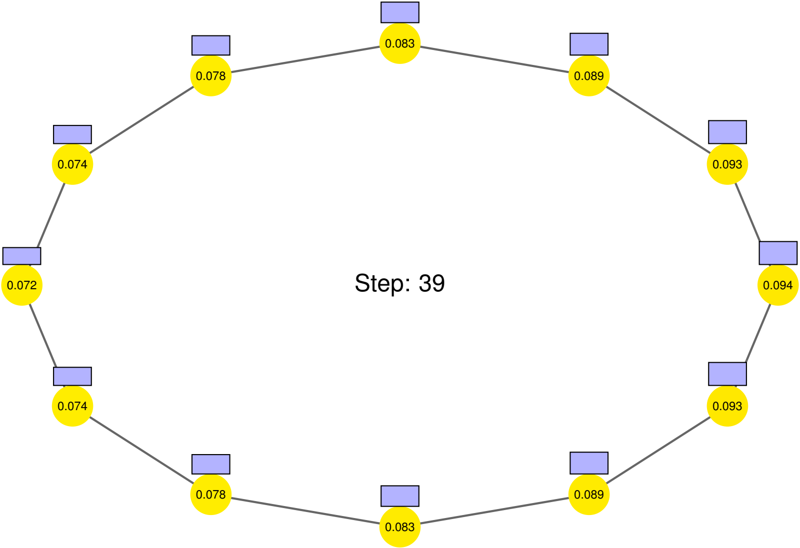

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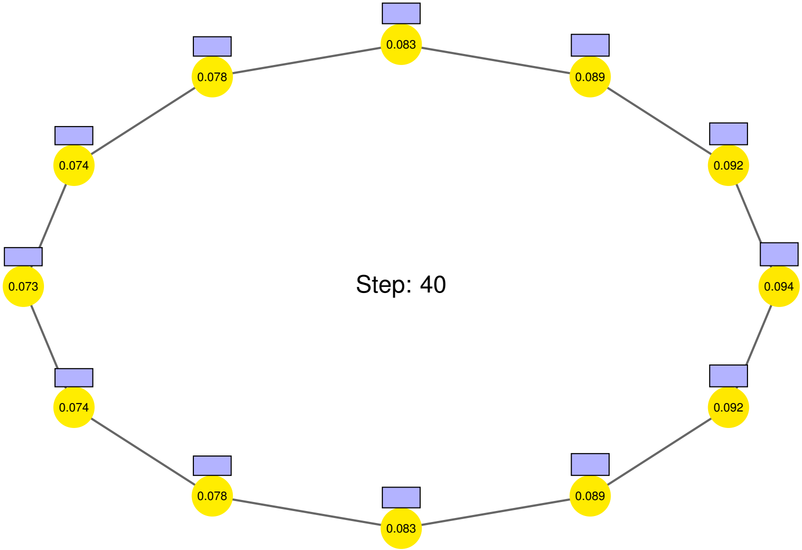

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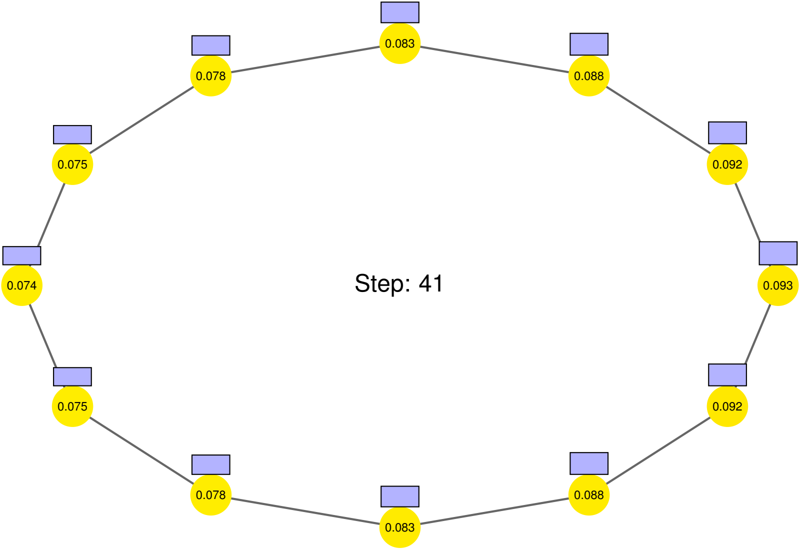

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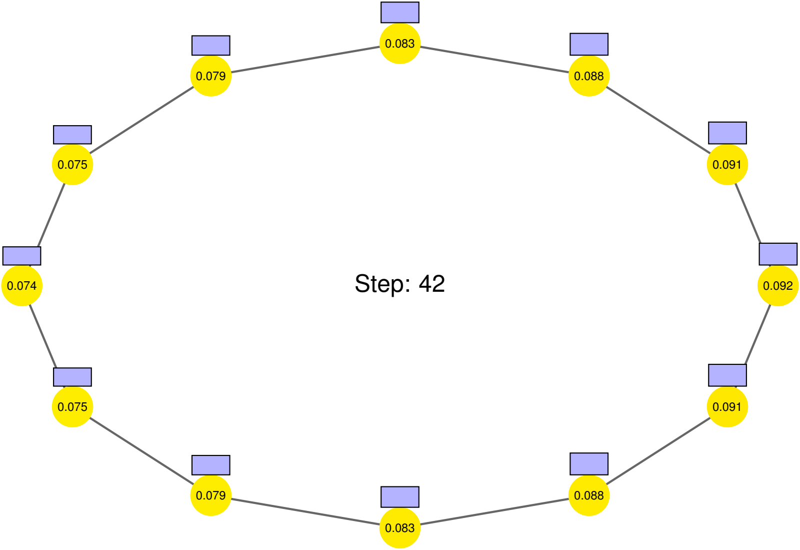

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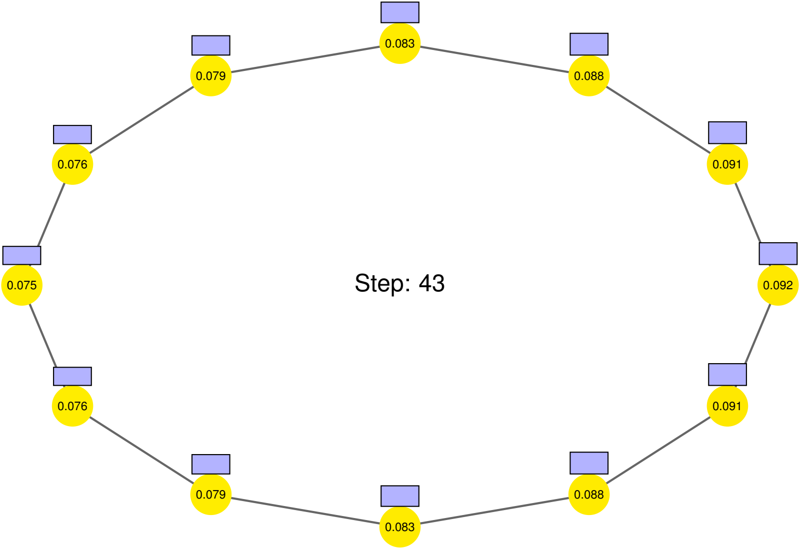

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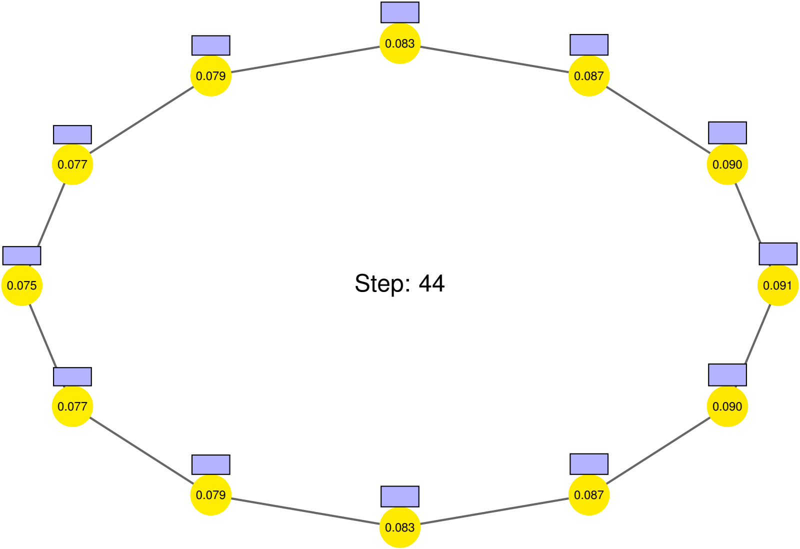

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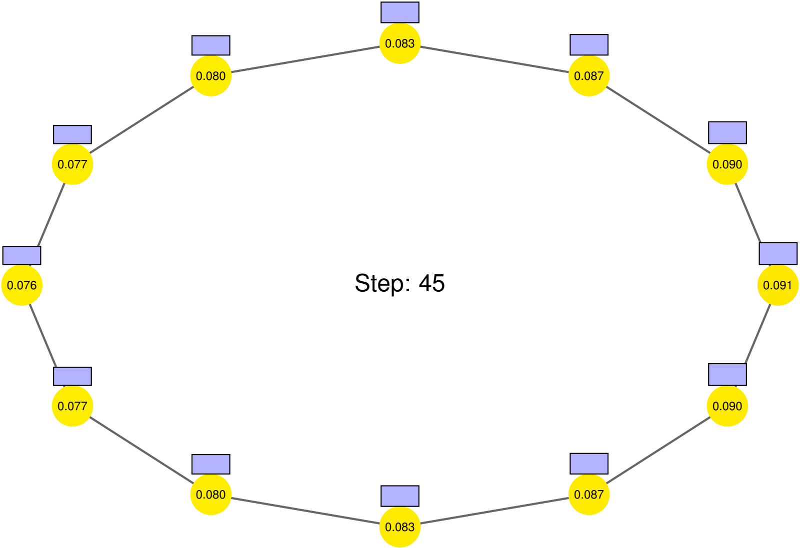

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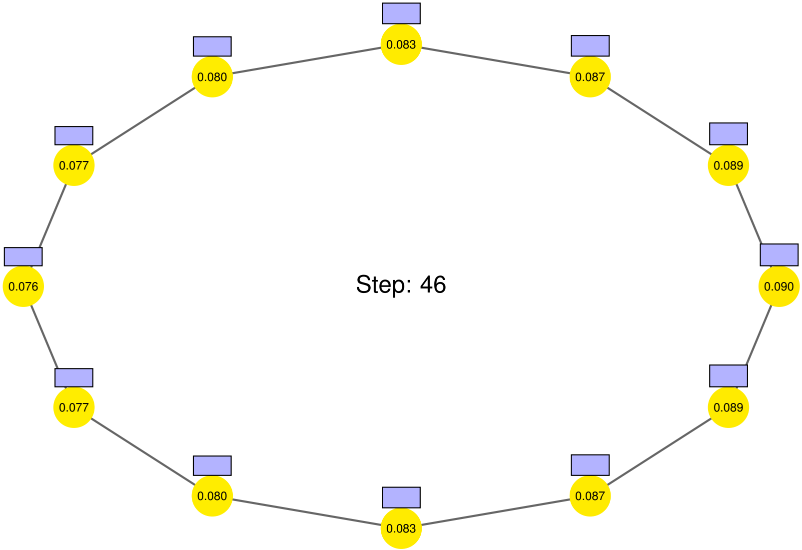

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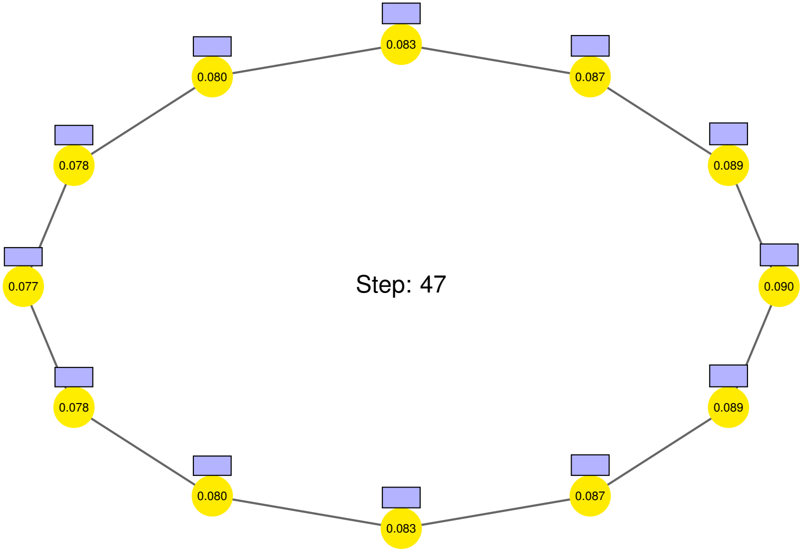

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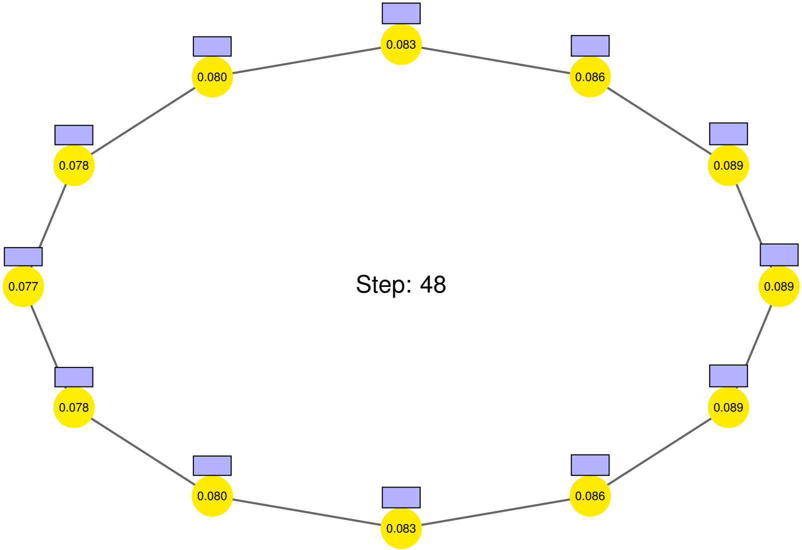

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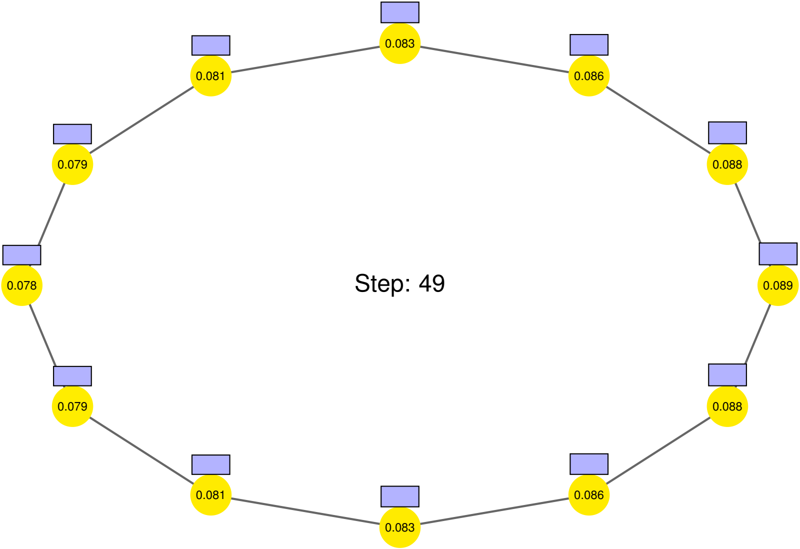

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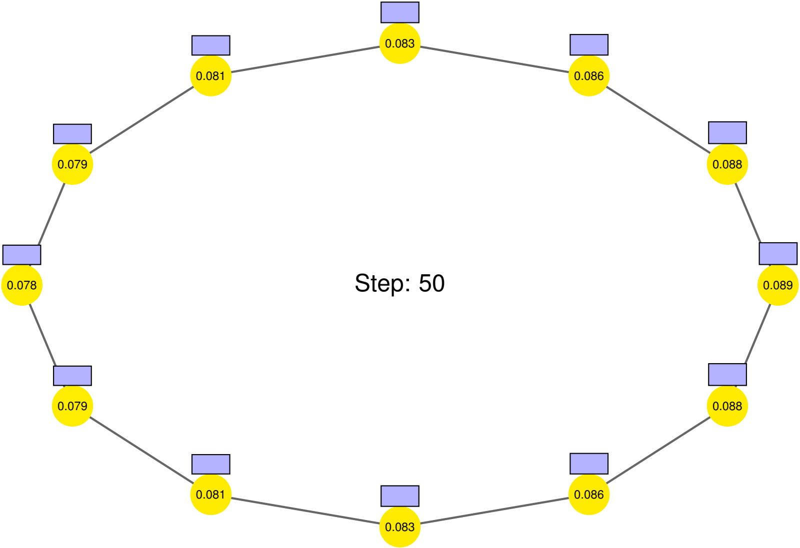

Diffusion of a Random Walk on a Static Cycle

- As long as the probability mass is concentrated on a small set of vertices, substantial progress in the ℓ_2-norm
- More precisely, $\|p_{u,.}^t - \frac{1}{n}\|_2^2 \sim 1/\sqrt{t}$
- This property only requires each graph G^t to be connected (& regular) at each time

Mixing in Dynamic Graphs: Definition

Sequence of graphs $\mathcal{G} = \{G^{(t)}\}_{t=1}^{\infty}$ on V with transition matrices $\{P^{(t)}\}_{t=1}^{\infty}$

- $\pi P^{(t)} = \pi$ for any t

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$$t_{mix}(\mathcal{G}) = \min\left\{ t \;\middle|\; \sum_{y \in V} \left(P^{[0,t]}(x,y) - \frac{1}{n} \right)^2 \leq \frac{1}{10n} \quad \forall\, x \in V \right\}.$$

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extends to non-regular in a natural way

A Bound on the ℓ_2-Decrease

Key Lemma

Let P be the transition matrix of a random walk on a connected, regular graph $G = (V, E)$. Then for any probability distribution σ,

$$\sum_{u,v \in V} (\sigma(u) - \sigma(v))^2 \cdot P_{u,v} \gtrsim \sum_{u \in V} \left(\sigma(u) - \frac{1}{n} \right)^2 .$$

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$\Rightarrow$ Let ℓ be the length of such path. Then,

$$\sum_{u,v \in V} (\sigma(u) - \sigma(v))^2 P_{u,v} \geq \frac{(\sigma(x^\star) - \sigma(y))^2}{2\ell} \text{ is large} \qquad \square$$

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- solves a conjecture by Aldous and Fill, which was proved by Benjamini and Kozma (Combinatorica'05) for static graphs

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Outline

Intro

Random Walks on Sequences of Connected Graphs

Random Walks on Sequences of (Possibly) Disconnected Graphs

Conclusion

What happens when the connectivity properties of the graph change over time?

How to bound mixing when connectivity is intermittent

- In static graphs, the eigenvalues of the individual transition matrices give a good bound on mixing:

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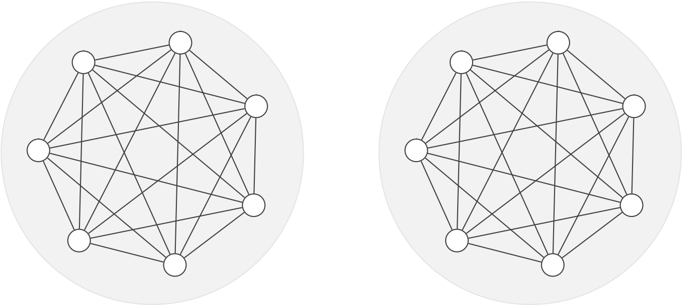

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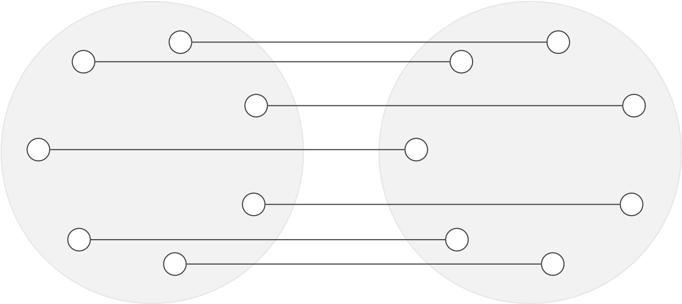

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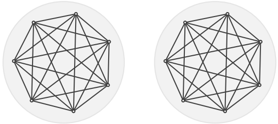

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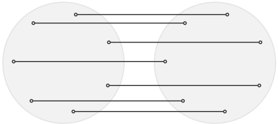

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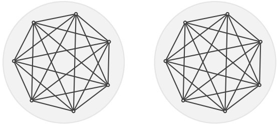

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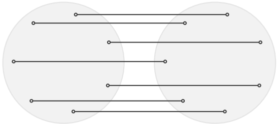

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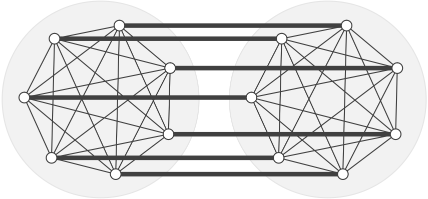

$1 - \lambda(\overline{P}) = \Omega(1)$

Mixing based on average connectivity properties

Theorem

Consider a sequence $\mathcal{G}$ with transition matrices $\{P^{(t)}\}_{t=1}^{\infty}$ such that

1. $\pi P^{(t)} = \pi$ for any t
2. there exists a time window $T \geq 1$ such that, for any $i \geq 0$, $\overline{P}^{[i \cdot T+1,(i+1) \cdot T]}$ is ergodic with spectral gap greater or equal than $1 - \lambda$

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Suppose that for any time window $\mathcal{I} = [i \cdot T + 1, (i+1) \cdot T]$ and any subset of vertices $A \subseteq V$ there exists $i \in \mathcal{I}$ such that $\Phi_{P(i)}(A) \geq \phi$. Then,

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Since $t_{hit}(\mathcal{G}) = O(t_{mix}(\mathcal{G})/\pi_*)$, does polynomial mixing time imply polynomial hitting times?

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Then, $t_{mix}(\mathcal{G}) = O(T^2 \log(1/\pi_*)/(1-\lambda))$

Corollary

Suppose that for any time window $\mathcal{I} = [i \cdot T + 1, (i+1) \cdot T]$ and any subset of vertices $A \subseteq V$ there exists $i \in \mathcal{I}$ such that $\Phi_{P(i)}(A) \geq \phi$. Then,

$$t_{mix}(\mathcal{G}) = O(T^3 \log(1/\pi_*)/\phi^2)$$

Since $t_{hit}(\mathcal{G}) = O(t_{mix}(\mathcal{G})/\pi_*)$, does polynomial mixing time imply polynomial hitting times?

- NO! When the graphs are disconnected, π_* can be exponentially small

Mixing based on average connectivity properties

Theorem

Consider a sequence $\mathcal{G}$ with transition matrices $\{P^{(t)}\}_{t=1}^{\infty}$ such that

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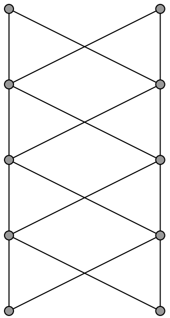
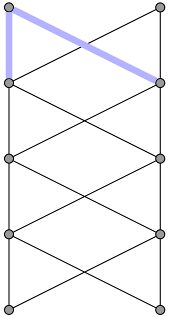
Since $t_{hit}(\mathcal{G}) = O(t_{mix}(\mathcal{G})/\pi_*)$, does polynomial mixing time imply polynomial hitting times?

- NO! When the graphs are disconnected, π_* can be exponentially small
- Why? We can simulate a random walk on a directed graph:

Simulating a Directed Graph using Dynamic Graphs

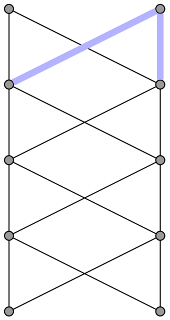


Simulating a Directed Graph using Dynamic Graphs



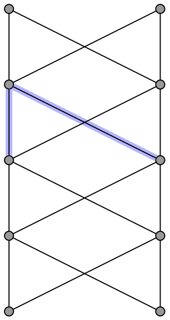
$t = 1$

Simulating a Directed Graph using Dynamic Graphs



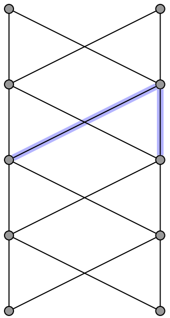
$t = 2$

Simulating a Directed Graph using Dynamic Graphs



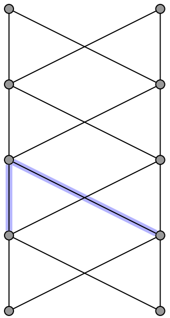
$t = 3$

Simulating a Directed Graph using Dynamic Graphs



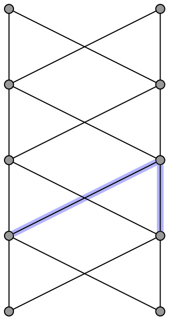
$t = 4$

Simulating a Directed Graph using Dynamic Graphs



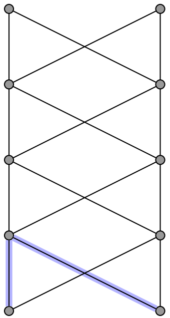
$t = 5$

Simulating a Directed Graph using Dynamic Graphs



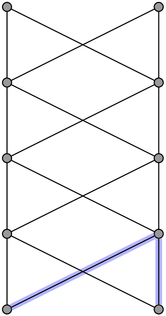
$t = 6$

Simulating a Directed Graph using Dynamic Graphs



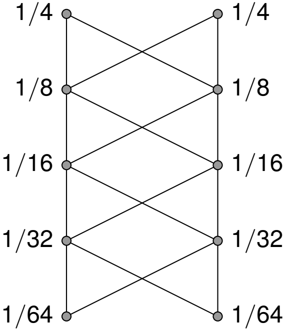
$t = 7$

Simulating a Directed Graph using Dynamic Graphs



$t = 8$

Simulating a Directed Graph using Dynamic Graphs



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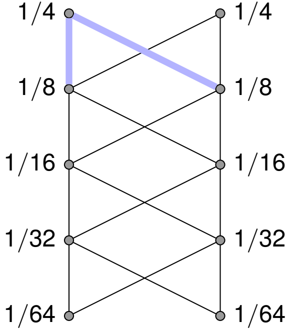

$t = 1$

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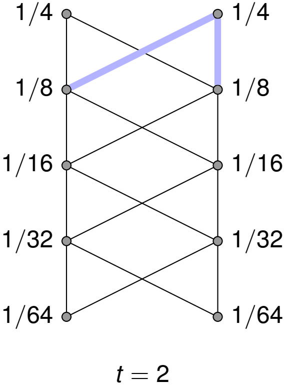

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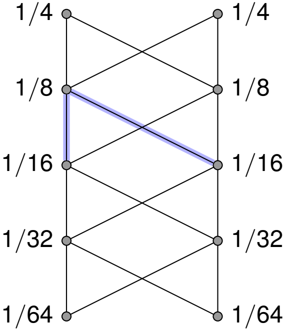

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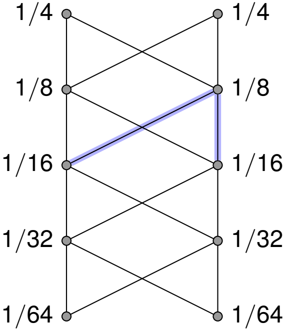

$t = 4$

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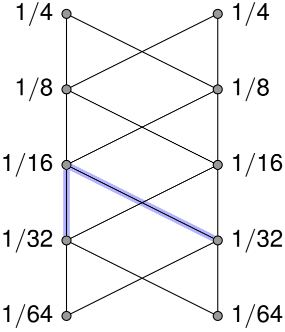

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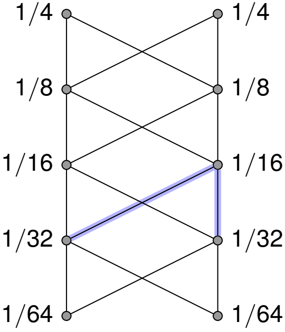

$t = 6$

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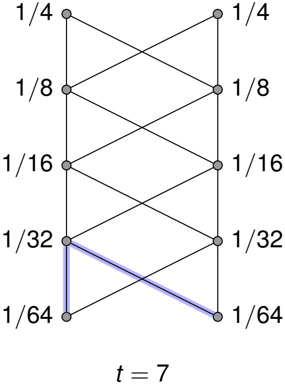

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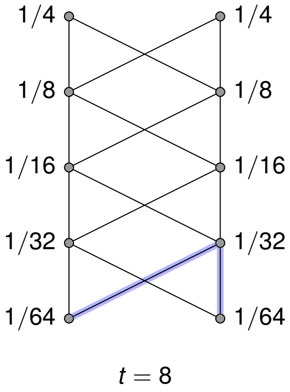

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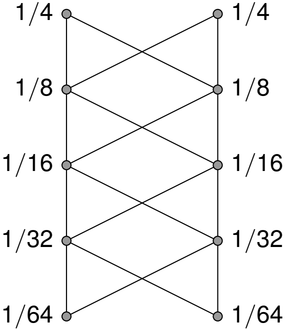
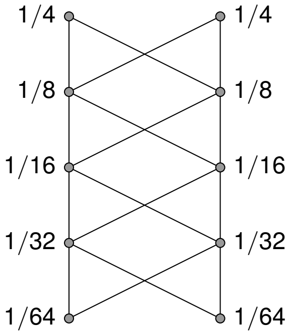


Random Walk Behaviour:

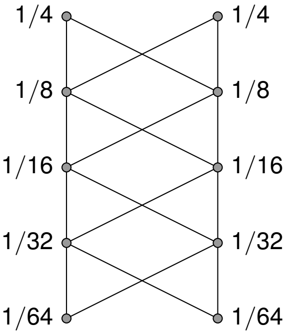
Simulating a Directed Graph using Dynamic Graphs



Random Walk Behaviour:

- Since the stationary distribution is exponentially small for the vertices at the bottom, hitting time is exponential in n

Simulating a Directed Graph using Dynamic Graphs



Random Walk Behaviour:

- Since the stationary distribution is exponentially small for the vertices at the bottom, hitting time is exponential in n
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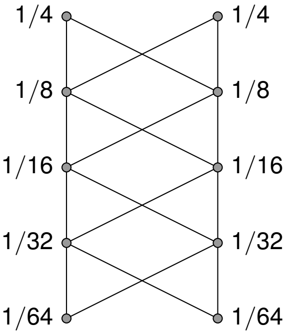
Simulating a Directed Graph using Dynamic Graphs



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- $\Rightarrow$ mixing time polynomial in n by our theorem!

Outline

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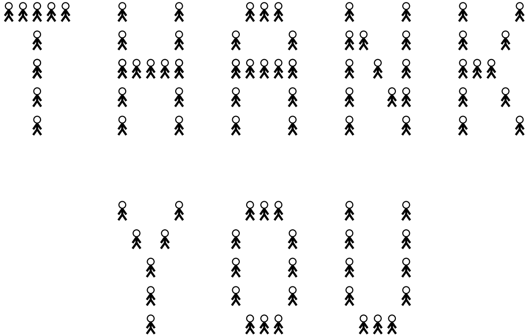
- Can our methods be applied to settings where the graph changes randomly?

The End



The End