When is Coalescing as fast as Meeting?

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Outline

Introduction

Relating Coalescing Time to the Mixing and Meeting Time

Conclusion
Random Walk Notation

- **$P$ transition matrix** of a lazy walk on an undirected, connected graph $G$

  \[ p_{u,v} = \begin{cases} 
  \frac{1}{2} & \text{if } u = v, \\
  \frac{1}{2 \deg(u)} & \text{if } \{u, v\} \in E(G), \\
  0 & \text{otherwise.} 
  \end{cases} \]

- **$\pi$** with $\pi_v = \frac{\deg(v)}{2|E|}$ is the **stationary distribution**
Random Walk Notation

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**Fundamental Quantities**

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Random Walk Notation

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**Fundamental Quantities**

- **mixing time**: $t_{\text{mix}}\left(\frac{1}{e}\right) = \min\{t \in \mathbb{N}: \forall u \in V: \frac{1}{2} \sum_{v \in V} |p_{u,v}^t - \pi_v| \leq \frac{1}{e}\}$
Random Walk Notation

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- (maximum) hitting time: $t_{\text{hit}} = \max_{u,v \in V} E_u \left[ \min \{ t : X_t = v \} \right]$
Random Walk Notation

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**Focus of this talk**

- **meeting time:** $t_{\text{meet}} = \max_{u,v \in V} E_{u,v} \left[ \min \{ t: X_t = Y_t \} \right]$

- **coalescing time:** $t_{\text{coal}} = E_{1,2,\ldots,n} \left[ \ldots \right]$
Coalescing Random Walks (Example)

Time: 0
Particles: 16

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Particles: 16

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Coalescing Random Walks (Example)

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Particles: 1

Time: 27.5
Particles: 1

Time: 28
Particles: 1

Particles: 10

Time: 2.25

Coalescing Random Walks (Example)

Time: 0
Particles: 16

Time: 0.25
Particles: 16

Time: 0.5
Particles: 16

Time: 0.75
Particles: 16

Time: 1
Particles: 12

Time: 1.25
Particles: 12

Time: 1.5
Particles: 12

Time: 1.75
Particles: 12

Time: 2
Particles: 10

Time: 2.25
Particles: 10

Time: 2.5
Particles: 10

Time: 2.75
Particles: 10

Time: 3
Particles: 7

Time: 3.25
Particles: 7

Time: 3.5
Particles: 7

Time: 3.75
Particles: 7

Time: 4
Particles: 6

Time: 4.25
Particles: 6

Time: 4.5
Particles: 6

Time: 4.75
Particles: 6

Time: 5
Particles: 6

Time: 5.25
Particles: 6

Time: 5.5
Particles: 6

Time: 5.75
Particles: 6

Time: 6
Particles: 5

Time: 6.5
Particles: 5

Time: 7
Particles: 4

Time: 7.5
Particles: 4

Time: 8
Particles: 3

Time: 8.5
Particles: 3

Time: 9
Particles: 3

Time: 9.5
Particles: 3

Time: 10
Particles: 3

Time: 10.5
Particles: 3

Time: 11
Particles: 3

Time: 11.5
Particles: 3

Time: 12
Particles: 2

Time: 12.5
Particles: 2

Time: 13
Particles: 2

Time: 13.5
Particles: 2

Time: 14
Particles: 2

Time: 14.5
Particles: 2

Time: 15
Particles: 2

Time: 15.5
Particles: 2

Time: 16
Particles: 2

Time: 16.5
Particles: 2

Time: 17
Particles: 2

Time: 17.5
Particles: 2

Time: 18
Particles: 2

Time: 18.5
Particles: 2

Time: 19
Particles: 2

Time: 19.5
Particles: 2

Time: 20
Particles: 2

Time: 20.5
Particles: 2

Time: 21
Particles: 2

Time: 21.5
Particles: 2

Time: 22
Particles: 2

Time: 22.5
Particles: 2

Time: 23
Particles: 2

Time: 23.5
Particles: 2

Time: 24
Particles: 2

Time: 24.5
Particles: 2

Time: 25
Particles: 1

Time: 25
Particles: 1

Time: 25
Particles: 1

Time: 25.5
Particles: 1

Time: 26
Particles: 1

Time: 26.5
Particles: 1

Time: 27
Particles: 1

Time: 27.5
Particles: 1

Time: 28
Particles: 1
Coalescing Random Walks (Example)

Time: 0
Particles: 16

Time: 0.25
Particles: 16

Time: 0.5
Particles: 16

Time: 0.75
Particles: 16

Time: 1
Particles: 12

Time: 1.25
Particles: 12

Time: 1.5
Particles: 12

Time: 1.75
Particles: 12

Time: 2
Particles: 10

Time: 2.25
Particles: 10

Time: 2.5
Particles: 10

Time: 2.75
Particles: 10

Time: 3
Particles: 7

Time: 3.25
Particles: 7

Time: 3.5
Particles: 7

Time: 3.75
Particles: 7

Time: 4
Particles: 6

Time: 4.25
Particles: 6

Time: 4.5
Particles: 6

Time: 4.75
Particles: 6

Time: 5
Particles: 6

Time: 5.25
Particles: 6

Time: 5.5
Particles: 6

Time: 5.75
Particles: 6

Time: 6
Particles: 5

Time: 6.5
Particles: 5

Time: 7
Particles: 4

Time: 7.5
Particles: 4

Time: 8
Particles: 3

Time: 8.5
Particles: 3

Time: 9
Particles: 3

Time: 9.5
Particles: 3

Time: 10
Particles: 3

Time: 10.5
Particles: 3

Time: 11
Particles: 3

Time: 11.5
Particles: 3

Time: 12
Particles: 2

Time: 12.5
Particles: 2

Time: 13
Particles: 2

Time: 13.5
Particles: 2

Time: 14
Particles: 2

Time: 14.5
Particles: 2

Time: 15
Particles: 2

Time: 15.5
Particles: 2

Time: 16
Particles: 2

Time: 16.5
Particles: 2

Time: 17
Particles: 2

Time: 17.5
Particles: 2

Time: 18
Particles: 2

Time: 18.5
Particles: 2

Time: 19
Particles: 2

Time: 19.5
Particles: 2

Time: 20
Particles: 2

Time: 20.5
Particles: 2

Time: 21
Particles: 2

Time: 21.5
Particles: 2

Time: 22
Particles: 2

Time: 22.5
Particles: 2

Time: 23
Particles: 2

Time: 23.5
Particles: 2

Time: 24
Particles: 2

Time: 24.5
Particles: 2

Time: 25
Particles: 1

Time: 25
Particles: 1

Time: 25
Particles: 1

Time: 25.5
Particles: 1

Time: 26
Particles: 1

Time: 26.5
Particles: 1

Time: 27
Particles: 1

Time: 27.5
Particles: 1

Time: 28
Particles: 1
Coalescing Random Walks (Example)

Time: 0
Particles: 16

Time: 0.25
Particles: 16

Time: 0.5
Particles: 16

Time: 0.75
Particles: 16

Time: 1
Particles: 12

Time: 1.25
Particles: 12

Time: 1.5
Particles: 12

Time: 1.75
Particles: 12

Time: 2
Particles: 10

Time: 2.25
Particles: 10

Time: 2.5
Particles: 10

Time: 2.75
Particles: 10

Time: 3
Particles: 7

Time: 3.25
Particles: 7

Time: 3.5
Particles: 7

Time: 3.75
Particles: 7

Time: 4
Particles: 6

Time: 4.25
Particles: 6

Time: 4.5
Particles: 6

Time: 4.75
Particles: 6

Time: 5
Particles: 6

Time: 5.25
Particles: 6

Time: 5.5
Particles: 6

Time: 5.75
Particles: 6

Time: 6
Particles: 5

Time: 6.5
Particles: 5

Time: 7
Particles: 4

Time: 7.5
Particles: 4

Time: 8
Particles: 3

Time: 8.5
Particles: 3

Time: 9
Particles: 3

Time: 9.5
Particles: 3

Time: 10
Particles: 3

Time: 10.5
Particles: 3

Time: 11
Particles: 3

Time: 11.5
Particles: 3

Time: 12
Particles: 2

Time: 12.5
Particles: 2

Time: 13
Particles: 2

Time: 13.5
Particles: 2

Time: 14
Particles: 2

Time: 14.5
Particles: 2

Time: 15
Particles: 2

Time: 15.5
Particles: 2

Time: 16
Particles: 2

Time: 16.5
Particles: 2

Time: 17
Particles: 2

Time: 17.5
Particles: 2

Time: 18
Particles: 2

Time: 18.5
Particles: 2

Time: 19
Particles: 2

Time: 19.5
Particles: 2

Time: 20
Particles: 2

Time: 20.5
Particles: 2

Time: 21
Particles: 2

Time: 21.5
Particles: 2

Time: 22
Particles: 2

Time: 22.5
Particles: 2

Time: 23
Particles: 2

Time: 23.5
Particles: 2

Time: 24
Particles: 2

Time: 24.5
Particles: 2

Time: 25
Particles: 1

Time: 25
Particles: 1

Time: 25
Particles: 1

Time: 25.5
Particles: 1

Time: 26
Particles: 1

Time: 26.5
Particles: 1

Time: 27
Particles: 1

Time: 27.5
Particles: 1

Time: 28
Particles: 1
Coalescing Random Walks (Example)

Time: 3.25
Particles: 7
Coalescing Random Walks (Example)

Time: 0
Particles: 16

Time: 0.25
Particles: 16

Time: 0.5
Particles: 16

Time: 0.75
Particles: 16

Time: 1
Particles: 12

Time: 1.25
Particles: 12

Time: 1.5
Particles: 12

Time: 1.75
Particles: 12

Time: 2
Particles: 10

Time: 2.25
Particles: 10

Time: 2.5
Particles: 10

Time: 2.75
Particles: 10

Time: 3
Particles: 7

Time: 3.25
Particles: 7

Time: 3.5
Particles: 7

Time: 3.75
Particles: 7

Time: 4
Particles: 6

Time: 4.25
Particles: 6

Time: 4.5
Particles: 6

Time: 4.75
Particles: 6

Time: 5
Particles: 6

Time: 5.25
Particles: 6

Time: 5.5
Particles: 6

Time: 5.75
Particles: 6

Time: 6
Particles: 5

Time: 6.5
Particles: 5

Time: 7
Particles: 4

Time: 7.5
Particles: 4

Time: 8
Particles: 3

Time: 8.5
Particles: 3

Time: 9
Particles: 3

Time: 9.5
Particles: 3

Time: 10
Particles: 3

Time: 10.5
Particles: 3

Time: 11
Particles: 3

Time: 11.5
Particles: 3

Time: 12
Particles: 3

Time: 12.5
Particles: 3

Time: 13
Particles: 3

Time: 13.5
Particles: 3

Time: 14
Particles: 3

Time: 14.5
Particles: 3

Time: 15
Particles: 3

Time: 15.5
Particles: 3

Time: 16
Particles: 3

Time: 16.5
Particles: 3

Time: 17
Particles: 3

Time: 17.5
Particles: 3

Time: 18
Particles: 3

Time: 18.5
Particles: 3

Time: 19
Particles: 3

Time: 19.5
Particles: 3

Time: 20
Particles: 3

Time: 20.5
Particles: 3

Time: 21
Particles: 3

Time: 21.5
Particles: 3

Time: 22
Particles: 3

Time: 22.5
Particles: 3

Time: 23
Particles: 3

Time: 23.5
Particles: 3

Time: 24
Particles: 3

Time: 24.5
Particles: 3

Time: 25
Particles: 2

Time: 25.5
Particles: 2

Time: 26
Particles: 2

Time: 26.5
Particles: 2

Time: 27
Particles: 2

Time: 27.5
Particles: 2

Time: 28
Particles: 2

Particles: 7
Coalescing Random Walks (Example)

Time: 0
Particles: 16

Time: 0.25
Particles: 16

Time: 0.5
Particles: 16

Time: 0.75
Particles: 16

Time: 1
Particles: 12

Time: 1.25
Particles: 12

Time: 1.5
Particles: 12

Time: 1.75
Particles: 12

Time: 2
Particles: 10

Time: 2.25
Particles: 10

Time: 2.5
Particles: 10

Time: 2.75
Particles: 10

Time: 3
Particles: 7

Time: 3.25
Particles: 7

Time: 3.5
Particles: 7

Time: 3.75
Particles: 7

Time: 4
Particles: 6

Time: 4.25
Particles: 6

Time: 4.5
Particles: 6

Time: 4.75
Particles: 6

Time: 5
Particles: 6

Time: 5.25
Particles: 6

Time: 5.5
Particles: 6

Time: 5.75
Particles: 6

Time: 6
Particles: 5

Time: 6.5
Particles: 5

Time: 7
Particles: 4

Time: 7.5
Particles: 4

Time: 8
Particles: 3

Time: 8.5
Particles: 3

Time: 9
Particles: 3

Time: 9.5
Particles: 3

Time: 10
Particles: 3

Time: 10.5
Particles: 3

Time: 11
Particles: 3

Time: 11.5
Particles: 3

Time: 12
Particles: 2

Time: 12.5
Particles: 2

Time: 13
Particles: 2

Time: 13.5
Particles: 2

Time: 14
Particles: 2

Time: 14.5
Particles: 2

Time: 15
Particles: 2

Time: 15.5
Particles: 2

Time: 16
Particles: 2

Time: 16.5
Particles: 2

Time: 17
Particles: 2

Time: 17.5
Particles: 2

Time: 18
Particles: 2

Time: 18.5
Particles: 2

Time: 19
Particles: 2

Time: 19.5
Particles: 2

Time: 20
Particles: 2

Time: 20.5
Particles: 2

Time: 21
Particles: 2

Time: 21.5
Particles: 2

Time: 22
Particles: 2

Time: 22.5
Particles: 2

Time: 23
Particles: 2

Time: 23.5
Particles: 2

Time: 24
Particles: 2

Time: 24.5
Particles: 2

Time: 25
Particles: 1

Time: 25.5
Particles: 1

Time: 26
Particles: 1

Time: 26.5
Particles: 1

Time: 27
Particles: 1

Time: 27.5
Particles: 1

Time: 28
Particles: 1

Time: 25
Particles: 1

Time: 25.5
Particles: 1

Time: 26
Particles: 1

Time: 26.5
Particles: 1

Time: 27
Particles: 1

Time: 27.5
Particles: 1

Time: 28
Particles: 1

Particles: 7

Time: 3.75
Coalescing Random Walks (Example)

Time: 0
Particles: 16

Time: 0.25
Particles: 16

Time: 0.5
Particles: 16

Time: 0.75
Particles: 16

Time: 1
Particles: 12

Time: 1.25
Particles: 12

Time: 1.5
Particles: 12

Time: 1.75
Particles: 12

Time: 2
Particles: 10

Time: 2.25
Particles: 10

Time: 2.5
Particles: 10

Time: 2.75
Particles: 10

Time: 3
Particles: 7

Time: 3.25
Particles: 7

Time: 3.5
Particles: 7

Time: 3.75
Particles: 7

Time: 4
Particles: 6

Time: 4.25
Particles: 6

Time: 4.5
Particles: 6

Time: 4.75
Particles: 6

Time: 5
Particles: 6

Time: 5.25
Particles: 6

Time: 5.5
Particles: 6

Time: 5.75
Particles: 6

Time: 6
Particles: 5

Time: 6.5
Particles: 5

Time: 7
Particles: 4

Time: 7.5
Particles: 4

Time: 8
Particles: 3

Time: 8.5
Particles: 3

Time: 9
Particles: 3

Time: 9.5
Particles: 3

Time: 10
Particles: 3

Time: 10.5
Particles: 3

Time: 11
Particles: 3

Time: 11.5
Particles: 3

Time: 12
Particles: 2

Time: 12.5
Particles: 2

Time: 13
Particles: 2

Time: 13.5
Particles: 2

Time: 14
Particles: 2

Time: 14.5
Particles: 2

Time: 15
Particles: 2

Time: 15.5
Particles: 2

Time: 16
Particles: 2

Time: 16.5
Particles: 2

Time: 17
Particles: 2

Time: 17.5
Particles: 2

Time: 18
Particles: 2

Time: 18.5
Particles: 2

Time: 19
Particles: 2

Time: 19.5
Particles: 2

Time: 20
Particles: 2

Time: 20.5
Particles: 2

Time: 21
Particles: 2

Time: 21.5
Particles: 2

Time: 22
Particles: 2

Time: 22.5
Particles: 2

Time: 23
Particles: 2

Time: 23.5
Particles: 2

Time: 24
Particles: 2

Time: 24.5
Particles: 2

Time: 25
Particles: 1

Time: 25.5
Particles: 1

Time: 26
Particles: 1

Time: 26.5
Particles: 1

Time: 27
Particles: 1

Time: 27.5
Particles: 1

Time: 28
Particles: 1
Coalescing Random Walks (Example)

Time: 0
Particles: 16

Time: 0.25
Particles: 16

Time: 0.5
Particles: 16

Time: 0.75
Particles: 16

Time: 1
Particles: 12

Time: 1.25
Particles: 12

Time: 1.5
Particles: 12

Time: 1.75
Particles: 12

Time: 2
Particles: 10

Time: 2.25
Particles: 10

Time: 2.5
Particles: 10

Time: 2.75
Particles: 10

Time: 3
Particles: 7

Time: 3.25
Particles: 7

Time: 3.5
Particles: 7

Time: 3.75
Particles: 7

Time: 4
Particles: 6

Time: 4.25
Particles: 6

Time: 4.5
Particles: 6

Time: 4.75
Particles: 6

Time: 5
Particles: 6

Time: 5.25
Particles: 6

Time: 5.5
Particles: 6

Time: 5.75
Particles: 6

Time: 6
Particles: 5

Time: 6.5
Particles: 5

Time: 7
Particles: 4

Time: 7.5
Particles: 4

Time: 8
Particles: 3

Time: 8.5
Particles: 3

Time: 9
Particles: 3

Time: 9.5
Particles: 3

Time: 10
Particles: 3

Time: 10.5
Particles: 3

Time: 11
Particles: 3

Time: 11.5
Particles: 3

Time: 12
Particles: 2

Time: 12.5
Particles: 2

Time: 13
Particles: 2

Time: 13.5
Particles: 2

Time: 14
Particles: 2

Time: 14.5
Particles: 2

Time: 15
Particles: 2

Time: 15.5
Particles: 2

Time: 16
Particles: 2

Time: 16.5
Particles: 2

Time: 17
Particles: 2

Time: 17.5
Particles: 2

Time: 18
Particles: 2

Time: 18.5
Particles: 2

Time: 19
Particles: 2

Time: 19.5
Particles: 2

Time: 20
Particles: 2

Time: 20.5
Particles: 2

Time: 21
Particles: 2

Time: 21.5
Particles: 2

Time: 22
Particles: 2

Time: 22.5
Particles: 2

Time: 23
Particles: 2

Time: 23.5
Particles: 2

Time: 24
Particles: 2

Time: 24.5
Particles: 2

Time: 25
Particles: 1

Time: 25
Particles: 1

Time: 25.5
Particles: 1

Time: 26
Particles: 1

Time: 26.5
Particles: 1

Time: 27
Particles: 1
Coalescing Random Walks (Example)

Time: 0
Particles: 16

Time: 0.25
Particles: 16

Time: 0.5
Particles: 16

Time: 0.75
Particles: 16

Time: 1
Particles: 12

Time: 1.25
Particles: 12

Time: 1.5
Particles: 12

Time: 1.75
Particles: 12

Time: 2
Particles: 10

Time: 2.25
Particles: 10

Time: 2.5
Particles: 10

Time: 2.75
Particles: 10

Time: 3
Particles: 7

Time: 3.25
Particles: 7

Time: 3.5
Particles: 7

Time: 3.75
Particles: 7

Time: 4
Particles: 6

Time: 4.25
Particles: 6

Time: 4.5
Particles: 6

Time: 4.75
Particles: 6

Time: 5
Particles: 6

Time: 5.25
Particles: 6

Time: 5.5
Particles: 6

Time: 5.75
Particles: 6

Time: 6
Particles: 5

Time: 6.5
Particles: 5

Time: 7
Particles: 4

Time: 7.5
Particles: 4

Time: 8
Particles: 3

Time: 8.5
Particles: 3

Time: 9
Particles: 3

Time: 9.5
Particles: 3

Time: 10
Particles: 3

Time: 10.5
Particles: 3

Time: 11
Particles: 3

Time: 11.5
Particles: 3

Time: 12
Particles: 2

Time: 12.5
Particles: 2

Time: 13
Particles: 2

Time: 13.5
Particles: 2

Time: 14
Particles: 2

Time: 14.5
Particles: 2

Time: 15
Particles: 2

Time: 15.5
Particles: 2

Time: 16
Particles: 2

Time: 16.5
Particles: 2

Time: 17
Particles: 2

Time: 17.5
Particles: 2

Time: 18
Particles: 2

Time: 18.5
Particles: 2

Time: 19
Particles: 2

Time: 19.5
Particles: 2

Time: 20
Particles: 2

Time: 20.5
Particles: 2

Time: 21
Particles: 2

Time: 21.5
Particles: 2

Time: 22
Particles: 2

Time: 22.5
Particles: 2

Time: 23
Particles: 2

Time: 23.5
Particles: 2

Time: 24
Particles: 2

Time: 24.5
Particles: 2

Time: 25
Particles: 1

Time: 25
Particles: 1

Time: 25
Particles: 1

Time: 25.5
Particles: 1

Time: 26
Particles: 1

Time: 26.5
Particles: 1

Time: 27
Particles: 1

Time: 27.5
Particles: 1

Time: 28
Particles: 1

Particles: 6
Coalescing Random Walks (Example)

Time: 0
Particles: 16

Time: 0.25
Particles: 16

Time: 0.5
Particles: 16

Time: 0.75
Particles: 16

Time: 1
Particles: 12

Time: 1.25
Particles: 12

Time: 1.5
Particles: 12

Time: 1.75
Particles: 12

Time: 2
Particles: 10

Time: 2.25
Particles: 10

Time: 2.5
Particles: 10

Time: 2.75
Particles: 10

Time: 3
Particles: 7

Time: 3.25
Particles: 7

Time: 3.5
Particles: 7

Time: 3.75
Particles: 7

Time: 4
Particles: 6

Time: 4.25
Particles: 6

Time: 4.5
Particles: 6

Time: 4.75
Particles: 6

Time: 5
Particles: 6

Time: 5.25
Particles: 6

Time: 5.5
Particles: 6

Time: 5.75
Particles: 6

Time: 6
Particles: 5

Time: 6.5
Particles: 5

Time: 7
Particles: 4

Time: 7.5
Particles: 4

Time: 8
Particles: 3

Time: 8.5
Particles: 3

Time: 9
Particles: 3

Time: 9.5
Particles: 3

Time: 10
Particles: 3

Time: 10.5
Particles: 3

Time: 11
Particles: 3

Time: 11.5
Particles: 3

Time: 12
Particles: 2

Time: 12.5
Particles: 2

Time: 13
Particles: 2

Time: 13.5
Particles: 2

Time: 14
Particles: 2

Time: 14.5
Particles: 2

Time: 15
Particles: 2

Time: 15.5
Particles: 2

Time: 16
Particles: 2

Time: 16.5
Particles: 2

Time: 17
Particles: 2

Time: 17.5
Particles: 2

Time: 18
Particles: 2

Time: 18.5
Particles: 2

Time: 19
Particles: 2

Time: 19.5
Particles: 2

Time: 20
Particles: 2

Time: 20.5
Particles: 2

Time: 21
Particles: 2

Time: 21.5
Particles: 2

Time: 22
Particles: 2

Time: 22.5
Particles: 2

Time: 23
Particles: 2

Time: 23.5
Particles: 2

Time: 24
Particles: 2

Time: 24.5
Particles: 2

Time: 25
Particles: 1

Time: 25.5
Particles: 1

Time: 26
Particles: 1

Time: 26.5
Particles: 1

Time: 27
Particles: 1

Time: 27.5
Particles: 1

Time: 28
Particles: 1

Time: 4.75
Particles: 6

Time: 25
Particles: 1
Coalescing Random Walks (Example)
Coalescing Random Walks (Example)

Time: 0
Particles: 16

Time: 0.25
Particles: 16

Time: 0.5
Particles: 16

Time: 0.75
Particles: 16

Time: 1
Particles: 12

Time: 1.25
Particles: 12

Time: 1.5
Particles: 12

Time: 1.75
Particles: 12

Time: 2
Particles: 10

Time: 2.25
Particles: 10

Time: 2.5
Particles: 10

Time: 2.75
Particles: 10

Time: 3
Particles: 7

Time: 3.25
Particles: 7

Time: 3.5
Particles: 7

Time: 3.75
Particles: 7

Time: 4
Particles: 6

Time: 4.25
Particles: 6

Time: 4.5
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Time: 4.75
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Coalescing Random Walks (Example)

Time: 0
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Time: 0.75
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Time: 1
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Particles: 12

Time: 1.5
Particles: 12

Time: 1.75
Particles: 12

Time: 2
Particles: 10

Time: 2.25
Particles: 10

Time: 2.5
Particles: 10

Time: 2.75
Particles: 10

Time: 3
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Particles: 7

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Particles: 7

Time: 3.75
Particles: 7

Time: 4
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Particles: 6

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Particles: 6

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Coalescing Random Walks (Example)

Time: 5.75
Particles: 6
Coalescing Random Walks (Example)

Time: 0
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Coalescing Random Walks (Example)

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Coalescing Random Walks (Example)

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Coalescing Random Walks (Example)

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Coalescing Random Walks (Example)

Time: 0
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Coalescing Random Walks (Example)
Coalescing Random Walks (Example)

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Coalescing Random Walks (Example)

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Coalescing Random Walks (Example)

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Coalescing Random Walks (Example)

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Coalescing Random Walks (Example)
Coalescing Random Walks (Example)

Time: 11.5

Particles: 3
Coalescing Random Walks (Example)

Time: 0
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Particles: 7

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Particles: 7

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Particles: 7

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Particles: 6

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Coalescing Random Walks (Example)

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Coalescing Random Walks (Example)

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Coalescing Random Walks (Example)

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Coalescing Random Walks (Example)

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Coalescing Random Walks (Example)

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Coalescing Random Walks (Example)

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Coalescing Random Walks (Example)

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Coalescing Random Walks (Example)

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Time: 27
Particles: 1

Time: 27.5
Particles: 1

Time: 28
Particles: 1
Coalescing Random Walks (Example)

Time: 0
Particles: 16

Time: 0.25
Particles: 16

Time: 0.5
Particles: 16

Time: 0.75
Particles: 16

Time: 1
Particles: 12

Time: 1.25
Particles: 12

Time: 1.5
Particles: 12

Time: 1.75
Particles: 12

Time: 2
Particles: 10

Time: 2.25
Particles: 10

Time: 2.5
Particles: 10

Time: 2.75
Particles: 10

Time: 3
Particles: 7

Time: 3.25
Particles: 7

Time: 3.5
Particles: 7

Time: 3.75
Particles: 7

Time: 4
Particles: 6

Time: 4.25
Particles: 6

Time: 4.5
Particles: 6

Time: 4.75
Particles: 6

Time: 5
Particles: 6

Time: 5.25
Particles: 6

Time: 5.5
Particles: 6

Time: 5.75
Particles: 6

Time: 6
Particles: 5

Time: 6.5
Particles: 5

Time: 7
Particles: 4

Time: 7.5
Particles: 4

Time: 8
Particles: 3

Time: 8.5
Particles: 3

Time: 9
Particles: 3

Time: 9.5
Particles: 3

Time: 10
Particles: 3

Time: 10.5
Particles: 3

Time: 11
Particles: 3

Time: 11.5
Particles: 3

Time: 12
Particles: 2

Time: 12.5
Particles: 2

Time: 13
Particles: 2

Time: 13.5
Particles: 2

Time: 14
Particles: 2

Time: 14.5
Particles: 2

Time: 15
Particles: 2

Time: 15.5
Particles: 2

Time: 16
Particles: 1

Time: 16.5
Particles: 1

Time: 17
Particles: 1

Time: 17.5
Particles: 1

Time: 18
Particles: 1

Time: 18.5
Particles: 1

Time: 19
Particles: 1

Time: 19.5
Particles: 1

Time: 20
Particles: 1

Time: 20.5
Particles: 1

Time: 21
Particles: 1

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Time: 22
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Particles: 1

Time: 25
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Particles: 1

Time: 26
Particles: 1

Time: 26.5
Particles: 1

Time: 27
Particles: 1

Time: 27.5
Particles: 1

Time: 28
Particles: 1

Time: 16.5
Particles: 2

Time: 16.5
Particles: 2
Coalescing Random Walks (Example)

Time: 0
Particles: 16

Time: 0.25
Particles: 16

Time: 0.5
Particles: 16

Time: 0.75
Particles: 16

Time: 1
Particles: 12

Time: 1.25
Particles: 12

Time: 1.5
Particles: 12

Time: 1.75
Particles: 12

Time: 2
Particles: 10

Time: 2.25
Particles: 10

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Particles: 10

Time: 2.75
Particles: 10

Time: 3
Particles: 7

Time: 3.25
Particles: 7

Time: 3.5
Particles: 7

Time: 3.75
Particles: 7

Time: 4
Particles: 6

Time: 4.25
Particles: 6

Time: 4.5
Particles: 6

Time: 4.75
Particles: 6

Time: 5
Particles: 6

Time: 5.25
Particles: 6

Time: 5.5
Particles: 6

Time: 5.75
Particles: 6

Time: 6
Particles: 5

Time: 6.5
Particles: 5

Time: 7
Particles: 4

Time: 7.5
Particles: 4

Time: 8
Particles: 3

Time: 8.5
Particles: 3

Time: 9
Particles: 3

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Particles: 3

Time: 10
Particles: 3

Time: 10.5
Particles: 3

Time: 11
Particles: 3

Time: 11.5
Particles: 3

Time: 12
Particles: 2

Time: 12.5
Particles: 2

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Time: 25.5
Particles: 1

Time: 26
Particles: 1

Time: 26.5
Particles: 1

Time: 27
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Time: 27.5
Particles: 1

Time: 28
Particles: 1
Coalescing Random Walks (Example)

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Time: 0.5
Particles: 16

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Particles: 16

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Time: 1.75
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Particles: 10

Time: 2.75
Particles: 10

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Particles: 7

Time: 3.5
Particles: 7

Time: 3.75
Particles: 7

Time: 4
Particles: 6

Time: 4.25
Particles: 6

Time: 4.5
Particles: 6

Time: 4.75
Particles: 6

Time: 5
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Time: 5.25
Particles: 6

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Particles: 6

Time: 5.75
Particles: 6

Time: 6
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Particles: 5

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Particles: 4

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Particles: 3

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Time: 10
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Particles: 3

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Particles: 3

Time: 11.5
Particles: 3

Time: 12
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Particles: 2

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Particles: 2

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Time: 22
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Particles: 2

Time: 23
Particles: 2

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Particles: 2

Time: 24
Particles: 2

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Particles: 2

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Particles: 1

Time: 25.5
Particles: 1
Coalescing Random Walks (Example)

Particles: 16

Time: 0

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Particles: 16

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Particles: 16

Time: 0.75

Particles: 12

Time: 1

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Time: 1.75

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Time: 2

Particles: 7

Time: 2.25

Particles: 7

Time: 2.5

Particles: 7

Time: 2.75

Particles: 7

Time: 3

Particles: 6

Time: 3.25

Particles: 6

Time: 3.5

Particles: 6

Time: 3.75

Particles: 6

Time: 4

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Time: 4.25

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Time: 4.75

Particles: 6

Time: 5

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Time: 5.25

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Time: 9

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Particles: 3

Time: 10

Particles: 3

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Particles: 3

Time: 11

Particles: 3

Time: 11.5

Particles: 3

Time: 12

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Time: 13

Particles: 2

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Particles: 2

Time: 24

Particles: 2

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Particles: 2

Time: 25

Particles: 1

Time: 25

Particles: 1

Time: 25

Particles: 1

Time: 25.5

Particles: 1

Time: 26

Particles: 1

Time: 26.5

Particles: 1

Time: 27

Particles: 1

Time: 27.5

Particles: 1
Coalescing Random Walks (Example)
Coalescing Random Walks (Example)

Time: 0
Particles: 16

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Particles: 16

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Particles: 16

Time: 0.75
Particles: 16

Time: 1
Particles: 12

Time: 1.25
Particles: 12

Time: 1.5
Particles: 12

Time: 1.75
Particles: 12

Time: 2
Particles: 10

Time: 2.25
Particles: 10

Time: 2.5
Particles: 10

Time: 2.75
Particles: 10

Time: 3
Particles: 7

Time: 3.25
Particles: 7

Time: 3.5
Particles: 7

Time: 3.75
Particles: 7

Time: 4
Particles: 6

Time: 4.25
Particles: 6

Time: 4.5
Particles: 6

Time: 4.75
Particles: 6

Time: 5
Particles: 6

Time: 5.25
Particles: 6

Time: 5.5
Particles: 6

Time: 5.75
Particles: 6

Time: 6
Particles: 5

Time: 6.5
Particles: 5

Time: 7
Particles: 4

Time: 7.5
Particles: 4

Time: 8
Particles: 3

Time: 8.5
Particles: 3

Time: 9
Particles: 3

Time: 9.5
Particles: 3

Time: 10
Particles: 3

Time: 10.5
Particles: 3

Time: 11
Particles: 3

Time: 11.5
Particles: 3

Time: 12
Particles: 2

Time: 12.5
Particles: 2

Time: 13
Particles: 2

Time: 13.5
Particles: 2

Time: 14
Particles: 2

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Particles: 2

Time: 15
Particles: 2

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Particles: 2

Time: 16
Particles: 2

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Particles: 2

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Particles: 2

Time: 18
Particles: 2

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Particles: 2

Time: 19
Particles: 2

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Particles: 2

Time: 20
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Time: 20.5
Particles: 2

Time: 21
Particles: 2

Time: 21.5
Particles: 2

Time: 22
Particles: 2

Time: 22.5
Particles: 2

Time: 23
Particles: 2

Time: 23.5
Particles: 2

Time: 24
Particles: 2

Time: 24.5
Particles: 2

Time: 25
Particles: 1

Time: 25.5
Particles: 1

Time: 26
Particles: 1

Time: 26.5
Particles: 1

Time: 27
Particles: 1

Time: 27.5
Particles: 1

Time: 28
Particles: 1

Time: 19
Particles: 2
Coalescing Random Walks (Example)

Time: 0
Particles: 16

Time: 0.25
Particles: 16

Time: 0.5
Particles: 16

Time: 0.75
Particles: 16

Time: 1
Particles: 12

Time: 1.25
Particles: 12

Time: 1.5
Particles: 12

Time: 1.75
Particles: 12

Time: 2
Particles: 10

Time: 2.25
Particles: 10

Time: 2.5
Particles: 10

Time: 2.75
Particles: 10

Time: 3
Particles: 7

Time: 3.25
Particles: 7

Time: 3.5
Particles: 7

Time: 3.75
Particles: 7

Time: 4
Particles: 6

Time: 4.25
Particles: 6

Time: 4.5
Particles: 6

Time: 4.75
Particles: 6

Time: 5
Particles: 6

Time: 5.25
Particles: 6

Time: 5.5
Particles: 6

Time: 5.75
Particles: 6

Time: 6
Particles: 5

Time: 6.5
Particles: 5

Time: 7
Particles: 4

Time: 7.5
Particles: 4

Time: 8
Particles: 3

Time: 8.5
Particles: 3

Time: 9
Particles: 3

Time: 9.5
Particles: 3

Time: 10
Particles: 3

Time: 10.5
Particles: 3

Time: 11
Particles: 3

Time: 11.5
Particles: 3

Time: 12
Particles: 2

Time: 12.5
Particles: 2

Time: 13
Particles: 2

Time: 13.5
Particles: 2

Time: 14
Particles: 2

Time: 14.5
Particles: 2

Time: 15
Particles: 2

Time: 15.5
Particles: 2

Time: 16
Particles: 2

Time: 16.5
Particles: 2

Time: 17
Particles: 2

Time: 17.5
Particles: 2

Time: 18
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Time: 18.5
Particles: 2

Time: 19
Particles: 2

Time: 19.5
Particles: 2

Time: 20
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Particles: 2

Time: 21
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Time: 22
Particles: 2

Time: 22.5
Particles: 2

Time: 23
Particles: 2

Time: 23.5
Particles: 2

Time: 24
Particles: 2

Time: 24.5
Particles: 2

Time: 25
Particles: 1

Time: 25.5
Particles: 1

Time: 26
Particles: 1

Time: 26.5
Particles: 1

Time: 27
Particles: 1

Time: 27.5
Particles: 1

Time: 28
Particles: 1

Time: 28.5
Particles: 1

Time: 29
Particles: 1

Time: 29.5
Particles: 1

Time: 30
Particles: 1

Time: 30.5
Particles: 1

Time: 31
Particles: 1

Time: 31.5
Particles: 1

Time: 32
Particles: 1

Time: 32.5
Particles: 1

Time: 33
Particles: 1

Time: 33.5
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Time: 34
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Time: 35
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Time: 36
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Time: 36.5
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Time: 37
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Time: 37.5
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Time: 38
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Time: 38.5
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Time: 39
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Time: 40
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Time: 41
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Time: 41.5
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Time: 42
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Time: 43
Particles: 1

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Time: 44
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Time: 45
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Time: 46
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Time: 46.5
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Time: 47
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Time: 51
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Time: 52
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Time: 53
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Time: 54
Particles: 1
Coalescing Random Walks (Example)

Time: 0
Particles: 16

Time: 0.25
Particles: 16

Time: 0.5
Particles: 16

Time: 0.75
Particles: 16

Time: 1
Particles: 12

Time: 1.25
Particles: 12

Time: 1.5
Particles: 12

Time: 1.75
Particles: 12

Time: 2
Particles: 10

Time: 2.25
Particles: 10

Time: 2.5
Particles: 10

Time: 2.75
Particles: 10

Time: 3
Particles: 7

Time: 3.25
Particles: 7

Time: 3.5
Particles: 7

Time: 3.75
Particles: 7

Time: 4
Particles: 6

Time: 4.25
Particles: 6

Time: 4.5
Particles: 6

Time: 4.75
Particles: 6

Time: 5
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Particles: 6

Time: 5.5
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Time: 5.75
Particles: 6

Time: 6
Particles: 5

Time: 6.5
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Time: 7
Particles: 4

Time: 7.5
Particles: 4

Time: 8
Particles: 3

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Time: 9
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Time: 9.5
Particles: 3

Time: 10
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Time: 10.5
Particles: 3

Time: 11
Particles: 3

Time: 11.5
Particles: 3

Time: 12
Particles: 2

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Particles: 2

Time: 13
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Time: 28
Coalescing Random Walks (Example)

Time: 20.5
Particles: 2
Coalescing Random Walks (Example)

Time: 0
Particles: 16

Time: 0.25
Particles: 16

Time: 0.5
Particles: 16

Time: 0.75
Particles: 16

Time: 1
Particles: 12

Time: 1.25
Particles: 12

Time: 1.5
Particles: 12

Time: 1.75
Particles: 12

Time: 2
Particles: 10

Time: 2.25
Particles: 10

Time: 2.5
Particles: 10

Time: 2.75
Particles: 10

Time: 3
Particles: 7

Time: 3.25
Particles: 7

Time: 3.5
Particles: 7

Time: 3.75
Particles: 7

Time: 4
Particles: 6

Time: 4.25
Particles: 6

Time: 4.5
Particles: 6

Time: 4.75
Particles: 6

Time: 5
Particles: 6

Time: 5.25
Particles: 6

Time: 5.5
Particles: 6

Time: 5.75
Particles: 6

Time: 6
Particles: 5

Time: 6.5
Particles: 5

Time: 7
Particles: 4

Time: 7.5
Particles: 4

Time: 8
Particles: 3

Time: 8.5
Particles: 3

Time: 9
Particles: 3

Time: 9.5
Particles: 3

Time: 10
Particles: 3

Time: 10.5
Particles: 3

Time: 11
Particles: 3

Time: 11.5
Particles: 3

Time: 12
Particles: 2

Time: 12.5
Particles: 2

Time: 13
Particles: 2

Time: 13.5
Particles: 2

Time: 14
Particles: 2

Time: 14.5
Particles: 2

Time: 15
Particles: 2

Time: 15.5
Particles: 2

Time: 16
Particles: 2

Time: 16.5
Particles: 2

Time: 17
Particles: 2

Time: 17.5
Particles: 2

Time: 18
Particles: 2

Time: 18.5
Particles: 2

Time: 19
Particles: 2

Time: 19.5
Particles: 2

Time: 20
Particles: 2

Time: 20.5
Particles: 2

Time: 21
Particles: 2

Time: 21.5
Particles: 2

Time: 22
Particles: 2

Time: 22.5
Particles: 2

Time: 23
Particles: 2

Time: 23.5
Particles: 2

Time: 24
Particles: 2

Time: 24.5
Particles: 2

Time: 25
Particles: 1

Time: 25.5
Particles: 1

Time: 26
Particles: 1

Time: 26.5
Particles: 1

Time: 27
Particles: 1

Time: 27.5
Particles: 1

Time: 28
Particles: 1

Particles: 2
Coalescing Random Walks (Example)

Time: 0
Particles: 16

Time: 0.25
Particles: 16

Time: 0.5
Particles: 16

Time: 0.75
Particles: 16

Time: 1
Particles: 12

Time: 1.25
Particles: 12

Time: 1.5
Particles: 12

Time: 1.75
Particles: 12

Time: 2
Particles: 10

Time: 2.25
Particles: 10

Time: 2.5
Particles: 10

Time: 2.75
Particles: 10

Time: 3
Particles: 7

Time: 3.25
Particles: 7

Time: 3.5
Particles: 7

Time: 3.75
Particles: 7

Time: 4
Particles: 6

Time: 4.25
Particles: 6

Time: 4.5
Particles: 6

Time: 4.75
Particles: 6

Time: 5
Particles: 6

Time: 5.25
Particles: 6

Time: 5.5
Particles: 6

Time: 5.75
Particles: 6

Time: 6
Particles: 5

Time: 6.5
Particles: 5

Time: 7
Particles: 4

Time: 7.5
Particles: 4

Time: 8
Particles: 3

Time: 8.5
Particles: 3

Time: 9
Particles: 3

Time: 9.5
Particles: 3

Time: 10
Particles: 3

Time: 10.5
Particles: 3

Time: 11
Particles: 3

Time: 11.5
Particles: 3

Time: 12
Particles: 2

Time: 12.5
Particles: 2

Time: 13
Particles: 2

Time: 13.5
Particles: 2

Time: 14
Particles: 2

Time: 14.5
Particles: 2

Time: 15
Particles: 2

Time: 15.5
Particles: 2

Time: 16
Particles: 2

Time: 16.5
Particles: 2

Time: 17
Particles: 2

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Particles: 2

Time: 18
Particles: 2

Time: 18.5
Particles: 2

Time: 19
Particles: 2

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Particles: 2

Time: 20
Particles: 2

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Particles: 2

Time: 21
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Time: 22
Particles: 2

Time: 22.5
Particles: 2

Time: 23
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Particles: 2

Time: 24
Particles: 2

Time: 24.5
Particles: 2

Time: 25
Particles: 1

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Particles: 1

Time: 25
Particles: 1

Time: 25.5
Particles: 1

Time: 26
Particles: 1

Time: 26.5
Particles: 1

Time: 27
Particles: 1

Time: 27.5
Particles: 1

Time: 28
Particles: 1
Coalescing Random Walks (Example)

Time: 0
Particles: 16

Time: 0.25
Particles: 16

Time: 0.5
Particles: 16

Time: 0.75
Particles: 16

Time: 1
Particles: 12

Time: 1.25
Particles: 12

Time: 1.5
Particles: 12

Time: 1.75
Particles: 12

Time: 2
Particles: 10

Time: 2.25
Particles: 10

Time: 2.5
Particles: 10

Time: 2.75
Particles: 10

Time: 3
Particles: 7

Time: 3.25
Particles: 7

Time: 3.5
Particles: 7

Time: 3.75
Particles: 7

Time: 4
Particles: 6

Time: 4.25
Particles: 6

Time: 4.5
Particles: 6

Time: 4.75
Particles: 6

Time: 5
Particles: 6

Time: 5.25
Particles: 6

Time: 5.5
Particles: 6

Time: 5.75
Particles: 6

Time: 6
Particles: 5

Time: 6.5
Particles: 5

Time: 7
Particles: 4

Time: 7.5
Particles: 4

Time: 8
Particles: 3

Time: 8.5
Particles: 3

Time: 9
Particles: 3

Time: 9.5
Particles: 3

Time: 10
Particles: 3

Time: 10.5
Particles: 3

Time: 11
Particles: 3

Time: 11.5
Particles: 3

Time: 12
Particles: 2

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Particles: 2

Time: 13
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Particles: 2

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Particles: 2

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Particles: 2

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Particles: 2

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Particles: 2

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Particles: 2

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Time: 23.5
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Time: 24
Particles: 2
Coalescing Random Walks (Example)

Time: 0
Particles: 16

Time: 0.25
Particles: 16

Time: 0.5
Particles: 16

Time: 0.75
Particles: 16

Time: 1
Particles: 12

Time: 1.25
Particles: 12

Time: 1.5
Particles: 12

Time: 1.75
Particles: 12

Time: 2
Particles: 10

Time: 2.25
Particles: 10

Time: 2.5
Particles: 10

Time: 2.75
Particles: 10

Time: 3
Particles: 7

Time: 3.25
Particles: 7

Time: 3.5
Particles: 7

Time: 3.75
Particles: 7

Time: 4
Particles: 6

Time: 4.25
Particles: 6

Time: 4.5
Particles: 6

Time: 4.75
Particles: 6

Time: 5
Particles: 6

Time: 5.25
Particles: 6

Time: 5.5
Particles: 6

Time: 5.75
Particles: 6

Time: 6
Particles: 5

Time: 6.5
Particles: 5

Time: 7
Particles: 4

Time: 7.5
Particles: 4

Time: 8
Particles: 3

Time: 8.5
Particles: 3

Time: 9
Particles: 3

Time: 9.5
Particles: 3

Time: 10
Particles: 3

Time: 10.5
Particles: 3

Time: 11
Particles: 3

Time: 11.5
Particles: 3

Time: 12
Particles: 2

Time: 12.5
Particles: 2

Time: 13
Particles: 2

Time: 13.5
Particles: 2

Time: 14
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Particles: 2

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Time: 15.5
Particles: 2

Time: 16
Particles: 2

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Particles: 2

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Particles: 2

Time: 18
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Particles: 2

Time: 19
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Particles: 2

Time: 20
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Particles: 2

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Particles: 2

Time: 22
Particles: 2

Time: 22.5
Particles: 2

Time: 23
Particles: 2

Time: 23.5
Particles: 2

Time: 24
Particles: 2

Time: 24.5
Particles: 2

Time: 25
Particles: 1

Time: 25
Particles: 1

Time: 25
Particles: 1

Time: 25.5
Particles: 1

Time: 26
Particles: 1

Time: 26.5
Particles: 1

Time: 27
Particles: 1

Time: 27.5
Particles: 1

Time: 28
Particles: 1
Coalescing Random Walks (Example)

Time: 0
Particles: 16

Time: 0.25
Particles: 16

Time: 0.5
Particles: 16

Time: 0.75
Particles: 16

Time: 1
Particles: 12

Time: 1.25
Particles: 12

Time: 1.5
Particles: 12

Time: 1.75
Particles: 12

Time: 2
Particles: 10

Time: 2.25
Particles: 10

Time: 2.5
Particles: 10

Time: 2.75
Particles: 10

Time: 3
Particles: 7

Time: 3.25
Particles: 7

Time: 3.5
Particles: 7

Time: 3.75
Particles: 7

Time: 4
Particles: 6

Time: 4.25
Particles: 6

Time: 4.5
Particles: 6

Time: 4.75
Particles: 6

Time: 5
Particles: 6

Time: 5.25
Particles: 6

Time: 5.5
Particles: 6

Time: 5.75
Particles: 6

Time: 6
Particles: 5

Time: 6.5
Particles: 5

Time: 7
Particles: 4

Time: 7.5
Particles: 4

Time: 8
Particles: 3

Time: 8.5
Particles: 3

Time: 9
Particles: 3

Time: 9.5
Particles: 3

Time: 10
Particles: 3

Time: 10.5
Particles: 3

Time: 11
Particles: 3

Time: 11.5
Particles: 3

Time: 12
Particles: 2

Time: 12.5
Particles: 2

Time: 13
Particles: 2

Time: 13.5
Particles: 2

Time: 14
Particles: 2

Time: 14.5
Particles: 2

Time: 15
Particles: 2

Time: 15.5
Particles: 2

Time: 16
Particles: 2

Time: 16.5
Particles: 2

Time: 17
Particles: 2

Time: 17.5
Particles: 2

Time: 18
Particles: 2

Time: 18.5
Particles: 2

Time: 19
Particles: 2

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Particles: 2

Time: 20
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Particles: 2

Time: 21
Particles: 2

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Particles: 2

Time: 22
Particles: 2

Time: 22.5
Particles: 2

Time: 23
Particles: 2

Time: 23.5
Particles: 2

Time: 24
Particles: 2

Time: 24.5
Particles: 2

Time: 25
Particles: 1

Time: 25
Particles: 1

Time: 25
Particles: 1

Time: 25.5
Particles: 1

Time: 26
Particles: 1

Time: 26.5
Particles: 1

Time: 27
Particles: 1

Time: 27.5
Particles: 1

Time: 28
Particles: 1
Coalescing Random Walks (Example)

Time: 23.5
Particles: 2
Coalescing Random Walks (Example)

Time: 0
Particles: 16

Time: 0.25
Particles: 16

Time: 0.5
Particles: 16

Time: 0.75
Particles: 16

Time: 1
Particles: 12

Time: 1.25
Particles: 12

Time: 1.5
Particles: 12

Time: 1.75
Particles: 12

Time: 2
Particles: 10

Time: 2.25
Particles: 10

Time: 2.5
Particles: 10

Time: 2.75
Particles: 10

Time: 3
Particles: 7

Time: 3.25
Particles: 7

Time: 3.5
Particles: 7

Time: 3.75
Particles: 7

Time: 4
Particles: 6

Time: 4.25
Particles: 6

Time: 4.5
Particles: 6

Time: 4.75
Particles: 6

Time: 5
Particles: 6

Time: 5.25
Particles: 6

Time: 5.5
Particles: 6

Time: 5.75
Particles: 6

Time: 6
Particles: 5

Time: 6.5
Particles: 5

Time: 7
Particles: 4

Time: 7.5
Particles: 4

Time: 8
Particles: 3

Time: 8.5
Particles: 3

Time: 9
Particles: 3

Time: 9.5
Particles: 3

Time: 10
Particles: 3

Time: 10.5
Particles: 3

Time: 11
Particles: 3

Time: 11.5
Particles: 3

Time: 12
Particles: 2

Time: 12.5
Particles: 2

Time: 13
Particles: 2

Time: 13.5
Particles: 2

Time: 14
Particles: 2

Time: 14.5
Particles: 2

Time: 15
Particles: 2

Time: 15.5
Particles: 2

Time: 16
Particles: 2

Time: 16.5
Particles: 2

Time: 17
Particles: 2

Time: 17.5
Particles: 2

Time: 18
Particles: 2

Time: 18.5
Particles: 2

Time: 19
Particles: 2

Time: 19.5
Particles: 2

Time: 20
Particles: 2

Time: 20.5
Particles: 2

Time: 21
Particles: 2

Time: 21.5
Particles: 2

Time: 22
Particles: 2

Time: 22.5
Particles: 2

Time: 23
Particles: 2

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Particles: 2

Time: 24
Particles: 2

Time: 24.5
Particles: 2

Time: 25
Particles: 1

Time: 25
Particles: 1

Time: 25
Particles: 1

Time: 25.5
Particles: 1

Time: 26
Particles: 1

Time: 26.5
Particles: 1

Time: 27
Particles: 1

Time: 27.5
Particles: 1

Time: 28
Particles: 1

Particles: 2
Coalescing Random Walks (Example)

Time: 0
Particles: 16

Time: 0.25
Particles: 16

Time: 0.5
Particles: 16

Time: 0.75
Particles: 16

Time: 1
Particles: 12

Time: 1.25
Particles: 12

Time: 1.5
Particles: 12

Time: 1.75
Particles: 12

Time: 2
Particles: 10

Time: 2.25
Particles: 10

Time: 2.5
Particles: 10

Time: 2.75
Particles: 10

Time: 3
Particles: 7

Time: 3.25
Particles: 7

Time: 3.5
Particles: 7

Time: 3.75
Particles: 7

Time: 4
Particles: 6

Time: 4.25
Particles: 6

Time: 4.5
Particles: 6

Time: 4.75
Particles: 6

Time: 5
Particles: 6

Time: 5.25
Particles: 6

Time: 5.5
Particles: 6

Time: 5.75
Particles: 6

Time: 6
Particles: 5

Time: 6.5
Particles: 5

Time: 7
Particles: 4

Time: 7.5
Particles: 4

Time: 8
Particles: 3

Time: 8.5
Particles: 3

Time: 9
Particles: 3

Time: 9.5
Particles: 3

Time: 10
Particles: 3

Time: 10.5
Particles: 3

Time: 11
Particles: 3

Time: 11.5
Particles: 3

Time: 12
Particles: 2

Time: 12.5
Particles: 2

Time: 13
Particles: 2

Time: 13.5
Particles: 2

Time: 14
Particles: 2

Time: 14.5
Particles: 2

Time: 15
Particles: 2

Time: 15.5
Particles: 2

Time: 16
Particles: 2

Time: 16.5
Particles: 2

Time: 17
Particles: 2

Time: 17.5
Particles: 2

Time: 18
Particles: 2

Time: 18.5
Particles: 2

Time: 19
Particles: 2

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Particles: 2

Time: 20
Particles: 2

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Particles: 2

Time: 21
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Particles: 2

Time: 22
Particles: 2

Time: 22.5
Particles: 2

Time: 23
Particles: 2

Time: 23.5
Particles: 2

Time: 24
Particles: 2

Time: 24.5
Particles: 2

Time: 25
Particles: 1

Time: 25
Particles: 1

Time: 25.5
Particles: 1

Time: 26
Particles: 1

Time: 26.5
Particles: 1

Time: 27
Particles: 1

Time: 27.5
Particles: 1

Time: 28
Particles: 1

Particles: 2
Coalescing Random Walks (Example)

Time: 0
Particles: 16

Time: 0.25
Particles: 16

Time: 0.5
Particles: 16

Time: 0.75
Particles: 16

Time: 1
Particles: 12

Time: 1.25
Particles: 12

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Particles: 12

Time: 1.75
Particles: 12

Time: 2
Particles: 10

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Particles: 10

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Particles: 10

Time: 2.75
Particles: 10

Time: 3
Particles: 7

Time: 3.25
Particles: 7

Time: 3.5
Particles: 7

Time: 3.75
Particles: 7

Time: 4
Particles: 6

Time: 4.25
Particles: 6

Time: 4.5
Particles: 6

Time: 4.75
Particles: 6

Time: 5
Particles: 6

Time: 5.25
Particles: 6

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Particles: 6

Time: 6
Particles: 5

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Particles: 5

Time: 7
Particles: 4

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Particles: 4

Time: 8
Particles: 3

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Particles: 3

Time: 9
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Particles: 3

Time: 10
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Particles: 3

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Particles: 3

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Particles: 3

Time: 12
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Particles: 2

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Particles: 1

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Particles: 1

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Particles: 1

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Particles: 1

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Coalescing Random Walks (Example)

Time: 0
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Coalescing Random Walks (Example)

Time: 0
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Particles: 2

Time: 23
Particles: 2

Time: 23.5
Particles: 2

Time: 24
Particles: 2

Time: 24.5
Particles: 2

Time: 25
Particles: 1

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Particles: 1

Particles: 1
Coalescing Random Walks (Example)

Time: 0
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Time: 0.75
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Particles: 12

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Particles: 10

Time: 3
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Time: 3.5
Particles: 7

Time: 3.75
Particles: 7

Time: 4
Particles: 6

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Time: 4.75
Particles: 6

Time: 5
Particles: 6

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Time: 25
Particles: 1
Coalescing Random Walks (Example)

Time: 0
Particles: 16

Time: 0.25
Particles: 16

Time: 0.5
Particles: 16

Time: 0.75
Particles: 16

Time: 1
Particles: 12

Time: 1.25
Particles: 12

Time: 1.5
Particles: 12

Time: 1.75
Particles: 12

Time: 2
Particles: 10

Time: 2.25
Particles: 10

Time: 2.5
Particles: 10

Time: 2.75
Particles: 10

Time: 3
Particles: 7

Time: 3.25
Particles: 7

Time: 3.5
Particles: 7

Time: 3.75
Particles: 7

Time: 4
Particles: 6

Time: 4.25
Particles: 6

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Particles: 6

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Time: 25
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Particles: 1

Particles: 1
Coalescing Random Walks (Example)

Time: 0
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Particles: 16

Time: 0.75
Particles: 16

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Particles: 12

Time: 1.75
Particles: 12

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Particles: 10

Time: 2.75
Particles: 10

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Time: 3.5
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Time: 3.75
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Particles: 1

Time: 25.5
Particles: 1

Time: 26
Particles: 1

Time: 26.5
Particles: 1

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Particles: 1

Particles: 1

Time: 27
Coalescing Random Walks (Example)

Time: 27.5
Particles: 1
Coalescing Random Walks (Example)

Time: 0
Particles: 16

Time: 0.25
Particles: 16

Time: 0.5
Particles: 16

Time: 0.75
Particles: 16

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Particles: 2

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Time: 25
Particles: 1

Time: 25
Particles: 1

Time: 25.5
Particles: 1

Time: 26
Particles: 1

Time: 26.5
Particles: 1

Time: 27
Particles: 1

Time: 27.5
Particles: 1

Time: 28
Particles: 1
Motivation: Voter Model

Voter Model

- Given a graph $G = (V, E)$ with $n$ nodes, each with a different opinion.
- At each round, each node "pulls" w.p. $1/2$ the opinion of a random neighbor, otherwise keeps his current opinion.

Time to reach consensus = Time for $n$ coalescing particles to merge.

Duality (voting) time (coalescence) time
Motivation: Voter Model

- **Voter Model**
  - Given a graph \( G = (V, E) \) with \( n \) nodes, each with a different opinion.
  - At each round, each node "pulls" w.p. 1/2 the opinion of a random neighbor, otherwise keeps his current opinion.

- **Duality**
  
  Time to reach consensus = Time for \( n \) coalescing particles to merge.
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Duality

Time to reach consensus = Time for $n$ coalescing particles to merge.
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  - Given a graph $G = (V, E)$ with $n$ nodes, each with a different opinion.
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**Duality**

Time to reach consensus = Time for $n$ coalescing particles to merge.
Some Related Work and the Agenda of this Talk

For the discrete-time variant:

For any graph, $t_{\text{coal}} = \frac{t_{\text{uni}} \cdot \log n}{\text{Hassin, Peleg, DIST'01}}$

For a random $d$-regular graph (non-lazy walks),

$t_{\text{coal}} = \left(2 + o\left(1\right)\right) \cdot d^{-1}d^{-2} \cdot n^{\text{Cooper, Frieze, Radzik, SIAM J. Discrete Math.'09}}$

For any graph,

$t_{\text{coal}} = \frac{1 - \lambda / \log 4 \cdot n + 1 / \pi^2}{\text{Cooper, Elsässer, Ono and Radzik, SIAM J. Discrete Math.'13}}$

For any graph $t_{\text{coal}} = \frac{\Phi / E / \delta}{\text{Berenbrink, Giakkoupis, Kermarrec and Mallmann-Trenn, ICALP'16}}$, where $\delta$ is the minimum degree.

For the continuous-time variant:

For any graph $t_{\text{coal}} = \frac{t_{\text{hit}}}{\text{Oliveira, TAMS'12}}$

(simplified) For graphs with $t_{\text{mix}} / n$, $t_{\text{coal}}$ behaves like on a clique

$[\text{Oliveira, Ann. Prob.'12}]$

For many graphs, $t_{\text{coal}} = t_{\text{meet}}$ or even $t_{\text{coal}} = n$ (if $G$ is regular).

Under the premise that $t_{\text{mix}}$ and $t_{\text{meet}}$ are "simpler" quantities, when does $t_{\text{coal}}$ hold?

6
Some Related Work and the Agenda of this Talk

For the discrete-time variant:

- For any graph, \( t_{\text{coal}} \preceq t_{\text{meet}} \cdot \log n \) \[\text{[Hassin, Peleg, DIST'01]}\]

For the continuous-time variant:

- (simplified) For graphs with \( t_{\text{mix}} / n \), \( t_{\text{coal}} \) behaves like on a clique \[\text{[Oliveira, TAMS'12]}\]

Under the premise that \( t_{\text{mix}} \) and \( t_{\text{meet}} \) are "simpler" quantities, when does \( t_{\text{coal}} \) hold?
Some Related Work and the Agenda of this Talk

For the discrete-time variant:

- For any graph, \( t_{\text{coal}} \preceq t_{\text{meet}} \cdot \log n \) \[\text{[Hassin, Peleg, DIST'01]}\]
- For a random \( d \)-regular graph (non-lazy walks), \( t_{\text{coal}} = (2 + o(1)) \cdot \frac{d-1}{d-2} \cdot n \) \[\text{[Cooper, Frieze, Radzik, SIAM J. Discrete Math.'09]}\]
For the discrete-time variant:

- For any graph, $t_{\text{coal}} \leq t_{\text{meet}} \cdot \log n$  
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  \[ \text{[Cooper, Frieze, Radzik, SIAM J. Discrete Math.’09]} \]

- For any graph, $t_{\text{coal}} \leq \frac{1}{1-\lambda_2} \cdot \left( \log^4 n + \frac{1}{\|\pi\|_2^2} \right)$  
  \[ \text{[Cooper, Elsässer, Ono and Radzik, SIAM J. Discrete Math.’13]} \]

- For any graph $t_{\text{coal}} \leq \frac{1}{\Phi} \cdot \frac{|E|}{\delta}$, where $\delta$ is the minimum degree  
  \[ \text{[Berenbrink, Giakkoupis, Kermarrec and Mallmann-Trenn, ICALP’16]} \]
Some Related Work and the Agenda of this Talk

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- For many graphs, \( t_{\text{coal}} \asymp t_{\text{meet}} \) or even \( t_{\text{coal}} \asymp n \) (if \( G \) is regular)
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- Under the premise that $t_{\text{mix}}$ and $t_{\text{meet}}$ are “simpler” quantities, when does $t_{\text{coal}} \asymp t_{\text{meet}}$ hold?
Outline

Introduction

Relating Coalescing Time to the Mixing and Meeting Time

Conclusion
The Upper Bound and some Consequences

Theorem (Upper Bound)

For any graph $G = (V, E)$,

$$t_{\text{coal}} \leq t_{\text{meet}} \cdot \left(1 + \sqrt{\frac{t_{\text{mix}}}{t_{\text{meet}}} \cdot \log n}\right)$$
The Upper Bound and some Consequences

**Theorem (Upper Bound)**

For any graph \( G = (V, E) \),

\[
t_{\text{coal}} \preceq t_{\text{meet}} \cdot \left( 1 + \sqrt{\frac{t_{\text{mix}}}{t_{\text{meet}}} \cdot \log n} \right)
\]

- Whenever \( \frac{t_{\text{meet}}}{t_{\text{mix}}} \gtrsim (\log n)^2 \), we have \( t_{\text{coal}} \preceq t_{\text{meet}} \)
The Upper Bound and some Consequences

For any graph $G = (V, E)$,

$$t_{coal} \preceq t_{meet} \cdot \left( 1 + \sqrt{\frac{t_{mix}}{t_{meet}}} \cdot \log n \right)$$

- Whenever $\frac{t_{meet}}{t_{mix}} \gtrsim (\log n)^2$, we have $t_{coal} \preceq t_{meet}$
- If $\frac{t_{meet}}{t_{mix}} \prec 1$, our bound states $t_{coal} \preceq t_{meet} \cdot \log n$

$\Rightarrow$ bound can be viewed as a refinement of the basic $t_{coal} \preceq t_{meet} \cdot \log n$
The Upper Bound and some Consequences

For any graph $G = (V, E)$,

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$\Rightarrow$ bound can be viewed as a refinement of the basic $t_{\text{coal}} \preceq t_{\text{meet}} \cdot \log n$

Application to “Real World” Graph Models

If the max-degree satisfies $\Delta \preceq n/\log^3 n$ and $t_{\text{mix}} \preceq \log n$, then $t_{\text{coal}} \preceq t_{\text{meet}}$. 
The Upper Bound and some Consequences

**Theorem (Upper Bound)**

For any graph $G = (V, E)$,

$$t_{coal} \preceq t_{meet} \cdot \left(1 + \sqrt{\frac{t_{mix}}{t_{meet}} \cdot \log n}\right)$$

- Whenever $\frac{t_{meet}}{t_{mix}} \gtrsim (\log n)^2$, we have $t_{coal} \preceq t_{meet}$
- If $\frac{t_{meet}}{t_{mix}} \asymp 1$, our bound states $t_{coal} \preceq t_{meet} \cdot \log n$

$\Rightarrow$ bound can be viewed as a refinement of the basic $t_{coal} \preceq t_{meet} \cdot \log n$

**Application to “Real World” Graph Models**

If the max-degree satisfies $\Delta \preceq n/\log^3 n$ and $t_{mix} \preceq \log n$, then $t_{coal} \preceq t_{meet}$.

Unfortunately we are not able to determine $t_{meet}$ (it is conceivable though that $t_{meet} \asymp 1/\|\pi\|^2_2$).
Proof is a bit technical, and we will only glance over one challenging part.

Consider two random walks \((X_t)_{t \geq 0}, (Y_t)_{t \geq 0}\) starting from stationarity.

By a scaling argument,
\[
\Pr\left[ \int (X_t, Y_t, t_{\text{mix}}) \geq t_{\text{mix}} / 16 \right] = p,
\]

If we have \(j\) random walks \(Y_1, Y_2, \ldots, Y_j\), do we have
\[
\Pr\left[ \int (X_t, Y_1, \ldots, Y_j, t_{\text{mix}}) \geq 1 - (1 - p)^j \right]?
\]

Define for \(\tau\):
\[
C_1 := \left\{ (x_0, \ldots, x_\tau) \in \mathbb{T}_\tau : \Pr\left[ \int (x_t, Y_\tau) \geq p^3 \right] \right\},
\]

\[
C_2 := \left\{ (x_0, \ldots, x_\tau) \in \mathbb{T}_\tau : \Pr\left[ \int (x_t, Y_\tau) \geq \sqrt{p} \right] \right\}.
\]

Then,
\[
\Pr\left[ (X_t)_{t=0}^{\tau} \in C_1 \right] \geq \sqrt{p^3} \text{ or } \Pr\left[ (X_t)_{t=0}^{\tau} \in C_2 \right] \geq p^3.
\]

Suppose \(\Pr\left[ (X_t)_{t=0}^{\tau} \in C_2 \right] \geq p^3\). Then a \(p\)-fraction of all walks have a “good” trajectory that is hit by a stationary walk with probability at least \(\sqrt{p}\).

(Issue: Random walks coalesce and could therefore have terminated earlier!) This is of course wrong, since the events are not independent!
Proof is a bit technical, and we will only glance over one challenging part.

- Consider two random walks \((X_t)_{t \geq 0}, (Y_t)_{t \geq 0}\) starting from stationarity.
A Glimpse at the Proof of the Upper Bound

Proof is a bit technical, and we will only glance over one challenging part.

- Consider two random walks \((X_t)_{t \geq 0}, (Y_t)_{t \geq 0}\) starting from stationarity
- By a scaling argument,
  \[
  \Pr\left[ \text{int}(X, Y, t_{\text{mix}}) \right] \geq \frac{t_{\text{mix}}}{16t_{\text{meet}}} =: p,
  \]

... (Issue: Random walks coalesce and could therefore have terminated earlier!)
Proof is a bit technical, and we will only glance over one challenging part.

- Consider two random walks \((X_t)_{t \geq 0}, (Y_t)_{t \geq 0}\) starting from stationarity.
- By a scaling argument,
  \[
  \Pr[\text{int}(X, Y, t_{\text{mix}})] \geq \frac{t_{\text{mix}}}{16 t_{\text{meet}}} =: p,
  \]
- If we have \(j\) random walks \(Y^1, Y^2, \ldots, Y^j\), do we have
  \[
  \Pr\left[ \bigcup_{\ell=1}^{j} \text{int}(X, Y^\ell, t_{\text{mix}}) \right] \geq 1 - (1 - p)^j
  \]
Proof is a bit technical, and we will only glance over one challenging part.

- Consider two random walks \((X_t)_{t \geq 0}, (Y_t)_{t \geq 0}\) starting from stationarity.
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\]

- If we have \(j\) random walks \(Y^1, Y^2, \ldots, Y^j\), do we have

\[
\Pr\left[\bigcup_{\ell=1}^{j} \text{int}(X, Y^\ell, t_{\text{mix}})\right] \geq 1 - (1 - p)^j
\]

This is of course wrong, since the events are not independent!
A Glimpse at the Proof of the Upper Bound

Proof is a bit technical, and we will only glance over one challenging part.

- Consider two random walks \((X_t)_{t \geq 0}, \ (Y_t)_{t \geq 0}\) starting from stationarity
- By a scaling argument,
  \[
  \Pr[\text{int}(X, Y, t_{\text{mix}})] \geq \frac{t_{\text{mix}}}{16t_{\text{meet}}} =: p,
  \]

- Define for \(\tau := t_{\text{mix}}\),
  \[
  C_1 := \{(x_0, \ldots, x_\tau) \in T_\tau: \Pr[\text{int}(x, Y, \tau)] \geq \frac{p}{3}\}
  \]
  \[
  C_2 := \{(x_0, \ldots, x_\tau) \in T_\tau: \Pr[\text{int}(x, Y, \tau)] \geq \sqrt{p}\}.
  \]
A Glimpse at the Proof of the Upper Bound

Proof is a bit technical, and we will only glance over one challenging part.

- Consider two random walks \((X_t)_{t \geq 0}, (Y_t)_{t \geq 0}\) starting from stationarity
- By a scaling argument,

\[
\Pr[\text{int}(X, Y, t_{\text{mix}})] \geq \frac{t_{\text{mix}}}{16t_{\text{meet}}} =: p,
\]

- Define for \(\tau := t_{\text{mix}}\),

\[
C_1 := \{(x_0, \ldots, x_\tau) \in T_\tau : \Pr[\text{int}(x, Y, \tau)] \geq \frac{p}{3}\}
\]

\[
C_2 := \{(x_0, \ldots, x_\tau) \in T_\tau : \Pr[\text{int}(x, Y, \tau)] \geq \sqrt{p}\}.
\]

- Then, \(\Pr[ (X_t)_{t=0}^\tau \in C_1 ] \geq \frac{\sqrt{p}}{3}\) or \(\Pr[ (X_t)_{t=0}^\tau \in C_2 ] \geq \frac{p}{3}\).
A Glimpse at the Proof of the Upper Bound

Proof is a bit technical, and we will only glance over one challenging part.

- Consider two random walks \( (X_t)_{t \geq 0}, (Y_t)_{t \geq 0} \) starting from stationarity.
- By a scaling argument,
  \[ \Pr[\text{int}(X, Y, t_{\text{mix}})] \geq \frac{t_{\text{mix}}}{16t_{\text{meet}}} =: p, \]

- Define for \( \tau := t_{\text{mix}}, \)
  \[ C_1 := \{ (x_0, \ldots, x_\tau) \in \mathcal{T}_\tau : \Pr[\text{int}(x, Y, \tau)] \geq \frac{p}{3} \} \]
  \[ C_2 := \{ (x_0, \ldots, x_\tau) \in \mathcal{T}_\tau : \Pr[\text{int}(x, Y, \tau)] \geq \sqrt{p} \}. \]

- Then, \( \Pr[ (X_t)_{t=0}^\tau \in C_1] \geq \frac{\sqrt{p}}{3} \) or \( \Pr[ (X_t)_{t=0}^\tau \in C_2] \geq \frac{p}{3}. \)

clique (vertex-transitive graphs)
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- Then, \(\Pr \left[ (X_t)_{t=0}^\tau \in C_1 \right] \geq \frac{\sqrt{p}}{3}\) or \(\Pr \left[ (X_t)_{t=0}^\tau \in C_2 \right] \geq \frac{p}{3}\).
- Suppose \(\Pr \left[ (X_t)_{t=0}^\tau \in C_2 \right] \geq \frac{p}{3}\). Then a \(p\)-fraction of all walks have a “good” trajectory that is hit by a stationary walk with probability at least \(\sqrt{p}\) . . .
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- Suppose \(\Pr[\left(X_t\right)_{t=0}^{\tau} \in C_2] \geq \frac{p}{3}\). Then a \(p\)-fraction of all walks have a “good” trajectory that is hit by a stationary walk with probability at least \(\sqrt{p}\) …
- (Issue: Random walks coalesce and could therefore have terminated earlier!)
A Graph Demonstrating Tightness

Node $z^*$ is connected to one designated node in each $G_i$, $1 \leq i \leq \sqrt{n}$, and to $\sqrt{n}/\alpha$ distinct nodes in $G_2$. $G_2$ is a $\sqrt{n}$-regular Ramanujan graph on $n/\alpha$ nodes ($\alpha = t \text{ meet } t \text{ mix}$).
A Graph Demonstrating Tightness

Node $z^*$ is connected to one designated node in each $G_i^1$ and to $\sqrt{n}$ distinct nodes in $G_2^1$.
A Graph Demonstrating Tightness

- $G_i^i$, $1 \leq i \leq \sqrt{n}$ are cliques over $\sqrt{n}$ nodes
- $G_2$ is a $\sqrt{n}$-regular Ramanujan graph on $n/\sqrt{\alpha}$ nodes ($\alpha = t_{\text{meet}}/t_{\text{mix}}$)
- Node $z^*$ is connected to one designated node in each $G_i^i$ and to $\sqrt{n/\alpha}$ distinct nodes in $G_2$
Intuition of the Construction

- $G_1^i, 1 \leq i \leq \sqrt{n}$ are cliques over $\sqrt{n}$ nodes
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- $t_{\text{mix}} \asymp n$
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  - “$\leq$”: use principle of “Mixing-Time equal to Hitting-Time of Large Sets”
    
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  - very unlikely to meet outside $G_2$
  - After $t_{\text{mix}}$ steps, w.p. $(1/\sqrt{\alpha})^2$ both walks on $G_2 \Rightarrow$ meet w.c.p.
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    [Peres, Sousi, J. of Theor. Prob.’15]
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- $t_{\text{coal}} \gtrsim \sqrt{\alpha} n \log n$
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  - After $t_{\text{mix}}$ steps, w.p. $(1/\sqrt{\alpha})^2$ both walks on $G_2$ ⇒ meet w.c.p.

- $t_{\text{coal}} \geq \sqrt{\alpha} n \log n$
  - 1 walk starting from $G_1^i$ that doesn’t reach $G_2$ in $\sqrt{\alpha} n \log n$ steps
Contrasting the Example with the Upper Bound

For the example $t_{\text{mix}} \asymp \sqrt{n}$, $t_{\text{meet}} \asymp \alpha \sqrt{n}$ and $t_{\text{coal}} \gtrsim \sqrt{\alpha \cdot n \log n}$:
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**Theorem (Lower Bound)**

For any $\alpha = \frac{t_{\text{meet}}}{t_{\text{mix}}} \in [1, \log^2 n]$ there exists a family of almost-regular graphs such that:

$$t_{\text{coal}} \gtrsim t_{\text{meet}} \cdot \left(1 + \sqrt{\frac{t_{\text{mix}}}{t_{\text{meet}}} \cdot \log n}\right)$$
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**Theorem (Upper Bound)**

For any graph \( G = (V, E) \),

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  t_{\text{coal}} \preceq t_{\text{meet}} \cdot \left( 1 + \sqrt{\frac{t_{\text{mix}}}{t_{\text{meet}}} \cdot \log n} \right)
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For the example $t_{\text{mix}} \asymp \sqrt{n}$, $t_{\text{meet}} \asymp \alpha \sqrt{n}$ and $t_{\text{coal}} \gtrsim \sqrt{\alpha \cdot n \log n}$:

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**Theorem (Upper Bound)**

For any graph $G = (V, E)$,

$$t_{\text{coal}} \lesssim t_{\text{meet}} \cdot \left(1 + \sqrt{\frac{t_{\text{mix}}}{t_{\text{meet}}} \cdot \log n}\right)$$

- For almost-regular graphs, $t_{\text{coal}}$ might be as large as $t_{\text{meet}} \cdot \log n$
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Theorem (Upper Bound)

For any graph $G = (V, E)$,

$$t_{\text{coal}} \leq t_{\text{meet}} \cdot \left(1 + \sqrt{\frac{t_{\text{mix}}}{t_{\text{meet}}} \cdot \log n}\right)$$

- For almost-regular graphs, $t_{\text{coal}}$ might be as large as $t_{\text{meet}} \cdot \log n$
- However, for any vertex-transitive graph, $t_{\text{coal}} \asymp t_{\text{meet}} \asymp t_{\text{hit}}$
Improved Bounds on Hitting Times (and Meeting Times)

- For any regular graph, $t_{\text{hit}} \lesssim \frac{n}{1 - \lambda_2}$  
  
  [Broder, Karlin, FOCS’88]
\begin{itemize}
\item For any regular graph, \( t_{\text{hit}} \leq \frac{n}{1-\lambda_2} \) \hfill [Broder, Karlin, FOCS'88]
\item For any graph, \( t_{\text{hit}} \leq \frac{1/\pi_{\text{min}}}{\Phi \cdot \log \Phi} \) \hfill [Aldous, Fill]
\end{itemize}
Improved Bounds on Hitting Times (and Meeting Times)

- For any regular graph, $t_{\text{hit}} \lesssim \frac{n}{1-\lambda_2}$ \hspace{2em} [Broder, Karlin, FOCS’88]
- For any graph, $t_{\text{hit}} \lesssim \frac{1/\pi_{\text{min}}}{\Phi \cdot \log \Phi}$ \hspace{2em} [Aldous, Fill]

Theorem

For any regular graph,

$$t_{\text{meet}} \lesssim t_{\text{hit}} \lesssim \frac{n}{\sqrt{1 - \lambda_2}}.$$
For any regular graph, \( t_{\text{hit}} \lesssim \frac{n}{1-\lambda_2} \) \[ \text{[Broder, Karlin, FOCS'88]} \]

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**Theorem**

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\]

For any given \( 1/(1 - \lambda_2) \), there is a graph matching this bound up to constants
Improved Bounds on Hitting Times (and Meeting Times)

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  \[ \text{[Broder, Karlin, FOCS'88]} \]

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  \[ \text{[Aldous, Fill]} \]

Theorem

For any regular graph,

\[ t_{\text{meet}} \lesssim t_{\text{hit}} \lesssim \frac{n}{\sqrt{1 - \lambda_2}}. \]

- For any given \( 1/(1 - \lambda_2) \), there is a graph matching this bound up to constants.
- Applying Cheeger’s inequality, we obtain \( t_{\text{hit}} = O(n/\Phi) \).
Outline

Introduction

Relating Coalescing Time to the Mixing and Meeting Time

Conclusion
Application to Concrete Networks
Application to Concrete Networks

1D Grid

\[ t_{\text{mix}} \asymp n^2 \]
\[ t_{\text{hit}} \asymp t_{\text{meet}} \asymp n^2 \]
\[ t_{\text{coal}} \asymp n^2 \quad (\checkmark) \]

2D Grid

\[ t_{\text{mix}} \asymp n \]
\[ t_{\text{hit}} \asymp t_{\text{meet}} \asymp n \log n \]
\[ t_{\text{coal}} \asymp n \log n \quad (\checkmark) \]

3D Grid

\[ t_{\text{mix}} \asymp n^{2/3} \]
\[ t_{\text{hit}} \asymp t_{\text{meet}} \asymp n \]
\[ t_{\text{coal}} \asymp n \quad \checkmark \]
Application to Concrete Networks

1D Grid

- $t_{mix} \asymp n^2$
- $t_{hit} \asymp t_{meet} \asymp n^2$
- $t_{coal} \asymp n^2 \ (✓)$

2D Grid

- $t_{mix} \asymp n$
- $t_{hit} \asymp t_{meet} \asymp n \log n$
- $t_{coal} \asymp n \log n \ (✓)$

3D Grid

- $t_{mix} \asymp n^{2/3}$
- $t_{hit} \asymp t_{meet} \asymp n$
- $t_{coal} \asymp n \ (✓)$

Hypercube

- $t_{mix} \asymp \log n \log \log n$
- $t_{hit} \asymp t_{meet} \asymp n$
- $t_{coal} \asymp n \ (✓)$

Expander Graph

- $t_{mix} \asymp \log n$
- $t_{hit} \asymp t_{meet} \asymp n$
- $t_{coal} \asymp n \ (✓)$

Binary Tree

- $t_{mix} \asymp n$
- $t_{hit} \asymp t_{meet} \asymp n \log n$
- $t_{coal} \asymp n \log n \ (✓)$
Summary and Open Questions

Results

1. For arbitrary graphs, $t_{coal} \preceq t_{meet} \cdot \left(1 + \sqrt{\frac{t_{mix}}{t_{meet}}} \cdot \log n\right)$
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### Results

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3. For graphs with constant \( \Delta/d \), \( t_{mix} \preceq t_{meet} \preceq t_{coal} \preceq t_{hit} \preceq t_{cov} \)
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Open Questions
Summary and Open Questions

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- Can we prove $t_{\text{coal}} \preceq t_{\text{hit}}$ for all graphs?
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- Is it true that \( t_{coal}^{(disc)} \preceq t_{coal}^{(cont)} \) for any graph?
Summary and Open Questions

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Open Questions

- Can we prove \( t_{\text{coal}} \preceq t_{\text{hit}} \) for all graphs?
- Is it true that \( t_{\text{coal}}^{(\text{disc})} \preceq t_{\text{coal}}^{(\text{cont})} \) for any graph?
- Reduce the number of walks to some threshold \( \kappa \in [1, n] \).
  Conjecture:
  - For any (regular) graph, no. walks can be reduced to \( \sqrt{n} \) in \( O(n) \) time.
  - More generally, it takes \( O((n/\kappa)^2) \) time to go from \( n \) to \( \kappa \).
The End

THANK YOU
The End

THANK YOU
Another Direction: Cat-and-Mouse Game

Definition

- The mouse picks a deterministic walk \((v_0, v_1, v_2, \ldots)\), unaware of the transitions of the cat.
Another Direction: Cat-and-Mouse Game

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- The **mouse** picks a deterministic walk 
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- The **cat** performs lazy random walk \((Y_t)_{t \geq 0}\) from \(u\)

Comments on the Cat-and-Mouse Game:

Easier to deal with in the sense there is only one random object (the cat!)

But do we have \(t_{\text{cat}} - t_{\text{mouse}} / t_{\text{hit}}\)?
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\[
t_{\text{cat-mouse}} := \max_{u, (v_0, v_1, \ldots)} E_u \left[ \min\{t \geq 0 : Y_t = v_t\} \right].
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- very similar version in Aldous and Fill (Section 4.3)
- we may assume w.l.o.g. that the cat starts from stationarity by simply letting the cat perform \(t_{\text{mix}}\) steps
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Comments on the Cat-and-Mouse Game:

- Easier to deal with in the sense there is only one random object (the cat!)
- Clearly, \(t_{\text{meet}} \leq t_{\text{cat-mouse}}\) and \(t_{\text{hit}} \leq t_{\text{cat-mouse}}\).
  But do we have \(t_{\text{cat-mouse}} \approx t_{\text{hit}}\)?