Complexity Theory

Lecture 7: Reductions beyond graphs

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http://www.cl.cam.ac.uk/teaching/2324/Complexity

Recap

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- A problem is $\mathcal{NP}$-complete if it is: (1) $\mathcal{NP}$-hard, (2) in $\mathcal{NP}$.
- Cook-Levin Theorem: 3SAT is $\mathcal{NP}$-complete.
- Using 3SAT, we can establish NP-completeness of many problems (e.g., IS, Clique, Hamiltonicity, TSP).

Protip

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What are the big questions at this stage?

k-Colourability

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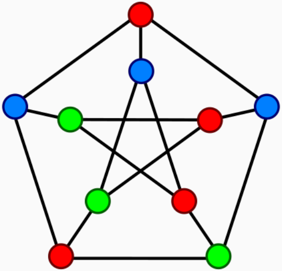

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For all $k > 2$, k-colourability is NP-complete.

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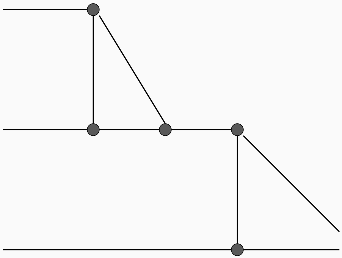
In addition, for each clause containing the literals l_1, l_2 and l_3 we have a gadget.

Gadget



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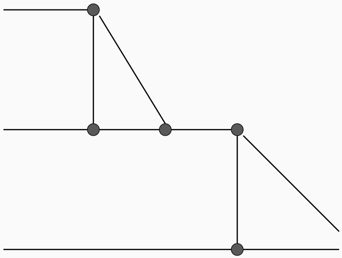

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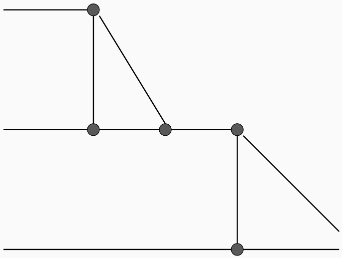

With a further edge from a to b.

Beyond graph problems

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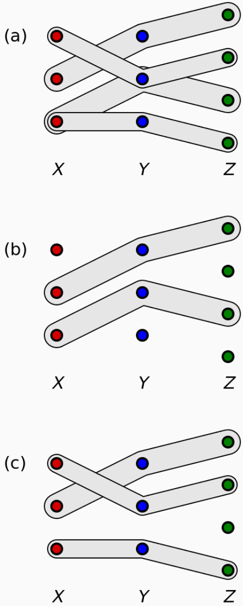
We now examine three more NP-complete problems, whose significance lies in that they have been used to prove a large number of other problems NP-complete, through reductions.

3D Matching



3D Matching

The decision problem of 3D Matching is defined as:

> Given three disjoint sets X, Y and Z, and a set of triples $M \subseteq X \times Y \times Z$, does M contain a matching?
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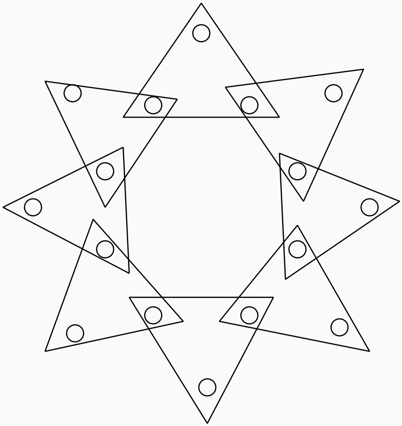
We can show that `3DM` is NP-complete by a reduction from `3SAT`.

Reduction

If a Boolean expression ϕ in 3CNF has n variables, and m clauses, we construct for each variable v the following gadget.



In addition, for every clause c, we have two elements x_c and y_c.

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Similarly, if $\neg v$ occurs in c, we include the triple

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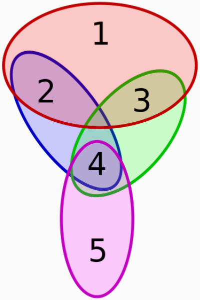
in M.

Finally, we include extra dummy elements in X and Y to make the numbers match up.

Set Cover



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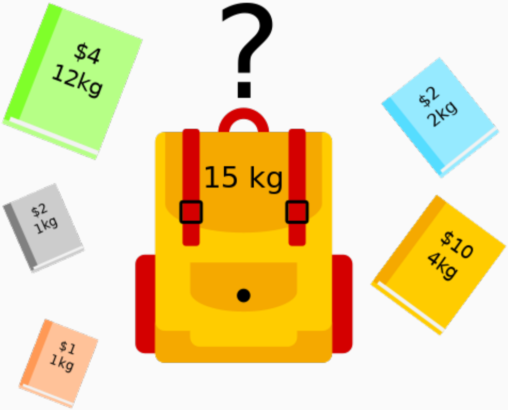
The reduction from 3DM simply takes $U = X \cup Y \cup Z$, and S to be the collection of three-element subsets resulting from M.

Set Covering

More generally, we have the Set Covering problem:

> Given a set U, a collection $S = \{S_1, \ldots, S_m\}$ of subsets of U and an integer budget B, is there a collection of B sets in S whose union is U?

Knapsack



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In the problem, we are given n items, each with a positive integer value v_i and weight w_i.

We are also given a maximum total weight W, and a minimum total value V.

> Can we select a subset of the items whose total weight does not exceed W, and whose total value is at least V?

Reduction

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Given a set $U = \{1, \ldots, 3n\}$ and a collection of 3-element subsets of U, $S = \{S_1, \ldots, S_m\}$.

We map this to an instance of KNAPSACK with m elements each corresponding to one of the S_i, and having weight and value

$$\sum_{j \in S_i} (m+1)^{j-1}$$

and set the target weight and value both to

$$\sum_{j=0}^{3n-1} (m+1)^j$$

Some examples of the kinds of scheduling tasks that have been proved NP-complete include:

Timetable Design

Given a set H of work periods, a set W of workers each with an associated subset of H (available periods), a set T of tasks and an assignment $r : W \times T \to \mathbb{N}$ of required work, is there a mapping $f : W \times T \times H \to \{0, 1\}$ which completes all tasks?

Sequencing with Deadlines

Given a set T of tasks and for each task a length $l \in \mathbb{N}$, a release time $r \in \mathbb{N}$ and a deadline $d \in \mathbb{N}$, is there a work schedule which completes each task between its release time and its deadline?

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Job Scheduling

Given a set T of tasks, a number $m \in \mathbb{N}$ of processors a length $l \in \mathbb{N}$ for each task, and an overall deadline $D \in \mathbb{N}$, is there a multi-processor schedule which completes all tasks by the deadline?

Food for thought:

Outside of P, is everything NP-hard?