Complexity Theory

Lecture 7: Reductions beyond graphs

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http://www.cl.cam.ac.uk/teaching/2324/Complexity
Recap

- A problem is $\mathcal{NP}$-hard if any language in $\mathcal{NP}$ is reducible to it.
- A problem is $\mathcal{NP}$-complete if it is: (1) $\mathcal{NP}$-hard, (2) in $\mathcal{NP}$.
- Cook-Levin Theorem: 3SAT is $\mathcal{NP}$-complete.
- Using 3SAT, we can establish $\mathcal{NP}$-completeness of many problems (e.g., IS, Clique, Hamiltonicity, TSP).
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Protip

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What are the big questions at this stage?
A graph $G = (V, E)$ is $k$-colourable, if there is a function

$$\chi : V \rightarrow \{1, \ldots, k\}$$

such that, for each $u, v \in V$, if $(u, v) \in E$,

$$\chi(u) \neq \chi(v)$$
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For all $k > 2$, $k$-colourability is $NP$-complete.
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For each variable \( x \), we have two vertices \( x, \bar{x} \) which are connected in a triangle with the vertex \( a \) (common to all variables).
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For each variable $x$, we have two vertices $x, \bar{x}$ which are connected in a triangle with the vertex $a$ (common to all variables).

In addition, for each clause containing the literals $l_1, l_2$ and $l_3$ we have a gadget.
Gadget
With a further edge from a to b.
Beyond graph problems
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We now examine three more \( \text{NP} \)-complete problems, whose significance lies in that they have been used to prove a large number of other problems \( \text{NP} \)-complete, through reductions.
3D Matching

(a)

(b)

(c)
3D Matching

The decision problem of 3D Matching is defined as:
Given three disjoint sets $X$, $Y$ and $Z$, and a set of triples $M \subseteq X \times Y \times Z$, does $M$ contain a matching?
I.e. is there a subset $M' \subseteq M$, such that each element of $X$, $Y$ and $Z$ appears in exactly one triple of $M'$?
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We can show that 3DM is NP-complete by a reduction from 3SAT.
If a Boolean expression $\phi$ in 3CNF has $n$ variables, and $m$ clauses, we construct for each variable $v$ the following gadget.
In addition, for every clause $c$, we have two elements $x_c$ and $y_c$. If the literal $v$ occurs in $c$, we include the triple

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Similarly, if $\neg v$ occurs in $c$, we include the triple

$$(x_c, y_c, \bar{z}_{vc})$$

in $M$.

Finally, we include extra dummy elements in $X$ and $Y$ to make the numbers match up.
Set Cover
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**Exact Cover by 3-Sets** is defined by:

Given a set \( U \) with \( 3n \) elements, and a collection \( S = \{S_1, \ldots, S_m\} \) of three-element subsets of \( U \), is there a sub-collection containing exactly \( n \) of these sets whose union is all of \( U \)?
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The reduction from 3DM simply takes \( U = X \cup Y \cup Z \), and \( S \) to be the collection of three-element subsets resulting from \( M \).
More generally, we have the **Set Covering** problem:

Given a set $U$, a collection $S = \{S_1, \ldots, S_m\}$ of subsets of $U$ and an integer budget $B$, is there a collection of $B$ sets in $S$ whose union is $U$?
Knapsack
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**KNAPSACK** is a problem which generalises many natural scheduling and optimisation problems, and through reductions has been used to show many such problems **NP**-complete.
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In the problem, we are given $n$ items, each with a positive integer value $v_i$ and weight $w_i$.

We are also given a maximum total weight $W$, and a minimum total value $V$.

Can we select a subset of the items whose total weight does not exceed $W$, and whose total value is at least $V$?
The proof that \textsc{Knapsack} is \textsc{NP}-complete is by a reduction from the problem of \textsc{Exact Cover by 3-Sets}.
Reduction

The proof that **KNAPSACK** is **NP**-complete is by a reduction from the problem of **Exact Cover by 3-Sets**.

Given a set $U = \{1, \ldots, 3n\}$ and a collection of 3-element subsets of $U$, $S = \{S_1, \ldots, S_m\}$.

We map this to an instance of **KNAPSACK** with $m$ elements each corresponding to one of the $S_i$, and having weight and value

$$\sum_{j \in S_i} (m + 1)^{j-1}$$

and set the target weight and value both to

$$\sum_{j=0}^{3n-1} (m + 1)^j$$
Some examples of the kinds of scheduling tasks that have been proved \textit{NP}-complete include:

**Timetable Design**

Given a set $H$ of work periods, a set $W$ of workers each with an associated subset of $H$ (available periods), a set $T$ of tasks and an assignment $r : W \times T \to \mathbb{N}$ of required work, is there a mapping $f : W \times T \times H \to \{0, 1\}$ which completes all tasks?
Sequencing with Deadlines

Given a set $T$ of tasks and for each task a length $l \in \mathbb{N}$, a release time $r \in \mathbb{N}$ and a deadline $d \in \mathbb{N}$, is there a work schedule which completes each task between its release time and its deadline?
Scheduling

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Job Scheduling

Given a set $T$ of tasks, a number $m \in \mathbb{N}$ of processors a length $l \in \mathbb{N}$ for each task, and an overall deadline $D \in \mathbb{N}$, is there a multi-processor schedule which completes all tasks by the deadline?
Food for thought:
Outside of P, is everything NP-hard?