# Complexity Theory

## Lecture 6: NP-Complete Problems

**Tom Gur**

http://www.cl.cam.ac.uk/teaching/2324/Complexity

# Preface: What do professors do all day?

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- In fact, so is CNF-SAT.
- And CNF-SAT is reducible to 3SAT:
  $(x_1 \vee x_2 \vee x_3 \vee x_4) \rightarrow (x_1 \vee x_2 \vee z_1) \wedge (\neg z_1 \vee x_3 \vee z_2) \wedge (\neg z_2 \vee x_4)$

# Composing Reductions

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Then

$$\text{SAT} \leq_P \text{3SAT} \leq_P A$$

Hence $A$ is also NP-complete.

**Let’s see some reductions!**

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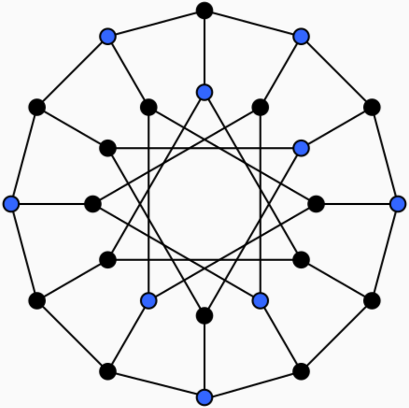

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To turn this *optimisation problem* into a *decision problem*, we define IS as:

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IS is clearly in NP. We now show it is NP-complete.

# Reduction

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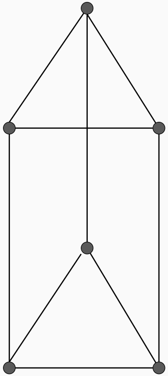
A Boolean expression $\phi$ in 3CNF with $m$ clauses is mapped by the reduction to the pair $(G, m)$, where $G$ is the graph obtained from $\phi$ as follows:

> *$G$ contains $m$ triangles, one for each clause of $\phi$, with each node representing one of the literals in the clause.*
> *Additionally, there is an edge between two nodes in different triangles if they represent literals where one is the negation of the other.*

# Example

$$(x_1 \vee x_2 \vee \neg x_3) \wedge (x_3 \vee \neg x_2 \vee \neg x_1)$$



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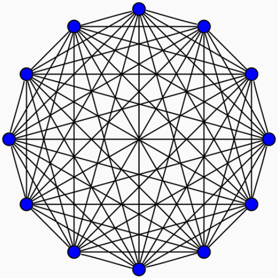

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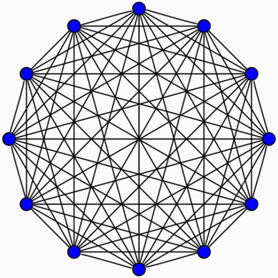

As with IS, we can define a decision problem:

CLIQUE is defined as:

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CLIQUE is NP-complete, since

IS $\leq_P$ CLIQUE

by the reduction that maps the pair $(G, K)$ to $(\bar{G}, K)$, where $\bar{G}$ is the complement graph of $G$.

# $k$-Colourability

A graph $G = (V, E)$ is $k$-colourable, if there is a function

$$\chi : V \to \{1, \dots, k\}$$

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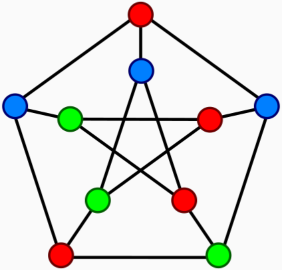

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For all $k > 2$, $k$-colourability is NP-complete.

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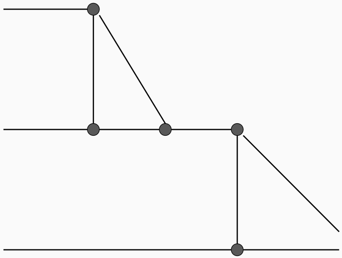
In addition, for each clause containing the literals $l_1$, $l_2$ and $l_3$ we have a gadget.

# Gadget



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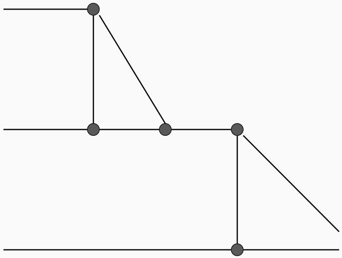

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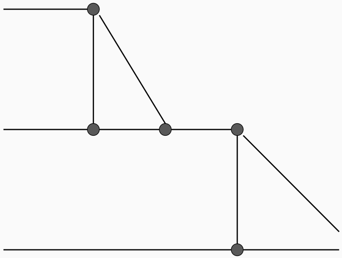

With a further edge from *a* to *b*.

# Hamiltonian Graphs

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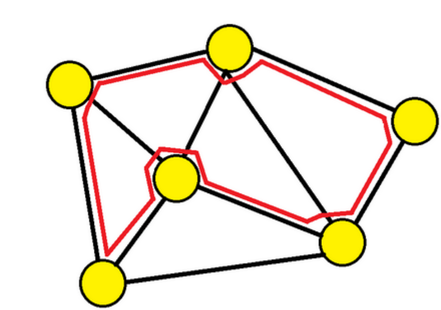

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The language HAM is the set of encodings of Hamiltonian graphs.

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This reduction is much more intricate than the one for IND.

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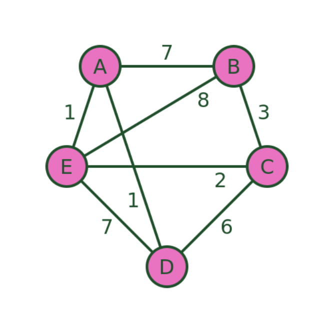

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The problem TSP consists of the set of triples

$$(V, c : V \times V \to \mathbb{N}, t)$$

such that there is a tour of the set of vertices $V$, which under the cost matrix $c$, has cost $t$ or less.

# Reduction

There is a simple reduction from HAM to TSP, mapping a graph $(V, E)$ to the triple $(V, c : V \times V \to \mathbb{N}, n)$, where

$$c(u, v) = \begin{cases} 1 & (u, v) \in E \\ 2 & (u, v) \notin E \end{cases}$$

and $n$ is the size of $V$.

# Bonus: Randomness and BPP