Complexity Theory

Lecture 5: Reductions

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http://www.cl.cam.ac.uk/teaching/2324/Complexity

Recap

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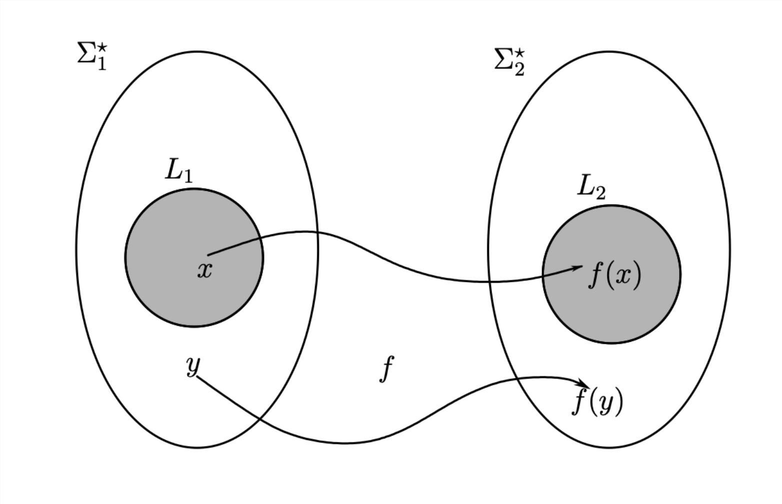
First superpower of complexity theory: solving one problem using another!

Reductions



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What is missing here?

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Why do we use the $\leq$ notation?

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We can get an algorithm to decide L_1 by first computing f, and then using the polynomial time algorithm for L_2.

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What languages are NP-complete?

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Since L is in NP, there is a nondeterministic Turing machine

$$M = (Q, \Sigma, s, \delta)$$

and a bound k such that a string x of length n is in L if, and only if, it is accepted by M within n^k steps.

Turing Machine Tableau

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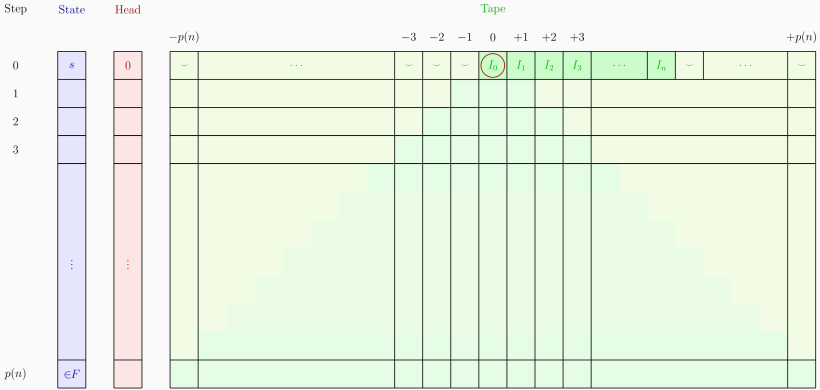

$f(x)$ has the following variables:

$$\begin{array}{ll} S_{i,q} & \text{for each } i \leq n^k \text{ and } q \in Q \\ T_{i,j,\sigma} & \text{for each } i,j \leq n^k \text{ and } \sigma \in \Sigma \\ H_{i,j} & \text{for each } i,j \leq n^k \end{array}$$

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We now have to see how to write the formula $f(x)$, so that it enforces these meanings.

Initialization

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The initial tape contents are x

$$\bigwedge_{j \leq n} T_{1,j,x_j} \wedge \bigwedge_{n<j} T_{1,j,\sqcup}$$

Consistency

The head is never in two places at once

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Each tape cell contains only one symbol

$$\bigwedge_i \bigwedge_j \bigwedge_\sigma (T_{i,j,\sigma} \rightarrow \bigwedge_{\sigma' \neq \sigma} (\neg T_{i,j,\sigma'}))$$

Computation

The tape does not change except under the head

$$\bigwedge_{i} \bigwedge_{j} \bigwedge_{j' \neq j} \bigwedge_{\sigma} (H_{i,j} \wedge T_{i,j',\sigma}) \to T_{i+1,j',\sigma}$$

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Each step is according to δ.

$$\bigwedge_{i} \bigwedge_{j} \bigwedge_{\sigma} \bigwedge_{q} (H_{i,j} \wedge S_{i,q} \wedge T_{i,j,\sigma}) \to \bigvee_{\Delta} (H_{i+1,j'} \wedge S_{i+1,q'} \wedge T_{i+1,j,\sigma'})$$

where Δ is the set of all triples (q', σ', D) such that $((q, \sigma), (q', \sigma', D)) \in \delta$ and

$$j' = \begin{cases} j & \text{if } D = S \\ j - 1 & \text{if } D = L \\ j + 1 & \text{if } D = R \end{cases}$$

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Finally, the accepting state is reached

$$\bigvee_i S_{i,\text{acc}}$$

CNF

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However, CNF-SAT, the collection of satisfiable CNF expressions, is NP-complete.

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3SAT is NP-complete, as there is a polynomial time reduction from CNF-SAT to 3SAT.

Composing Reductions

Polynomial time reductions are clearly closed under composition.

So, if $L_1 \leq_P L_2$ and $L_2 \leq_P L_3$, then we also have $L_1 \leq_P L_3$.

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If we show, for some problem A in NP that

$$\text{SAT} \leq_P A$$

or

$$\text{3SAT} \leq_P A$$

it follows that A is also NP-complete.

Questions?