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Recap

- Goal: understand the complexity of computational problems
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Is \textit{Composite} \(\in\) \textit{P}?

Clearly, the answer is yes if, and only if, \textit{Prime} \(\in\) \textit{P}.

Is there a conceptual difference between the two?
Hamiltonian Graphs

Given a graph $G = (V, E)$, a Hamiltonian cycle in $G$ is a path in the graph, starting and ending at the same node, such that every node in $V$ appears on the cycle exactly once.
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Given a graph $G = (V, E)$, a *Hamiltonian cycle* in $G$ is a path in the graph, starting and ending at the same node, such that every node in $V$ appears on the cycle *exactly once*.

The first of these graphs is not Hamiltonian, but the second one is.
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Is HAM ∈ P?
Graph Isomorphism

Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, is there a bijection

$$\pi : V_1 \rightarrow V_2$$

such that for every $u, v \in V_1$,

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In each case, there is a search space of possible solutions.

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- truth assignments to the variables of $\phi$;
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The size of the search is *exponential* in the length of the input.

Given a potential solution in the search space, it is *easy* to check whether or not it is a solution.
A verifier $V$ for a language $L$ is an algorithm such that

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Many natural examples arise, whenever we have to construct a solution to some design constraints or specifications.
Nondeterminism

If, in the definition of a Turing machine, we relax the condition on $\delta$ being a function and instead allow an arbitrary relation, we obtain a **nondeterministic Turing machine**.

$$\delta \subseteq (Q \times \Sigma) \times ((Q \cup \{\text{acc, rej}\}) \times \Sigma \times \{R, L, S\}).$$

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We still define the language accepted by $M$ by:

\[
\{x \mid (s, \triangleright, x) \rightarrow^*_M (\text{acc}, w, u) \text{ for some } w \text{ and } u\}
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though, for some $x$, there may be computations leading to accepting as well as rejecting states.
Computation Trees

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Nondeterministic Complexity Classes

We have already defined $\text{TIME}(f)$ and $\text{SPACE}(f)$. 

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\( \text{NTIME}(f) \) is defined as the class of those languages \( L \) which are accepted by a *nondeterministic* Turing machine \( M \), such that for every \( x \in L \), there is an accepting computation of \( M \) on \( x \) of length \( O(f(n)) \), where \( n \) is the length of \( x \).
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Theorem

A language $L$ is polynomially verifiable if, and only if, it is in $NP$. 
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The following describes a *nondeterministic algorithm* that accepts $L$:

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1. input $x$ of length $n$
2. nondeterministically guess $c$ of length $\leq p(n)$
3. run $V$ on $(x, c)$
In the other direction, suppose $M$ is a nondeterministic machine that accepts a language $L$ in time $n^k$. 

We define the deterministic algorithm $V$ which on input $(x, c)$ simulates $M$ on input $x$. At the $i$th nondeterministic choice point, $V$ looks at the $i$th character in $c$ to decide which branch to follow. If $M$ accepts then $V$ accepts, otherwise it rejects. $V$ is a polynomial verifier for $L$. 
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Why NP and not EXP?