# Complexity Theory

## Lecture 4: The class NP

**Tom Gur**

http://www.cl.cam.ac.uk/teaching/2324/Complexity

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Today we will go beyond tractable computation!

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Is there a conceptual difference between the two?

# Hamiltonian Graphs

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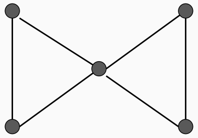

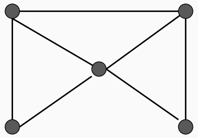

The first of these graphs is not Hamiltonian, but the second one is.

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Is HAM $\in$ P?

# Graph Isomorphism

Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, is there a *bijection*

$$\pi : V_1 \to V_2$$

such that for every $u, v \in V_1$,

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Is $\mathrm{Graph\ Isomorphism} \in \mathsf{P}$?

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Given a potential solution in the search space, it is *easy* to check whether or not it is a solution.

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Many natural examples arise, whenever we have to construct a solution to some design constraints or specifications.

# Nondeterminism

If, in the definition of a Turing machine, we relax the condition on $\delta$ being a function and instead allow an arbitrary relation, we obtain a *nondeterministic Turing machine*.

$$\delta \subseteq (Q \times \Sigma) \times ((Q \cup \{\mathsf{acc}, \mathsf{rej}\}) \times \Sigma \times \{R, L, S\}).$$

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We still define the language accepted by $M$ by:

$$\{x \mid (s, \triangleright, x) \rightarrow_M^\star (\mathrm{acc}, w, u) \text{ for some } w \text{ and } u\}$$

though, for some $x$, there may be computations leading to accepting as well as rejecting states.

# Computation Trees

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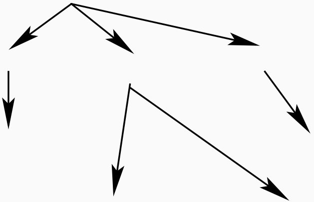
With a nondeterministic machine, each configuration gives rise to a tree of successive configurations.



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$$\mathsf{NP} = \bigcup_{k=1}^{\infty} \mathsf{NTIME}(n^k)$$

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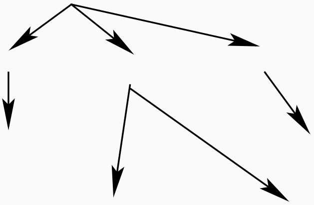

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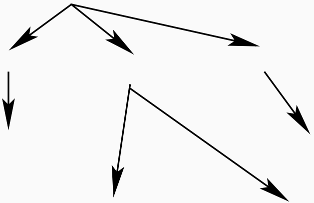

For a language in $\mathsf{NTIME}(f)$, the height of the tree can be bounded by $f(n)$ when the input is of length $n$.

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3. run $V$ on $(x, c)$

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$V$ is a polynomial verifier for $L$.

# Why NP and not EXP?