Complexity Theory

Lecture 2: Abstracting algorithms via Turing machines

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http://www.cl.cam.ac.uk/teaching/2324/Complexity

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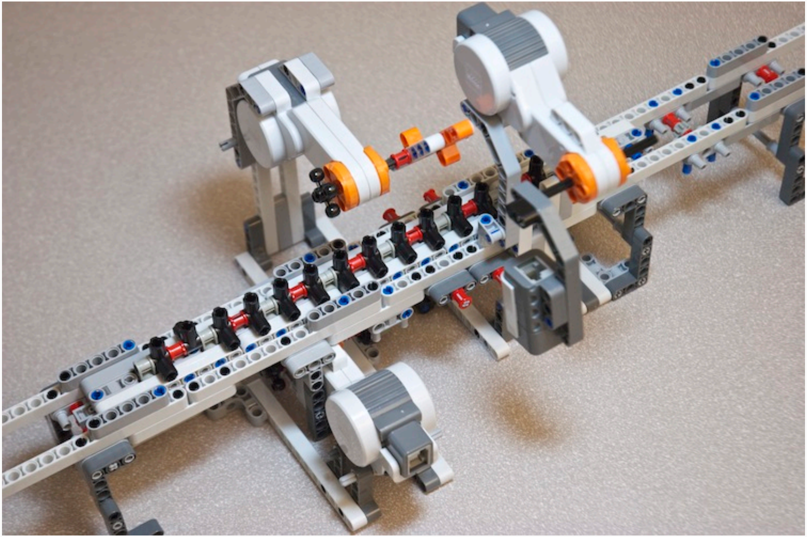
We will use the Turing machine.

> The simplicity of the Turing machine means it's not useful for actually expressing algorithms, but very well suited for proofs about all algorithms.

Turing Machines



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- Σ — a finite set of symbols, including $\sqcup$ and $\triangleright$.
- $s \in Q$ — an initial state;
- $\delta : (Q \times \Sigma) \to (Q \cup \{acc, rej\}) \times \Sigma \times \{L, R, S\}$
  A transition function that specifies, for each state and symbol a next state (or accept acc or reject rej), a symbol to overwrite the current symbol, and a direction for the tape head to move (L – left, R – right, or S - stationary)

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The configuration of a machine completely determines the future behaviour of the machine.

Computations

Given a machine $M = (Q, \Sigma, s, \delta)$ we say that a configuration (q, w, u) yields in one step (q', w', u'), written

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- $\delta(q, a) = (q', b, D)$; and
- either $D = L$ and $w' = v$ and $u' = bu$
  or $D = S$ and $w' = vb$ and $u' = u$
  or $D = R$ and $w' = vbc$ and $u' = x$, where $u = cx$. If u is empty, then $w' = vb\sqcup$ and u' is empty.

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The language $L(M) \subseteq \Sigma^\star$ accepted by the machine M is the set of strings

$$\{x \mid (s, \triangleright, x) \to_M^\star (acc, w, u) \text{ for some } w \text{ and } u\}$$

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A machine M is said to halt on input x if for some w and u, either $(s, \triangleright, x) \rightarrow_M^\star (acc, w, u)$ or $(s, \triangleright, x) \rightarrow_M^\star (rej, w, u)$

Example

Consider the machine with δ given by:

	$\triangleright$	0	1	$\sqcup$
s	$\mathrm{s}, \triangleright, \mathrm{R}$	$\mathrm{rej}, 0, \mathrm{S}$	$\mathrm{rej}, 1, \mathrm{S}$	$\mathrm{q}, \sqcup, \mathrm{R}$
q	$\mathrm{rej}, \triangleright, \mathrm{R}$	$\mathrm{q}, 1, \mathrm{R}$	$\mathrm{q}, 1, \mathrm{R}$	$\mathrm{q}', 0, \mathrm{R}$
q'	$\mathrm{rej}, \triangleright, \mathrm{R}$	$\mathrm{rej}, 0, \mathrm{S}$	$\mathrm{q}', 1, \mathrm{L}$	$\mathrm{acc}, \sqcup, \mathrm{S}$

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Consider the machine with δ given by:

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s	$s, \triangleright, R$	$rej, 0, S$	$rej, 1, S$	$q, \sqcup, R$
q	$rej, \triangleright, R$	$q, 1, R$	$q, 1, R$	$q', 0, R$
q'	$rej, \triangleright, R$	$rej, 0, S$	$q', 1, L$	$acc, \sqcup, S$

This machine, when started in configuration $(s, \triangleright, \sqcup 1^n 0)$ eventually halts in configuration $(acc, \triangleright \sqcup 1^{n+1} 0 \sqcup, \varepsilon)$.

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Hence, the model does not matter. We can use whichever is most convenient.

To date, the only widely accepted contender to the Extended Church-Turing thesis is Quantum Computing.

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Similarly, a configuration is of the form:

$$(q, w_1, u_1, \ldots, w_k, u_k)$$

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A language L is semi-decidable if it is recursively enumerable.

A function $f : \Sigma^\star \to \Sigma^\star$ is computable, if there is a machine M, such that for all x, $(s, \triangleright, x) \to_M^\star (acc, \triangleright f(x), \varepsilon)$

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$r(n)$ is defined to be the largest value R such that there is a string x of length n so that the computation of M starting with configuration $(s, \triangleright, x)$ is of length R (i.e. has R successive configurations in it) and ends with an accepting configuration.

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We let $r(n) = 0$ if M does not accept any inputs of length n.

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In defining space complexity, we assume a machine M, which has a read-only input tape, and a separate work tape. We only count cells on the work tape towards the complexity.

Questions?