Correction to the formula for the London moment of a rotating superconductor

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Abstract

This paper gives a full quantum-mechanical analysis of the magnetic field (first discussed by London) which appears spontaneously when a sample of superconductor is set into rotation. It shows that, for slow rotation velocities and using certain approximations, the field $\underline{B}$ threading a cavity within a superconductor which rotates at angular velocity $\underline{\omega}$, is given by $e\,\underline{B} = 2\,(m_0 - W/c^2)\,\underline{\omega}$, where $-e$ is the charge on the electron, m_0 is the free electron mass, W is the work-function of the superconductor, and c is the velocity of light. In this calculation effects which are second-order in the rotation velocity have been ignored, and the result is only strictly valid at the zero of temperature.

The application of this result to experiments using practical, non-ideal apparatus is then illustrated for a simple geometry.

1 Introduction

When a sample of superconductor is set into rotation, a magnetic field is generated spontaneously by currents flowing in the surface of the superconductor. This field is called the London field, and the following analysis is based upon the work of F. London. [1], [2]. The local canonical momentum of the electron pairs in a superconductor is related to their velocity $\underline{u}$ and to the magnetic vector potential $\underline{A}$ which they experience:

$$\underline{p} = m^* \underline{u} + e^* \underline{A} \tag{1.1}$$

where m* and e* are the effective mass and charge associated with an electron pair. (We shall see that m* cannot be identified with the band effective mass.)

London showed that the velocity of the electron pairs deep within a sample of superconductor is just the velocity of the lattice, so that for a rotating superconductor $\underline{u} = \underline{\omega} \times \underline{r}$, where $\underline{\omega}$ is the rotation vector and $\underline{r}$ is the vector from the axis of rotation. The quantization condition that the line integral of the canonical momentum around any loop must vanish can now be applied (we do not consider here singular situations such as would occur if for example vortices were present). This condition yields:

$$\oint m^* \underline{u}.d\underline{l} = \oint -e^* \underline{A}.d\underline{l}$$

$$m^* \,\mathrm{curl}\, \underline{u} = -e^* \,\mathrm{curl}\, \underline{A} \tag{1.2}$$

$$2\, m^* \underline{\omega} = -e^* \underline{B}$$

The last of these equations gives the London formula for the magnetic field $\underline{B}$.

This treatment is unsatisfactory in several respects. It is assumed that the superconductor can be analysed by analogy with the properties of a pair of electrons. Even if this analogy is accepted, it is unclear what value should be used for the effective mass m^* of the pair: London's analysis used simply twice the rest mass of the electron, but it is not clear from this analysis whether lattice interactions and other effects may modify this mass. (There was in fact a stray factor of two in London's original analysis because it was not realized at the time that pairs rather than single electrons were involved. We have inserted appropriate factors of two where necessary.)

Experimental measurements upon the London moment in several different materials and for several different geometries have shown agreement with the London value $m^* = 2\, m_0$, where m_0 is the rest mass of the electron. (3) This agreement has been verified to of order 1% . However, experiments have been proposed (4) to use the London moment in a measurement of the mass of the electron, which is known at present to $5\frac{1}{2}$ significant figures. It is therefore important to have a full analysis of the value of the quantity m^* which enters into the London moment. The purpose of this analysis is to obtain a precise value for this mass m^* .

Our analysis will refer to the following, ideal apparatus. The London field $\underline{B}$ is measured within a cavity which is completely enclosed by the superconductor, where the thickness of all walls is much greater than the magnetic penetration depth in the material. In this way the field is screened from the influence of external fields. There is no apparatus in the cavity which might induce capacitive charges anywhere on the walls of the cavity: such charges would constitute currents as they rotated with the walls and so would influence the magnetic field. There is no apparatus within the cavity which might cause currents to flow in the walls of the cavity: if there were such currents then the magnetic penetration depth of the material would enter into the formula for the field. (It can be noted that the current in the walls of a cavity within a rotating superconductor is zero unless some perturbation is applied: this follows from the observation that the London field (1.2) is a constant field and so can be set up entirely by currents flowing at the outer surfaces of the superconductor. We give later a more detailed discussion of this property.)

In section 2 we give an analysis of the value for the effective mass m^* , which is based upon London's assumption that the analogy with the properties of a pair of electrons is valid. We shall assume that our pair of electrons is at the Fermi level in the metal. Although this analysis is not rigorous and is based upon assumptions which cannot easily be justified, it does show the physical origins behind the corrections to the effective mass. In section 3 a fuller, quantum mechanical analysis of the problem is given, which does not require these assumptions to be made. Section 4 illustrates the way in which the above restrictions can be relaxed, so that the analysis can be applied to apparatus in which there may be capacitive charges inside the cavity, and in which currents may be induced in the walls of the cavity.

2 Semi-classical analysis of the London moment

In this section we follow through the consequences of London's assumption that the London moment can be analysed by analogy with the properties of a pair of electrons. This analysis will show the physical basis behind the corrections to London's value for the effective mass m^*, but it should again be emphasized that this section is not rigorous and that a full and rigorous analysis is given in the next section.

This section is therefore based upon an analogy with the properties of a pair of electrons. It is important to know what is the energy of the pair, and we analyse this first. Since superconductivity is a phenomenon associated with the electrons within a few millielectronvolts of the Fermi level in the superconductor (5), we shall assume that our pair has an energy appropriate to this level. In order to evaluate this energy, consider an experiment in which an electron is knocked out from the Fermi level into a large cavity within the metal, so that the electron is at rest in the cavity. The energy required to do this is W, the work-function of the metal: note that the value of W defined in this way depends upon the metal used, but since there can be no electric fields within the cavity (other than those due to the electron itself) then the value of W does not depend upon other conditions.* The work-function is therefore a well-defined quantity. The electron within the cavity behaves exactly like a free electron, and it therefore has rest mass energy $m_0 c^2$. (We have chosen an electromagnetic gauge (6) in which the absolute potential within the cavity is zero, and so there are no electrostatic contributions to this rest-mass energy.) By subtraction it will be clear that the energy of a pair of electrons at the Fermi level is $2 (m_0 c^2 - W)$.

The energy of our pair can be split into three terms: the rest-mass energy, the kinetic energy, and the electrostatic potential energy. We shall for the sake of concreteness imagine that the electrons comprising our pair are orbiting around one another with kinetic energy KE per particle, and that each electron is subjected to a potential V. The energy of each electron $m_0 c^2 + KE - e V = m_0 c^2 - W$ is of course a constant in space,

* The cavity must be large enough that mirror charges do not significantly affect the electron's energy.

although the individual kinetic and potential energy contributions could well vary rapidly in space. For example, the potential V includes contributions from all charge distributions in the metal, such as surface dipoles lining the walls of the cavity, the screening hole surrounding each electron, and charge inhomogeneities associated with atomic cores and valence elecrons. In our analysis of the London moment in this section we shall consider separately the effects of kinetic and of potential energy terms at each point in space upon the effective mass of a pair.

Consider first the rest-mass and kinetic energy terms. It is well known in relativity that the mass of a system with rest mass $2\ m_o$ and kinetic energy 2 KE is:

$$m^*_{kinetic} = 2\ (\ m_o + KE/c^2\ ) \tag{2.1}$$

To be precise, this mass $m^*_{kinetic}$ is defined so that, if the whole system is subjected to a Lorentz transformation so that it moves past an observer with a (low) velocity $\underline{u}$, and if there is no electromagnetic field, then the momentum of the system is increased by $m^*_{kinetic}\ \underline{u}$. [7] We apply this to the centre-of-mass motion of an electron pair deep within the material of the superconductor, which moves with the lattice velocity $\underline{u}$. If in addition the magnetic vector potential $\underline{A}$ is taken into account, then the local momentum of an electron pair which is deep within the metal of a superconductor whose lattice moves at local velocity $\underline{u}$, is given by:

$$\underline{p} = 2\ (\ m_o + KE/c^2\ )\underline{u} - 2\ e\ \underline{A} \tag{2.2}$$

It should be noted that the concept of band effective mass does not enter into consideration here, since the electron pairs are considered to remain stationary with respect to the lattice.

The local vector potential $\underline{A}$ which is defined within the material of the metal, is however not the quantity of interest to an experimenter. We wish to obtain the value of the field which is within the cavity, $\underline{A}_{external}$. This will differ from the field within the material of the superconductor because of the presence of the charge distributions which set up the potential V experienced by the electrons in our pair: the charges move with the lattice as it rotates and so constitute currents which create electromagnetic fields. (We shall show later using a self-consistency argument that these electromagnetic fields due to the moving charges are not screened by supercurrents.)

In the following analysis we shall make the approximation that all the dimensions of the apparatus are large compared to the electrostatic penetration depth in the material. This approximation is of great computational convenience, because the velocity of motion of the lattice as it rotates can be regarded as constant in the region of the wall where the potential V seen by the electrons is changing rapidly. Since the electrostatic penetration depth is very small, typically a few angstroms, this is a good approximation and we consider no further any corrections on account of the finite penetration depth. We therefore now consider a small region of the wall of the cavity, which is rotating with a tangential velocity $\underline{u} = \underline{\omega} \times \underline{r}$. This velocity $\underline{u}$ is constant in this small region, according to our approximation.

If the local charge density in the metal is ρ , then the rotating superconductor has a current density on account of the moving charge, whose magnitude is $\rho \, \underline{u} = \underline{j}$. We shall assume for the moment that the fields generated by these moving charges are not screened by supercurrents. (We shall later demonstrate the self-consistency of this assumption). Maxwell's equations for the magnetic vector potential $\underline{A}$ can now be applied:

$$\nabla^2 \underline{A} = \mu_0 \underline{j} = \mu_0 \rho \underline{u} = \mu_0 \epsilon_0 \underline{u} \nabla^2 V$$

$$= (\underline{u}/c^2) \nabla^2 V \qquad (2.3)$$

where we have used the equation $\mu_0 \epsilon_0 c^2 = 1$. This equation can now be integrated across the wall of the cavity. If the field within the cavity is $\underline{A}_{external}$, close to the wall, then the field within the material of the superconductor just across the wall, $\underline{A}$, is given by:

$$\underline{A} = \underline{A}_{external} + (\underline{u}/c^2) V \qquad (2.4)$$

where we have used the fact that we have chosen a gauge where the absolute potential V inside the cavity is zero. This equation can now be inserted into (2.2):

$$\underline{p} = 2(m_0 + KE/c^2 - eV/c^2)\underline{u} - 2e\underline{A}_{external}$$

$$= 2(m_0 - W/c^2)\underline{u} - 2e\underline{A}_{external} \qquad (2.5)$$

It will be clear by comparison with equation (1.1), that the value of the effective mass of our pair, m^*, is simply

$$m^* = 2(m_0 - W/c^2) \qquad (2.6)$$

This value for m^* is the principal result in this section. It was obtained assuming that the magnetic fields due to the moving charge densities in the metal are not screened by supercurrents; we now show that this assumption was indeed justified, using a self-consistency argument.

Consider our small section of the wall of the cavity, using for simplicity a gauge where the canonical momentum of the electron pairs just inside the material of the superconductor is zero; that is, in equation (2.5) $\underline{p} = \underline{0}$. Using equation (2.5), we deduce that the value of the magnetic vector potential just inside the cavity is given by $\underline{A}_{external} = (m_0 - W/c^2)\,\underline{u}\,/\,e$. In this situation, consider the part of the wavefunction of our electron pair, which extends beyond the metal and a short distance into the cavity itself. Using a semi-classical analysis, one notes that the part of the pair in this region has no potential energy, $V = 0$, and it has negative kinetic energy, $KE = -W$. Equation (2.5) can be applied to this part of the system, and it can be seen that the canonical momentum associated with the part of the electron pair's wavefunction which extends into the cavity is zero, $\underline{p} = \underline{0}$. By a similar argument, it is easy to see that $\underline{p} = 0$ holds everywhere in the region of wall under consideration. If however supercurrents were to flow so as to screen out the magnetic fields on account of the moving charges, this condition would no longer hold and the quantization of momentum (1.2) would no longer be valid for electron pairs in different parts of the region of the wall under consideration. It can therefore be concluded that there is a self-consistent solution to the equations in which the fields of the rotating charges are not screened by supercurrents.

To summarize this section, we have assumed that the London moment can be analysed by analogy with the properties of a pair of electrons, each of which has the Fermi energy in the superconductor. We have used the approximation that the electrostatic penetration depth is very short compared to the dimensions of our experiment. We have ignored centrifugal effects, and the influence of any excitations in the superconductor which may be mobile (so that we are restricted to the case where $T \ll T_c$, where there are few excitations). The principal result is that the effective mass m^* entering into London's formula (1.2) is $m^* = 2 (m_o - W / c^2)$, where W is the work-function of the superconductor.

Though plausible, the first assumption made above cannot easily be justified. In the next section we give a full and rigorous quantum-mechanical analysis of the situation which does not require the first assumption made above.

3 Quantum analysis of the London moment

In his derivation of the effect which now bears his name, Josephson (8) emphasized the importance of the phase θ of the order parameters in a superconductor. We shall follow some of Josephson's discussion of phase, and apply some of the results derived there to our own problem. In particular we note that θ must be continuous and single-valued, so that the integral of the rate of change of phase with position, taken around any closed loop, must vanish. (As before, we do not consider singular behaviour such as would occur if flux lines were to thread the superconductor.) In other words, for any closed loop:

$$\oint \underline{\nabla}\theta . \underline{dl} \quad = \quad 0 \qquad (3.1)$$

This equation (3.1) is the basic equation behind our derivation in this section. We shall see that it is closely related to London's condition, that the line integral of the canonical momentum of an electron pair taken around any closed loop must vanish, i.e. $\oint \underline{p} \cdot \underline{dl} = 0$.

Before it is possible to interpret (3.1) in terms of velocities and magnetic fields, it is necessary to investigate some of the properties of the phase θ . Firstly, consider a sample of superconductor which is stationary, which is not rotating, and which has no magnetic field threading it. Equation (3.1) holds in this situation, so therefore it is possible to choose a gauge in which the phase θ does not vary in space, $\underline{\nabla}\,\theta = \underline{0}$ everywhere. We shall use this gauge in all our following analysis. (Of course, if the conditions affecting the sample are changed, by for example the application of a magnetic field or setting the sample into rotation, we would not change our frame of reference in our description of the system, and so it would no longer necessarily hold that $\underline{\nabla}\,\theta = \underline{0}$.)

In our nonrotating sample, therefore, the phase θ is invariant with spatial position. In order to complete our description of the phase, it is necessary to know the time dependence of this parameter. In appendix 1 we reproduce an analysis of this which is originally due to Josephson [8], although the notation has been changed somewhat and the result has been put into the gauge of our analysis, where the absolute electrostatic potential inside a cavity within the superconductor is zero. The result is that the phase θ varies with space and time according to:

$$\hbar\,[\partial\theta/c\partial t,\ \underline{\nabla}\theta] = [-2(m_oc^2-W)/c,\ \underline{0}] \qquad (3.2)$$

where m_o is the rest mass of the electron, c is the velocity of light, and W is the work-function of the superconductor used, according to the definition given in the previous section.

We now have a complete description of the spatial and temporal dependence of the phase of the superconducting order parameter θ in our stationary, nonrotating sample of superconductor. It is now possible to generate a description of the space-time dependence of the phase in a sample identical to the one described above, but which has been set into motion with uniform velocity $\underline{u}$.

Equation (3.2) is written in a special form. The phase θ is a scalar, whilst the derivative $[\partial/c\partial t,\ \underline{\nabla}]$ is a Lorentz four-vector. [7] The left hand side of (3.2) is therefore a Lorentz four-vector, and it transforms according to the usual relativistic transformation laws. (Of course, this would not be true if we had omitted any terms from (3.2). Note the importance of the rest mass energy term in (3.2).) In particular, one needs simply to apply the Lorentz transformation laws to (3.2) in order to generate a description of a sample which has been set in motion with velocity $\underline{u}$. This yields:

$$\hbar\,[\,\partial\theta/c\,\partial t\,,\ \underline{\nabla}\theta\,] \;=\; (m_o c^2 - W)\,[\,-2\gamma/c\,,\ 2\gamma\underline{u}\,] \qquad (3.3)$$

where $\gamma^{-2} = 1 - u^2/c^2$. It should be noted that this takes into account all effects, including for example the effect upon the phase of θ of the magnetic fields due to the charge densities within the metal which are moving with velocity $\underline{u}$. (As a matter of fact, the fields of the moving charge densities are confined to within the material of the superconductor, at least in the simple case where the sample is uncharged and in a region of zero electric field.)

In future analysis we shall make the approximation that all velocities of motion of the lattice are small compared to that of light. We shall therefore take $\gamma = 1$ to hold.

In apendix 1 there is also a discussion of the effect of the application of a constant magnetic vector potential $\underline{A}_{external}$ upon the phase in a sample of superconductor. Using the results of that appendix, the phase gradient of a sample of superconductor which moves with velocity $\underline{u}$ and which is also subjected to a constant magnetic vector potential $\underline{A}_{external}$, is:

$$\hbar\,\underline{\nabla}\theta \;=\; 2(\,m_o - W/c^2\,)\,\underline{u} \;-\; 2e\underline{A}_{external} \qquad (3.4)$$

Equation (3.4) is an equation written in local form. To within certain approximations which we shall discuss later, the state of a rotating superconductor is locally the same as that of an equivalent translating superconductor. In the case of rotation, the local velocity of motion of the lattice is simply $\underline{u} = \underline{\omega} \times \underline{r}$. Applying condition (3.1) to this, one deduces that:

$$\oint 2(m_0 - W/c^2)\, \underline{u}.d\underline{l} \quad = \quad \oint 2\, e\, \underline{A}_{external} \cdot d\underline{l}$$

$$(m_0 - W/c^2) \text{ curl } \underline{u} \quad = \quad e \text{ curl } \underline{A}_{external} \qquad (3.5)$$

$$2(m_0 - W/c^2)\, \underline{\omega} \quad = \quad e\, \underline{B}$$

The last of these equations gives the London field $\underline{B}$ which is in the cavity of a superconductor which rotates at angular velocity $\underline{\omega}$. It is of interest to compare this derivation with that given in section 1 (equations (1.1) and (1.2)). The quantity $\hbar \underline{\nabla} \theta$ replaces the canonical momentum $\underline{p}$ of section 1, but otherwise the derivations follow closely similar lines.

There are however a number of approximations which have to be made before it is valid to apply equation (3.4) to the case of a rotating sample of superconductor, as shown above. Firstly, the effects of centrifugal fields have been ignored: such fields could for example distort the lattice and cause the value of the work-function of the superconductor to change in some region, thereby affecting the magnitude of the London field. Secondly, the effect of excitations in the superconductor has not been be taken into account: excitations moving radially inwards or outwards will experience Coriolis and magnetic forces which will cause some motion of charges relative to the lattice. Although on average one might expect the net current on account of this to be zero, there may be second-order effects. (As a matter of fact, the Coriolis and magnetic forces have already been taken into account as far as they affect the order parameters of the ground-state wavefunction, since we have matched the boundary conditions for the order parameters in our rotating sample. See reference (9) for further discussion.) The third approximation results from the fact that the tangential velocity of rotation is not truly a constant, but varies with the radius from the axis of rotation. It was noted earlier that the transformation leading to equation (3.4) takes into account the magnetic field due to the charge densities in the metal, which are all assumed to be moving with the same, uniform velocity $\underline{u}$. In the case of rotation, however, these charge densities could well be at some different radius and so they need not necessarily move at the same velocity as the element under consideration. The magnitude of the resulting correction will depend upon the typical distance over which the potential V changes, i.e. the electrostatic penetration depth. (It is easy to show that the magnetic flux enclosed by the outer of two concentric, corotating, nonconducting long cylinders of charge with radii r and $r - \lambda$ and which support a constant voltage between them, is proportional to $1 - \lambda / r$ to first order in λ / r . The correction to the London moment would therefore be expected to depend in a similar fashion upon the electrostatic penetration depth λ and the radius of the experiment r .) Since the electrostatic penetration depth is typically a few angstroms, much shorter than the dimensions of any reasonable-sized experimental apparatus, then it is a good approximation to neglect the electrostatic penetration depth.

It is here that lie the most serious problems with our calculation of the London field. The corrections on account of the centrifugal field and on account of excitations in the superconductor could well be of consequence to the experimentalist. It is unlikely that the correction due to electrostatic penetration depth could be measured experimentally, on account of the very small size of the effect, though the effect throws up a number of interesting theoretical points which have not been analysed. In particular, it may be possible to treat more exactly the effect of the finite electrostatic penetration depth through the use of angular transformations rather than the linear ones used in this paper. Some discussion of these transformations is given in chapter 1.

To summarize this section, we have made the approximations that the velocity of rotation of a sample of superconductor is small, and that the centrifugal effects are therefore negligible; that the electrostatic penetration depth in the superconductor is very short compared to the dimensions of the sample; and that the effects of excitations in the superfluid can be neglected. The result of this section is that the mass m^* entering London's equation for the London field has value $m^* = 2 (m_0 - W / c^2)$, where W is the work-function of the superconductor, defined as the energy required to knock an electron out from the Fermi level in the metal, into a large cavity within the metal.

4 Practical apparatus

In the previous sections we have analysed the London field of a rotating superconductor, without consideration of the practical problems of measuring such a field. For example, suppose that the field within the cavity of our rotating sample is to be measured using a superconducting loop and superconducting ammeter (SQUID): if the loop itself rotates then it too will have a London field which will perturb the measurement; if the loop is made of some metal which has a work-function different from the work-function of the superconductor under investigation, there could be electric fields set up within the cavity to maintain the difference in work-functions, and the charges which maintain these fields would in turn create magnetic fields as they rotate with the apparatus, thus affecting the measurement; similarly, any voltages applied to the system could affect the measurement through the magnetic fields of the Coulomb charges set up as they rotate with the apparatus. In this section we illustrate how to analyse the corrections which result from these effects, using a cylindrical geometry in which the mathematics takes on a particularly simple form.

See figure 1, which shows a cross-section of the apparatus which will be investigated in depth in this section. A long cylinder with inner radius r_o is made from a sample of superconductor with uniform work-function W_o and with magnetic penetration depth λ_o. This cylinder can be rotated about its axis with angular velocity ω_o. Inside this is a second long cylinder, which is made from a superconductor with uniform work-function W_i, and magnetic penetration depth λ_i. The outer radius of this cylinder is r_i, and it can be rotated about its axis with rotation velocity ω_i.

A voltage V can be applied between the two cylinders. By the quantity V is meant the voltage which would be measured using a voltmeter attached between the two cylinders: that is, $V = D\mu / e$ where $D\mu$ is the difference in electrochemical potential between the electrons in the two cylinders. It can be noted that V is related to the difference in work-functions DW and to the electrostatic voltage between the cylinders $V_{ES} = \int \underline{E}\, \underline{dl}$, through the equation:

$$V = V_{ES} + DW / e \qquad (4.1)$$

The current flowing around the inner cylinder can be measured using a SQUID which is inserted into the cylinder. For example, there could be a slit extending the length of the cylinder, and the SQUID could be connected across the slit. In the following analysis we shall not consider the properties of the SQUID, assuming that it has small inductance so that it does not interrupt the flow of current around the loop, and assuming that it is physically small so that any London-like effects occurring within the SQUID itself as it rotates with the inner loop can be neglected. Later, we relax these assumptions.

In order to simplify the equations which occur in this section, we shall use the notation $r_o' = r_o + \lambda_o$, and $r_i' = r_i - \lambda_i$. These primed quantities are of use in considering the screening effect of one cylinder upon magnetic fields generated by the other: for example, suppose that currents in the outer cylinder cause a magnetic field $\underline{B}$ to appear in the space between the cylinders. The flux enclosed within a line taken around the loop with radius r_o' is, after allowance has been made for the screening effect of the inner cylinder, $\phi = \pi (r_o'^2 - r_i'^2) B$. In other words, the cylinders behave as though they had the primed radii as far as screening of fields is concerned.

To begin our analysis of the current flowing through the SQUID shown in the figure, we shall take the simple case where there is no electrostatic voltage between the two cylinders: that is, in equation (4.1), $V_{ES} = 0$. In this case there are no net Coulomb charges anywhere in the system, and so we can neglect the effect of such charges. Suppose now that the outer cylinder is caused to rotate at angular velocity ω_o , whilst the inner cylinder remains stationary. Consider a line taken around the cylinder at radius r_o' . From equation (3.5) (or, alternatively, from equations (1.2) and (2.6)), one deduces that the flux enclosed $\phi = \oint \underline{A}.\underline{dl}$ is:

$$e\,\phi = 2(\,m_o - W/c^2\,)\,\pi\, r_o'^2\,.\omega_o \qquad (4.2)$$

It now follows, using simple magnetic formulae for long cylinders, and using the remark made above about the screening effect of the inner cylinder, that the current flowing in the SQUID is:

$$\mu_o\, e\, I = 2\left(m_o - \frac{W_o}{c^2}\right) \frac{r_o'^2}{r_o'^2 - r_i'^2}\,\omega_o \qquad (4.3)$$

A similar analysis can be made for the case where the inner cylinder is made to rotate at angular velocity ω_i, whilst the outer cylinder is kept in an inertial frame (i.e. not rotating). The flux within a line taken around the cylinder at radius r_i' is given by equation (3.5), whilst the flux within the line taken around the cylinder at radius r_o' must be zero. The result of this calculation, when added to the result (4.3), gives the current flowing through the ammeter as a function of the work-functions, the radii and the angular velocities of the two cylinders; it is thus far restricted to the case where the electrostatic voltage between the cylinders is zero:

$$\mu_o e I = \left[\left(m_o - \frac{W_o}{c^2}\right)\omega_o - \left(m_o - \frac{W_i}{c^2}\right)\omega_i\right] \qquad (4.4)$$

$$\times \qquad \frac{2 r_o'^2}{r_o'^2 - r_i'^2}$$

We next consider the effect of the application of an electrostatic voltage between the cylinders. If such a voltage is applied, then the Coulomb charges appear within the electrostatic penetration depth of the surfaces of the cylinders; we shall make the approximation that this depth is very short, so that the charges appear at radii r_o and r_i. The question now arises: does the appearance of these charges alter the magnetic penetration depth in the superconductor by altering the number of available charge carriers within this depth ? Simple order-of-magnitude calculations show that the number of charge carriers within the magnetic penetration depth exceeds by many orders of magnitude the number of carriers which could be added or subtracted by electric fields of even several kilovolts per millimetre (the breakdown field of liquid helium), and so we shall not consider this effect.

The capacitance between the cylinders per unit length is given by the formula $C = 2\pi \epsilon_o / \log_e (r_o / r_i)$. If the outer cylinder rotates at angular velocity ω_o whilst there is an electrostatic voltage V_{ES}, then the motion of the charges constitutes a current (we consider later how this is screened by supercurrents) of magnitude I per unit length of cylinder, where:

$$I = (C V_{ES} / 2\pi r_o) r_o \omega_o$$

$$= \epsilon_o V_{ES} \omega_o / \log_e(r_o/r_i) \qquad (4.5)$$

This current is screened by supercurrents which we imagine to flow at radii r_o' and r_i'; the effect of this upon the current through the ammeter can be calculated in a similar way to the calculations above, and the effect of the electrostatic voltage is to cause a current to flow in the ammeter:

$$\mu_o e I = \frac{\mu_o \epsilon_o e V_{ES}}{\log_e (r_o - r_i)} \cdot \omega_o \cdot \frac{r_o^2 - r_i^2}{r_o^2 - r_i'^2} \tag{4.6}$$

In a similar fashion the extra current due to an applied voltage can be calculated for the case where the inner cylinder is caused to rotate. The sum total of this effect, and of the effects described by equations (4.6) and (4.3) is:

$$\mu_o e I = \left[\left(m_o - \frac{W_o}{c^2}\right)\omega_o - \left(m_o - \frac{W_i}{c^2}\right)\omega_i \right] \frac{2 r_o'^2}{r_o'^2 - r_i'^2} + \frac{e V_{ES}}{c^2 \log_e (r_o / r_i)} \left[\frac{r_o'^2 - r_o^2}{r_o'^2 - r_i^2}\, \omega_o - \frac{r_o^2 - r_i^2}{r_o^2 - r_i'^2}\, \omega_i \right] \tag{4.7}$$

where we have used the equation $\mu_o \epsilon_o c^2 = 1$ to simplify the equation.

This equation (4.7) is the principal result of this section. In order to see some of the consequences of this equation, we shall apply it to the simple case where the inner and outer cylinder are made of the same material, so that the work functions are identical, $W_o = W_i = W$, and the magnetic penetration depths are also the same, $\lambda_o = \lambda_i = \lambda$. We shall work in units where the mean radius of the cylinders is unity, $(r_o + r_i) / 2 = 1$, and we shall define the semi-distance between the cylinders to be d, that is, $(r_o - r_i) / 2 = d$. In this notation and in these units, (4.7) takes on a simpler form:

$$\mu_0 e I = 2\left(m_0 - \frac{W}{c^2}\right) \frac{(1+d+\lambda)^2}{2(d+\lambda)} \left[\omega_0 - \omega_i\right]$$

$$+ \frac{eV}{c^2(2d+\lambda)(2+\lambda)(2d)\log_e\left(\frac{1+d}{1-d}\right)} \left[\lambda(2+2d+\lambda)\omega_0 - 4d\omega_i\right] \tag{4.8}$$

This equation suggests a method of making a measurement of the magnetic penetration depth λ in the superconductor. There is a term in the equation which is proportional to $V (\lambda (2 + 2d + \lambda) \omega_o - 4d\omega_i)$, and by comparison of the current I for various values of the voltage and rotation rates, and knowing the value of the parameter d, then the magnetic penetration depth might be inferred.

In order to see further the significance of this equation, we shall make the approximation that the magnetic penetration depth is small and so can be neglected (or, if necessary, corrected for), and we shall expand only to first order in the semi-distance between the plates d. The result is:

$$2 d \mu_o e I = 2 (m_o - W / c^2) [\omega_o - \omega_i] [1 + 2d]$$

$$+ \quad e V / c^2 \quad [\quad \omega_i' \quad] \tag{4.9}$$

This form of the equation suggests a method of making a device which is sensitive to its angular velocity of rotation in space. A current is caused to flow through the ammeter which is proportional to the voltage, and which depends upon the angular velocity of rotation of the whole instrument. A fuller analysis and experimental results on this application are given in chapter 3.

This form of the equation also suggests a method of measuring the mass of the electron. By comparison of the current through the ammeter with and without an applied voltage, whilst, say, the inner cylinder only is caused to rotate, the ratio $(m_0 - W/c^2)/(eV)$ can be measured. This application will also be discussed in appendix 2.

Finally, we return to the assumption made earlier that the ammeter has low inductance and so does not significantly affect the current flowing around the inner cylinder. Provided that the apparatus has sufficient symmetry that the inductance of the inner loop does not change as the apparatus rotates, then the current I in the above equations is reduced by a factor $L / (L + L_{SQUID})$. Provided that the SQUID inductance is well-defined, then this is a constant factor which does not affect the results for the mass of the electron or for the magnetic penetration depth.

Appendix 1 Time dependence of the phase

In this appendix we follow Josephson's argument to derive the time dependence of the phase θ of the order parameter of a superconducting wavefunction (8).

The superconducting state is associated with the appearance of a wavefunction containing an indefinite number of particles (9):

$$\Psi = \sum_N \Psi_N \tag{A1.1}$$

where each part Ψ_N contains exactly N particles. The order parameter, which has macroscopic expectation value, may be written in the usual second quantization notation:

$$\begin{aligned} \Delta(r) &= V(r) \langle \hat{\Psi}(r) \hat{\Psi}(r) \rangle \\ &= A \exp(i\theta) \end{aligned} \tag{A1.2}$$

We shall follow Josephson and assume that there exists an operator S which has eigenvalue $s = \exp(i\theta)$ for a superconducting wavefunction with phase θ. Multiplication of the wavefunction by the operator $\exp(iN\theta)$ (N is the number operator) multiplies s by $\exp(2i\theta)$, (since the operator in (A1.2) annihilates two particles) so $\exp(-iN\theta)\, S \exp(iN\theta) = \exp(2i\theta)\, S$. Differentiating with respect to θ and putting $\theta = 0$, one obtains:

$$[S, N] = 2S \quad \text{(A1.3)}$$

Writing the Hamiltonian for the superconductor:

$$H = H_o + \mu N \quad \text{(A1.4)}$$

where H_o is some constant and μ is the electrochemical potential of the superelectrons, one can deduce from (A1.3) and (A1.4) that:

$$s = s_o \, e^{-2i\mu t/\hbar} \quad \text{(A1.5)}$$

In his analysis of the tunneling effects between superconductors which followed this, Josephson used an arbitrary origin of energy for μ . In our analysis, we have chosen a particular gauge (namely, that the electrostatic potential V inside a cavity within the superconductor is zero), and in this gauge the origin of energy for μ is well defined.

Consider a single electron which is within the large cavity in the superconductor. The electron behaves exactly like an ordinary free electron (provided that the cavity is large enough), and in particular it posesses rest mass energy $m_o c^2$. The energy required to knock this electron from the superconductor into the cavity is W , the work-function of the metal; therefore it is clear by subtraction that the electrochemical potential of the electrons in the material of the superconductor is:

$$\mu = m_o c^2 - W \qquad \text{(A1.6)}$$

Finally, combining (A1.6) with (A1.5), the phase of the order parameter of a superconductor progresses with time according to:

$$-\hbar \partial \theta / \partial t = 2(m_o c^2 - W) \qquad \text{(A1.7)}$$

This is the principal result of this appendix.

In this appendix, we have also set up the machinery to understand the effect of a transformation of gauge so that a constant magnetic vector potential $\underline{A}_{external}$ is added to whatever fields are present in the superconductor.

Since each part Ψ_N of Ψ is associated with charge $-Ne$, then application of a gauge transformation [6] to a frame with an extra magnetic vector potential $\underline{A}_{external}$, has the effect of multiplying each part Ψ_N by a phase factor appropriate to the charge in the wavefunction:

$$\Psi_N' = \Psi_N \, e^{Nie\underline{A}.\underline{x}/\hbar} \qquad \text{(A1.8)}$$

The order parameter Δ is therefore multiplied by the phase factor:

$$\Delta' = \Delta \, e^{2ie\underline{A}.\underline{x}/\hbar} \qquad (A1.9)$$

In other words, the application of a gauge transformation to a frame with an additional vector potential $\underline{A}_{external}$ (or, alternatively, the external application of such a vector potential with no change of gauge) causes the gradient of phase of the order parameter to increase by $2 e \underline{A}_{external} / \hbar$. This result is used in the main part of the text of section 3.

Appendix 2 Accurate measurements of the London moment

Experiments are at present under way at Stanford university towards an accurate measurement of the London monent[10]. At the present stage an accuracy of better than 1% has been achieved. In this appendix we describe this experiment in outline. We then compare an alternative method based upon the voltage dependent effect discussed in section 4.

The Stanford experiments are performed within a region of space which has an exceedingly low ambient magnetic field, less than 10^{-12} Tesla. This is achieved using the 'expandable lead balloon' technique[11]. In this way unwanted noise currents caused by mechanical motion such as vibration or thermal expansion can be virtually eliminated.

A quartz cylinder with accurately measured radius r has superconductor evaporated onto its outer surface. The thickness of the evaporation is much less than the magnetic penetration depth in the superconductor, so that the radius at which supercurrents flow is accurately defined. The magnetic field in the region of space within the cylinder can be measured using a SQUID system. The cylinder itself can be set into rotation, and provision is made to be able to heat the cylinder to allow flux quanta to enter it.

In the simplest method of operation which we shall now describe, the SQUID reading is taken first with the cylinder stationary. A flux quantum is then allowed to enter the cylinder, and the rotation velocity ω necessary to return the SQUID to its original reading is measured.

If the effective mass associated with the superconducting order parameter is m^* then the cylinder obeys the quantization condition that

$$\oint m^* \underline{u} \cdot \underline{dl} = nh \qquad \text{(A2.1)}$$

where $\underline{u}$ is the velocity of motion of the superelectrons and the integral is taken around the cylinder. Suppose that n is increased by one (when a flux quantum enters the cylinder) and the tangential velocity u is increased by $r\omega$ (by the rotation at angular velocity ω). One can verify by direct substitution that the quantization condition is unaffected provided that the following condition holds:

$$m^* r^2 \omega = \hbar \qquad \text{(A2.2)}$$

This equation therefore relates the the mass m^* to the radius r and the angular velocity measured in the experiment ω.

There are several corrections to be taken into account. These include the effects of any fixed charges on the quartz or the evaporated superconductor, which charges would generate magnetic fields as they rotated. The radius r and the angular velocity ω must be measured with high accuracy.

It may be difficult to write down a comprehensive theory to describe how the measured mass m^* is related to the free electron mass m_o in this experiment. In bulk superconductor they are related through the work function, as described in the main part of this chapter. However in thin films as used here the analysis is more complicated. For example, the work function of a thin film may differ from that in the bulk, and surface effects may become important as the surface to volume ratio increases. Stresses in the film may affect the kinetic energy levels in the metal, and interactions with the quartz may also affect the result.

An alternative method of measuring the London moment involves the apparatus shown in figure 1. The theory of the effect and the method of using the apparatus to measure the London moment is discussed in detail in section 4. Here we concentrate on two questions: are the effects large enough to be measured accurately; and what corrections to the theoretical result $m^* \doteq 2 (m_o - W/c^2)$ might there be? We do not discuss here the experimental problems in realizing the apparatus. Such discussions are probably best left until experience has been gained in both the Stanford experiment and in the operation of the superconducting gyroscope. The superconducting gyroscope is similar in concept to the experiment suggested here, and it is discussed in chapter 3. The preliminary nature of these calculations should therefore be stressed.

We concentrate first on the question of sensitivity: are the effects large enough to be measured accurately? We shall consider an apparatus built as shown in figure 1 with radii r_1 = 10cm, r_2 = 15cm and height 10cm. If vacuum is used as the insulator a field of 10^7 V m^{-1} could be applied. We shall assume that a SQUID with energy sensitivity $dE/d\Omega = 10^{-30}$ J Hz^{-1} is coupled into the system.

If such a system were used simply as a gyroscope then its sensitivity would be given by equation (3.4) of chapter 3. The reader is referred to that chapter for detailed discussion of the calculation. The sensitivity derived there depends upon a number of experimental problems being overcome, particularly mechanical deformation (vibration, thermal contraction Etc.) in the ambient magnetic field, and the problem of leakage of current through the dielectric. Substituting the above parameters of the apparatus into the equation, one obtains a sensitivity to rotation velocity

$$d\omega = 3 \cdot 10^{-6} \text{ radian/s} \qquad \text{(A2.3)}$$

The magnitude of the effect therefore appears to be large enough to be able to measure a rotation velocity of 3 radians per second, to an accuracy of one part in 10^6. At least in principle, the effects are large enough to be measured with good accuracy. It should again be stressed that these preliminary calculations do not take into account the experimental problems which there might be in realizing this apparatus.

We now consider the question: what corrections to the theoretical result $m^* = 2\,(m_o - W/c^2)$ might there be? Since bulk superconductor is involved, surface effects would have little effect,(although dirty surfaces might affect the magnetic penetration depth and lead to inaccuracies). Since a superconductor contracts upon cooling then the energy levels of the electrons will be affected by cooling. In particular the kinetic energies of the electron levels will be affected. One would therefore expect that the work function will depend upon temperature, and the value appropriate to the correct temperature should be used. If pure metal is used then there should not be significant strains set up upon cooling. However if mixtures of metals (such as solder) are used there may be significant corrections on account of strains.

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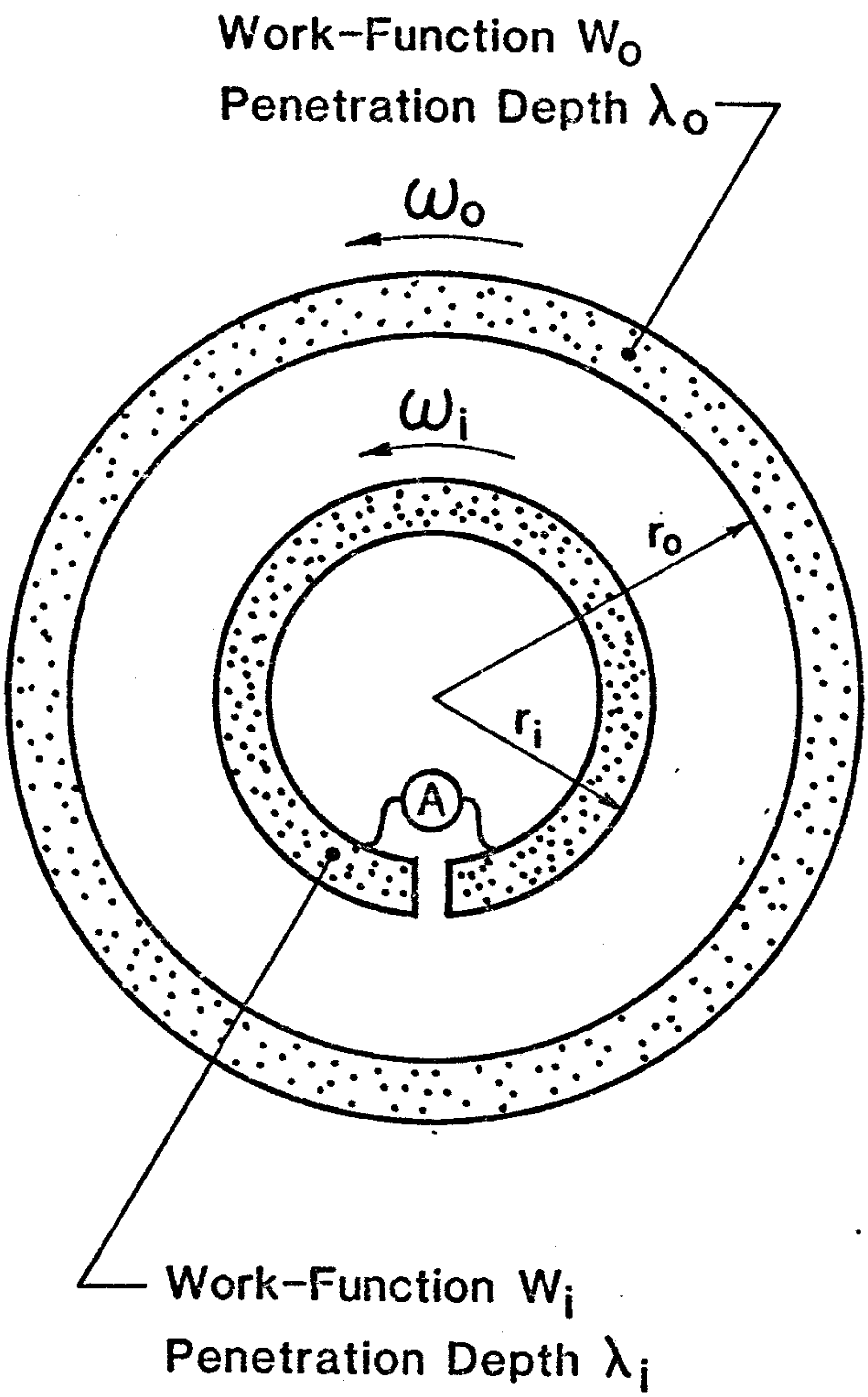

Figure 1 - The apparatus discussed in section 4 and appendix 2