Modelling Cubical Type Theory in Agda

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Computer Laboratory

Workshop on HoTT/UF ’16, Porto
Overview

This material in this talk is based on our paper:

Axioms for Modelling Cubical Type Theory in a Topos, CSL 2016
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- Our axiomatisation
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Covering:

- The internal type theory of a topos
- Translating this into Agda
- Our axiomatisation
- Why this is a good approach
The internal type theory of a topos
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- Standard interpretation of extensional MLTT in a category with families (CwF) associated with any topos $\mathcal{E}$ (with families over $X \simeq \mathcal{E}/X$).
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- The subobject classifier $\Omega$ becomes an impredicative universe of propositions with logical connectives, equality and quantifiers.
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- The subobject classifier $\Omega$ becomes an impredicative universe of propositions with logical connectives, equality and quantifiers.
- The universal property of $\Omega$ gives rise to comprehension subtypes...
Comprehension subtypes

For any type $\Gamma \vdash A$ we can form comprehension subtypes:

$$\Gamma, x : A \vdash \varphi(x) : \Omega$$

$$\Gamma \vdash \{x : A \mid \varphi(x)\}$$

whose terms are those $t : A$ for which $\varphi(t)$ is provable.
What do we need in Agda?

In order to apply the same reasoning that we use in the paper we need to extend Agda with:
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- Function extensionality
An impredicative universe of propositions

We use an idea of Martin Escardo\textsuperscript{1}:

\textsuperscript{1} \url{www.cs.bham.ac.uk/~mhe/impredicativity/}
An impredicative universe of propositions

We use an idea of Martin Escardo$^1$:

```agda
{-# OPTIONS --type-in-type #-}
-- the following definition relies on type-in-type,
-- which is switched on only in this module

record Ω : Set where
  constructor prop
  field
    prf : Set
    equ : (u v : prf) → u ≡ v
```

1. [www.cs.bham.ac.uk/~mhe/impredicativity/](http://www.cs.bham.ac.uk/~mhe/impredicativity/)
Comprehension subtypes

We simply form the sigma type:

$$\text{set} : (A : \text{Set})(P : A \to \Omega) \to \text{Set}$$

$$\text{set } A \ P = \Sigma \ x \in A \ , \ \text{prf } (P \ x)$$

**syntax**  \text{set } A \ (\lambda \ x \to P) = [ \ x \in A \mid P ]
Comprehension subtypes

For example:

\[ \text{Evens} : \text{Set} \]
\[ \text{Evens} = [ \, n \in \mathbb{N} \mid \exists \, m \in \mathbb{N} \, , \, 2 \cdot m \approx n \, ] \]

\[ \text{four} : \text{Evens} \]
\[ \text{four} = (4 \mid 2 \mid \text{refl} \mid 1) \]
Overview of axiomatisation

Elementary topos $\mathcal{E}$ (with a NNO), and
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- An internal full subtopos $\mathcal{U}$
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Paths in $A$ are maps $I \to A$
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Internalisation of the face lattice $\mathcal{F}$. Think of elements of $\text{Cof}$ as propositions such as $(i = 1)$, $(i = 0) \lor (j = 1)$, $(i = 0) \land (j = 0)$ etc.
Modelling Kan filling

How do we model the Kan filling operation from cubical type theory?

\[ \Gamma, \ i : I \vdash fill^i A [\varphi \mapsto u] \ a_0 : A \]
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\[ \Gamma, i : \mathbb{I} \vdash \text{fill}^i A [\varphi \mapsto u] a0 : A \]
Modelling partial terms/types

How do we model partial types?

\[ \Gamma, \varphi \vdash A \]

And partial terms?

\[ \Gamma, \varphi \vdash a : A \]
Comprehension subtypes again

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In particular we can take $A = 1$ to get:

$$[\varphi] \triangleq \{ \_ : 1 \mid \varphi \}$$
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[\varphi] \triangleq \{\_ : 1 \mid \varphi\}
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We will make extensive use of these types in connection with partial elements.
Partial elements

A partial element of a type $A$ is a pair:

- $\varphi : \Omega$, called the extent
- $f : [\varphi] \to A$. 
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Later we will want to talk about extending a partial element to a total one:
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We say that a partial element $(\varphi, f)$ extends to $a : A$ if the following relation holds:

$$(\varphi, f) \xrightarrow{a} \triangleq \forall (u : [\varphi]). f u = a$$
Filling in the internal TT

The notion of Kan filling in our internal type theory:
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The type of filling structures for $I$-indexed families of types, $\text{Fill} : (e : \{0, 1\})(A : I \rightarrow \mathcal{U}) \rightarrow \mathcal{U}$, is defined by

\[
\text{Fill}_e A \triangleq \\
(\varphi : \text{Cof})(f : [\varphi] \rightarrow \Pi_I A) \\
(a : \{a' : A e \mid (\varphi, f) \circ @ e \Rightarrow a'\}) \\
\rightarrow \\
\{g : \Pi_I A \mid (\varphi, f) \Rightarrow g \land g e = a\}
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\[ \rightarrow \]
\[ \{ g : \Pi I A \mid (\varphi, f) \rightarrow g \land g e = a \} \]

\[
\text{Fill} e A = \\
(\phi : \text{Cof})(f : [ \phi ] \rightarrow \Pi A) \\
(a : \llbracket a' \in A \langle e \rangle \mid (\phi, f) \cdot \langle e \rangle \rightarrow a' \rrbracket) \\
\rightarrow \\
\llbracket g \in \Pi A \mid ((\phi, f) \rightarrow g) \land (g \langle e \rangle \approx \text{fst} a) \rrbracket 
\]
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  - easily translate them into Agda
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  - postulate them in Agda
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  - easily translate them into Agda
- Look for models of these axioms
  - e.g. classifying topos for the theory
Thanks for listening!

Axioms for Modelling Cubical Type Theory in a Topos

Ian Orton and Andrew Pitts, CSL 2016

Paper and Agda:
http://www.cl.cam.ac.uk/~rio22/

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