Axioms for Modelling Cubical Type Theory in a Topos

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Overview
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- Express the constructions of the CCHM\textsuperscript{1} presheaf model in the internal type theory of an elementary topos using partial elements

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- Identify additional models

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- Identify additional models

A suggestion of Thierry Coquand

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The internal type theory of a topos
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The internal type theory of a topos

- Standard interpretation of extensional MLTT in a category with families (CwF) associated with any topos $\mathcal{E}$ (with families over $X \simeq \mathcal{E}/X$).
- The subobject classifier $\Omega$ becomes an impredicative universe of propositions with logical connectives, equality and quantifiers.
- The universal property of $\Omega$ gives rise to comprehension subtypes...
Comprehension subtypes

For any type $\Gamma \vdash A$ we can form comprehension subtypes:

$$
\Gamma, x : A \vdash \varphi(x) : \Omega
$$

$$
\Gamma \vdash \{ x : A \mid \varphi(x) \}
$$

whose terms are those $t : A$ for which $\varphi(t)$ is provable.
A crash course in Cubical Type Theory
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We have an interval object $I$ which models the real interval $[0, 1]$. 
A crash course in Cubical Type Theory

We have an interval object $I$ which models the real interval $[0, 1]$. From this we get:

\[
\begin{align*}
\vdash a : A \\
i : I \vdash a : A \\
i : I, j : I \vdash a : A
\end{align*}
\]

(points) (lines) (squares)
Partial terms

We can restrict to certain faces and edges, using face formulae.

$$\Gamma, \varphi \vdash a : A$$
Partial terms

We can restrict to certain faces and edges, using face formulae.

\[ \Gamma, \varphi \vdash a : A \]

E.g.

\[ \Gamma \triangleq i : I, \ j : I, \ k : I \]

\[ \varphi \triangleq (i = 0) \lor (j = 0) \lor (j = 1 \land k = 1) \]
A crash course in Kan filling

\[ \Gamma, i : \mathbb{I} \vdash \text{fill}^i A [\varphi \mapsto u] \ a_0 : A \]
A crash course in Kan filling

$$\Gamma, i : I \vdash fill^i A [\varphi \mapsto u] \ a_0 : A$$
A crash course in Kan filling

\[ \Gamma, \ i : I \vdash \text{fill}^i A \ [\varphi \mapsto u] \ a_0 : A \]
A crash course in Kan filling

\[ \Gamma, \ i : \ I \vdash \text{fill}^i A [\phi \mapsto u] \ a_0 : A \]
A crash course in Kan filling

\[ \Gamma, i: I \vdash \text{\textit{fill}^i A [\varphi \mapsto u] a_0 : A} \]
Modelling partial terms/types

How do we model partial terms?

\[ \Gamma, \varphi \vdash a : A \]
Comprehension subtypes again

For any type $\Gamma \vdash A$ we can form comprehension subtypes:

$$\begin{align*}
\Gamma, x : A & \vdash \varphi(x) : \Omega \\
\Gamma & \vdash \{ x : A \mid \varphi(x) \}
\end{align*}$$

whose terms are those $t : A$ for which $\varphi(t)$ is provable.
Comprehension subtypes again

For any type $\Gamma \vdash A$ we can form comprehension subtypes:

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\Gamma, x : A \vdash \varphi(x) : \Omega
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$$
\Gamma \vdash \{x : A \mid \varphi(x)\}
$$

whose terms are those $t : A$ for which $\varphi(t)$ is provable.

In particular we can take $A = 1$ to get:

$$
[\varphi] \triangleq \{_ : 1 \mid \varphi\}$$
Comprehension subtypes again

For any type $\Gamma \vdash A$ we can form comprehension subtypes:

$$\Gamma, x : A \vdash \varphi(x) : \Omega$$

$$\Gamma \vdash \{x : A \mid \varphi(x)\}$$

whose terms are those $t : A$ for which $\varphi(t)$ is provable.

In particular we can take $A = 1$ to get:

$$[\varphi] \triangleq \{\_ : 1 \mid \varphi\}$$

We will make extensive use of these types in connection with partial elements.
A partial element of a type $A$ is a pair:

- $\varphi : \Omega$, called the extent
- $f : [\varphi] \rightarrow A$. 
Partial elements

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- $f : [\varphi] \to A$.

Later we will want to talk about extending a partial element to a total one:
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- $\varphi : \Omega$, called the extent
- $f : [\varphi] \to A$.

Later we will want to talk about extending a partial element to a total one:

We say that a partial element $(\varphi, f)$ extends to $a : A$ if the following relation holds:

$$(\varphi, f) \rightsquigarrow a \triangleq \forall (u : [\varphi]). \ f \ u = a$$
Cofibrant propositions

Operations such as Kan filling and Kan composition have to do with extending maps from a subspace to a whole space,
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Cofibrant propositions

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In our setting we will consider extending partial elements generated by a collection of cofibrant propositions (cf. the face lattice $\mathcal{F}$).
A collection of cofibrant propositions is a subobject Cof \ni \Omega such that:

\[\begin{align*}
    i : I & \quad e : \{0, 1\} \\
    (i = e) & \in \text{Cof}
\end{align*}\]

\[\begin{align*}
    \varphi & \in \text{Cof} \quad \psi \in \text{Cof} \\
    \varphi \lor \psi & \in \text{Cof}
\end{align*}\]

\[\begin{align*}
    \varphi & \in \text{Cof} \quad \varphi \Rightarrow (\psi \in \text{Cof}) \\
    \varphi \land \psi & \in \text{Cof}
\end{align*}\]
Cofibrant propositions

A collection of cofibrant propositions is a subobject $\text{Cof} \rightarrow \Omega$ such that:

- $i : I \quad e : \{0, 1\}$
- $(i = e) \in \text{Cof}$

- $\varphi, \psi \in \text{Cof}$
- $\varphi \land \psi \in \text{Cof}$
- $\varphi \lor \psi \in \text{Cof}$

- $\varphi \in \text{Cof} \quad \varphi \Rightarrow (\psi \in \text{Cof})$

Only needed for definitional/strong identity types
Cofibrant propositions

A collection of cofibrant propositions is a subobject $\text{Cof} \ni \Omega$ such that:

- $i : I 
  e : \{0, 1\} 
  (i = e) \in \text{Cof}$
- $\varphi \in \text{Cof}$, $\psi \in \text{Cof}$
  $\varphi \lor \psi \in \text{Cof}$
- $\forall (i : I) \ (\varphi_i \in \text{Cof})$
  $(\forall (i : I) \varphi_i) \in \text{Cof}$

Required in the cubical sets model by Cohen et al. to get a univalent universe.
Cofibrant propositions

A collection of cofibrant propositions is a subobject \( \text{Cof} \twoheadrightarrow \Omega \) such that:

\[
\begin{align*}
    i : I & \quad e : \{0, 1\} \\
    (i = e) & \in \text{Cof} \\
    \phi \in \text{Cof} \quad \psi \in \text{Cof} & \quad \phi \lor \psi \in \text{Cof} \\
    \phi \in \text{Cof} \quad \phi \Rightarrow (\psi \in \text{Cof}) & \quad \phi \land \psi \in \text{Cof}
\end{align*}
\]
A cofibrant partial element is a partial element \((\varphi, f)\) such that \(\varphi\) is a cofibrant proposition. We write \(\Box A\) for the type of cofibrant partial elements of a type \(A\).

\[
\Box A \triangleq (\varphi : \text{Cof}) \times ([\varphi] \to A)
\]
We can now express a (simplified) notion of Kan filling in our internal type theory.
Filling

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The type of 0-filling structures for constant families, \(\text{Fill}_0\), is defined by

\[
\text{Fill}_0 A \triangleq \bigtriangleup \\
(a : A) \\
(f_\varphi : \{ f_\varphi : \Box (I \to A) \mid f_\varphi @ 0 \rightharpoonup a \} ) \\
\to \\
\{ g : I \to A \mid f_\varphi \rightharpoonup g \land g 0 = a \}
\]
\[ \text{Fill}_0 A = (a : A) \\
\{ f_\varphi : \{ f_\varphi : \Box (I \to A) \mid f @ 0 \to a \} \} \to \{ g : I \to A \mid f_\varphi \to g \land g 0 = a \} \]
Filling

An element at one end

\[\text{Fill}_0 A \triangleq (a : A) \quad (f_\varphi : \{f_\varphi : \square(I \to A) \mid f @ 0 \rightharpoonup a\}) \rightarrow \{g : I \to A \mid f_\varphi \nearrow g \land g 0 = a\}\]
A cofibrant (well behaved) partial path that agrees with the element

\[
\text{Fill } 0 A \triangleq (a : A) (f_{\varphi} : \{ f_{\varphi} : \square (I \to A) \mid f @ 0 \rhd a \}) \to \{ g : I \to A \mid f_{\varphi} \rhd g \land g 0 = a \}
\]
Filling

**Fill 0 A** ≜

\[(a : A) \leftarrow ((f_\varphi : \{f_\varphi : \Box(I \to A) \mid f @ 0 \rightarrow a \}) \to \{g : I \to A \mid f_\varphi \rightarrow g \land g 0 = a\})\]

An element at one end

A cofibrant (well behaved) partial path that agrees with the element

A total path that extends the partial path and agrees with \(a\) at the end
Towards univalence

Using the glueing construction we show:
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- Paths $\longrightarrow$ equivalences
Towards univalence

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- Paths $\rightarrow$ equivalences
- Equivalences $\rightarrow$ paths
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but...
Towards univalence

Using the glueing construction we show:

- Paths $\rightarrow$ equivalences
- Equivalences $\rightarrow$ paths
- These conversions are (in some sense) quasi-inverses

but we do not yet have an internalisation of a univalent universe.
Agda development
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- Add an impredicative universe of mere propositions via a method due to Martin Escardo
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- Define a proof relevant version of \{x : A | \phi\}
Agda development

- Add an impredicative universe of mere propositions via a method due to Martin Escardo
- Define a proof relevant version of $\{x : A \mid \phi\}$
- Postulate the axioms
Thanks for listening!

Summary:

- Any topos that satisfies the axioms will be a model of cubical type theory*

* without a universe object (for now).

"Axioms for Modelling Cubical Type Theory in a Topos"
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