Axioms for Modelling Cubical Type Theory in a Topos

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(joint work with Andrew Pitts)

UNIVERSITY OF CAMBRIDGE
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Workshop on HoTT and UF

Cubical type theory

Cubical type theory [1]

1. C. Cohen, T. Coquand, S. Huber, and A. Mörtberg. Cubical type theory: a constructive interpretation of the univalence axiom. Preprint, December 2015.

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- Types come equipped with a composition operation.
- The glueing construction gives a constructive interpretation of Voevodsky's univalence axiom.
- There is a (constructive) presheaf model.

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Overview

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- Express the constructions of the CCHM presheaf model in the internal type theory of an elementary topos using partial elements

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A suggestion of Thierry Coquand

The internal type theory of a topos

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- Standard interpretation of extensional MLTT in a category with families (CwF) associated with any topos $\mathcal{E}$ (with families over $X \simeq \mathcal{E}/X$).
- The subobject classifier Ω becomes an impredicative universe of propositions with logical connectives, equality and quantifiers.
- The universal property of Ω gives rise to comprehension subtypes...

Comprehension subtypes

For any type $\Gamma \vdash A$ we can form comprehension subtypes:

$$\frac{\Gamma, x : A \vdash \varphi(x) : \Omega}{\Gamma \vdash \{x : A \mid \varphi(x)\}}$$

whose terms are those $t : A$ for which $\varphi(t)$ is provable.

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We will make extensive use of these types in connection with partial elements.

Partial elements

A partial element of a type A is a pair:

- $\varphi : \Omega$, called the extent
- $f : [\varphi] \to A$.

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We say that a partial element (φ, f) extends to $a : A$ if the following relation holds:

$$(\varphi, f) \nearrow a \triangleq \forall (u : [\varphi]).\ f\,u = a$$

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- A subobject $\mathtt{Cof} \rightarrowtail \mathbf{\Omega}$ of “cofibrant" propositions.

The interval

We interpret equalities as paths, which we model as maps from an interval object.

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An interval object is:

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- and morphisms $\sqcap, \sqcup : \mathtt{I} \to \mathtt{I} \to \mathtt{I}$

such that $\mathtt{I}$ is connected, $\mathtt{0} \neq \mathtt{1}$ and $\sqcap, \sqcup$ form a path connection algebra.

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Define the path type associated with a type A as $\_ \sim \_ : A \to A \to \mathcal{U}$ where:

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- ✓ Reflexivity (`refl`)
- ✓ Singletons ($(a : A) \times a_0 \sim a$) are contractible
- × Transport $\longleftarrow$ need a Kan-style filling operation

Cofibrant propositions

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In our setting we will consider extending partial elements generated by a collection of cofibrant propositions (cf. the face lattice $\mathbb{F}$).

Cofibrant propositions

A collection of cofibrant propositions is a subobject $\mathtt{Cof} \rightarrowtail \boldsymbol{\Omega}$ such that:

$$\frac{i : \mathtt{I} \qquad e : \{0, 1\}}{(i = e) \in \mathtt{Cof}} \qquad \frac{\varphi \in \mathtt{Cof} \qquad \psi \in \mathtt{Cof}}{\varphi \vee \psi \in \mathtt{Cof}}$$

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Only needed for definitional/strong identity types

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$$\frac{\forall (i : \mathbb{I})\ (\varphi\, i \in \mathtt{Cof})}{(\forall (i : \mathbb{I})\, \varphi\, i) \in \mathtt{Cof}}$$

Required in the cubical sets model by Cohen et al. to get a univalent universe

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Cofibrant partial elements and paths

A cofibrant partial element is a partial element (φ, f) such that φ is a cofibrant proposition. We write $\Box A$ for the type of cofibrant partial elements of a type A.

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A cofibrant (dependent) partial path is an element of the type $\Box(\Pi_{\mathtt{I}} A)$. Given such a path $(\varphi, f) : \Box(\Pi_{\mathtt{I}} A)$ we can evaluate it at a point $i : \mathtt{I}$ to get a cofibrant partial element $(\varphi, f) @ i : \Box(A\, i)$:

$$(\varphi, f) @ i \triangleq (\varphi, \lambda(u : [\varphi]) \to f\, u\, i)$$

Filling

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The type of filling structures for $\mathbb{I}$-indexed families of types, $\mathtt{Fill} : (e : \{0, 1\})(A : \mathbb{I} \to \mathcal{U}) \to \mathcal{U}$, is defined by

$$
\begin{aligned}
&\mathtt{Fill}\, e\, A \triangleq \\
&\quad (\varphi : \mathtt{Cof})(f : [\varphi] \to \Pi_{\mathbb{I}} A) \\
&\quad (a : \{a' : A\, e \mid (\varphi, f) @ e \nearrow a'\}) \\
&\quad \to \\
&\quad \{g : \Pi_{\mathbb{I}} A \mid (\varphi, f) \nearrow g \ \wedge\ g\, e = a\}
\end{aligned}
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Filling

A cofibrant (well behaved) partial path

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$$\texttt{Fill}\, e\, A \triangleq (\varphi : \texttt{Cof})(f : [\varphi] \to \Pi_{\mathbb{I}} A)\, (a : \{a' : A\, e \mid (\varphi, f)\, @\, e \nearrow a'\}) \to \{g : \Pi_{\mathbb{I}} A \mid (\varphi, f) \nearrow g \wedge g\, e = a\}$$

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\end{aligned}
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A total path that extends the partial path and agrees with a at the end

Composition

A cofibrant (well behaved) partial path

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$$
\begin{aligned}
&\texttt{Comp}\,e\,A \triangleq \\
&\quad (\varphi : \texttt{Cof})(f : [\varphi] \to \Pi_{\mathbb{I}} A) \\
&\quad \{a_0 : A\,e \mid (\varphi, f) @ e \nearrow a_0\} \\
&\quad \to \\
&\quad \{a_1 : A\,\overline{e} \mid (\varphi, f) @ \overline{e} \nearrow a_1\}
\end{aligned}
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A total element that extends the partial path at the other end

Fibrations

The type of fibration structures for a family $A : \Gamma \to \mathcal{U}$ is given by:

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& (e : \{\mathtt{0}, \mathtt{1}\}) \\
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Theorems:

- Closed under $\Sigma, \Pi, \mathtt{Path}, \uplus$
- $\mathtt{N}$ is fibrant (a fibration over $\mathbf{1}$)
- $\mathtt{Id}$ types from $\mathtt{Path}$ via Swan's construction

Fibrations - why?

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...and hence interpret the J-eliminator for
`Path`/`Id` types.

Towards univalence

Using the glueing construction we show:

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but we do not yet have an internalisation of a univalent universe. In particular

$$\mathcal{U}' \triangleq (A : \mathcal{U}) \times \mathtt{Fib}(\lambda\_ : \mathbf{1} \to A)$$

does not work.

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- $\mathtt{Cof} \rightarrowtail \mathbf{\Omega_{dec}} \rightarrowtail \mathbf{\Omega}$ where

  $$\mathbf{\Omega_{dec}}(c) \triangleq \{S \text{ sieve on } c \mid \forall (f \in \mathbb{C}/c)\, f \in S \vee f \notin S\}$$

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Examples (within a constructive meta-theory):

- Cubical sets
- Simplicial sets

Agda development

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- Add an impredicative universe of mere propositions via a method due to Martin Escardo

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(Still in development)

Thanks for listening!

Summary:

- Any topos that satisfies the axioms will be a model of cubical type theory*

* without a universe object (for now).

"Axioms for Modelling Cubical Type Theory in a Topos"
Ian Orton and Andrew Pitts
Paper and Agda: http://www.cl.cam.ac.uk/~rio22/
Ian.Orton@cl.cam.ac.uk   Andrew.Pitts@cl.cam.ac.uk

Univalence without a universe

Introduce a new type $UA^i(f)$ with the following formation rule:

$$\frac{\Gamma \vdash f : \texttt{Equiv}\, A\, B}{\Gamma, i : \mathbb{I} \vdash UA^i(f)}$$

and definitional equalities:

$$\Gamma \vdash UA^i(f)(i0) \equiv A$$

$$\Gamma \vdash UA^i(f)(i1) \equiv B$$

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Observe:

$$\dfrac{\dfrac{\Gamma \vdash f : \mathtt{Equiv}\, A\, B}{\Gamma, i : \mathtt{I} \vdash UA^i(f) : \mathcal{U}}}{\Gamma \vdash \langle i \rangle \, UA^i(f) : \mathtt{Path}\, \mathcal{U}\, A\, B}$$