A version of monoidal categories (Szlachányi (2012))

Structural transformations need not be invertible:

\[ \alpha : (A \otimes B) \otimes C \to A \otimes (B \otimes C) \]
\[ \lambda : I \otimes A \to A \]
\[ \rho : A \to A \otimes I \]
Skew monoidal categories

A version of monoidal categories (Szlachányi (2012))

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\[ \lambda : I \otimes A \rightarrow A \]
\[ \rho : A \rightarrow A \otimes I \]

Example

- For \( C \) with coproducts, \( (X/C) \) with

\[ (X \overset{a}{\rightarrow} A) \oplus (X \overset{b}{\rightarrow} B) := X \overset{\text{inl}}{\rightarrow} X + X \overset{a+b}{\rightarrow} A + B \]

- For \( C \) cocomplete, \( \mathcal{[J,C]} \) with unit \( J \) and tensor

\[ F \star G := (\text{lan}_J F) \circ G \ (\text{Altenkirch et al. (2010))}. \]
A version of monoidal categories (Szlachányi (2012))

Structural transformations need not be invertible:

\[ \alpha : (A \otimes B) \otimes C \to A \otimes (B \otimes C) \]
\[ \lambda : I \otimes A \to A \]
\[ \rho : A \to A \otimes I \]

Recently studied very actively (*list not exhaustive!*):


Past work
Linton (’69), Kock (’71a, ’71b), Guitart (’80), Jacobs (’94), Seal (’13), …

\( \mathcal{C} \) monoidal
\( \mathbb{T} \) a monoidal monad
reflexive coequalizers in \( \mathcal{C} \) +
preservation conditions

\( \Rightarrow \)

\( \mathcal{C}^{\mathbb{T}} \) monoidal
Past work
Linton ('69), Kock ('71a, '71b), Guitart ('80), Jacobs ('94), Seal ('13), ...

\[ C \text{ monoidal} \]
\[ \mathbb{T} \text{ a monoidal monad} \]
\[ \Rightarrow \]
\[ C^{\mathbb{T}} \text{ monoidal} \]
reflexive coequalizers in \( C + \)
preservation conditions

This work
\[ C \text{ skew monoidal} \]
\[ \mathbb{T} \text{ a strong monad} \]
\[ \Rightarrow \]
\[ C^{\mathbb{T}} \text{ skew monoidal} \]
reflexive coequalizers in \( C + \)
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Past work
Linton (’69), Kock (’71a, ’71b), Guitart (’80), Jacobs (’94), Seal (’13), …

\[ C \text{ monoidal} \]
\[ \mathcal{T} \text{ a monoidal monad} \quad \Rightarrow \quad C^\mathcal{T} \text{ monoidal} \]

reflexive coequalizers in \( C \) + preservation conditions

This work
\[ C \text{ skew monoidal} \]
\[ \mathcal{T} \text{ a strong monad} \quad \Rightarrow \quad C^\mathcal{T} \text{ skew monoidal} \]

monoids are \( T \)-monoids
Monoidal case \((\mathcal{C}, \mathbb{T} \text{ monoidal})\)

**Definition (Kock (1971))**

For \((A, a), (B, b), (C, c) \in \mathcal{C}^\mathbb{T}\) a map \(h : A \otimes B \rightarrow C\) in \(\mathcal{C}\) is *bilinear* if it is linear in each argument:

\[
\begin{align*}
T(A) \otimes B & \xrightarrow{T(A) \otimes \eta} T(A) \otimes T(B) \xrightarrow{\kappa} T(A \otimes B) \xrightarrow{Th} TC \\
A \otimes B & \xrightarrow{h} C
\end{align*}
\]

\[
\begin{align*}
T(A) \otimes B & \xrightarrow{T(A) \otimes \eta} T(A) \otimes T(B) \xrightarrow{\kappa} T(A \otimes B) \xrightarrow{Th} TC \\
A \otimes B & \xrightarrow{h} C
\end{align*}
\]

\[
\begin{align*}
A \otimes T(B) & \xrightarrow{\eta \otimes T(B)} T(A) \otimes T(B) \xrightarrow{\kappa} T(A \otimes B) \xrightarrow{Th} TC \\
A \otimes b & \xrightarrow{h} C
\end{align*}
\]

\[
\begin{align*}
A \otimes T(B) & \xrightarrow{\eta \otimes T(B)} T(A) \otimes T(B) \xrightarrow{\kappa} T(A \otimes B) \xrightarrow{Th} TC \\
A \otimes b & \xrightarrow{h} C
\end{align*}
\]
Monoidal case \((\mathcal{C}, \top \text{ monoidal})\)

**Aim**

Construct \((-) \star (\_): \mathcal{C}^\top \times \mathcal{C}^\top \to \mathcal{C}^\top\) satisfying

1. \(\mathcal{C}^\top (A \star B, C) \cong \text{Bilin}_\mathcal{C}(A, B; C)\)

2. A suitable preservation property to guarantee coherence
Monoidal case \((\mathcal{C}, \top\mathrm{monoidal})\)

**Aim**

Construct \((-) \star (=) : \mathcal{C}^\top \times \mathcal{C}^\top \to \mathcal{C}^\top\) satisfying

1. \(\mathcal{C}^\top(A \star B, C) \cong \text{Bilin}_\mathcal{C}(A, B; C)\)

2. A suitable preservation property to guarantee coherence

**Construction (Linton 1969)**

Reflexive coequalizer in \(\mathcal{C}^\top\):

\[
\begin{align*}
T(T(A) \otimes T(B)) & \xrightarrow{T\kappa} T^2(A \otimes B) \xrightarrow{\mu} T(A \otimes B) \xrightarrow{\text{coeq.}} A \star B \\
& \Downarrow T(a \otimes b)
\end{align*}
\]

NB: \(U : \mathcal{C}^\top \to \mathcal{C}\) creates reflexive coequalizers if \(T\) preserves them
Monoidal case \((\mathcal{C}, \mathbb{T} \text{ monoidal})\)

**Aim**

Construct \((-) \star (=) : \mathcal{C}^\mathbb{T} \times \mathcal{C}^\mathbb{T} \rightarrow \mathcal{C}^\mathbb{T}\) satisfying

1. \(\mathcal{C}^\mathbb{T}(A \star B, C) \cong \text{Bilin}_\mathcal{C}(A, B; C)\)

2. if every \((-) \otimes X\) and \(X \otimes (-)\) preserve reflexive coequalizers, so do \((-) \star (A, a)\) and \((A, a) \star (-)\)

**Construction (Linton 1969)**

Reflexive coequalizer in \(\mathcal{C}^\mathbb{T}\):

\[
\begin{array}{ccc}
T(T(A) \otimes T(B)) & \xrightarrow{T \kappa} & T^2(A \otimes B) & \xrightarrow{\mu} & T(A \otimes B) \\
& \xrightarrow{T(a \otimes b)} & T(a \otimes b)
\end{array}
\]

\(\text{coeq.}\) \(\rightarrow\) \(A \star B\)

**NB:** \(U : \mathcal{C}^\mathbb{T} \rightarrow \mathcal{C}\) creates reflexive coequalizers if \(T\) preserves them
Monoidal case \((\mathcal{C}, \top \text{ monoidal})\)

Proposition (Guitart ('80), Seal ('13))

Suppose that

- \(\mathcal{C}\) has all reflexive coequalizers,
- \(T\) preserves reflexive coequalizers,
- Every \((-) \otimes X\) and \(X \otimes (-)\) preserves reflexive coequalizers

Then \((\mathcal{C}^\top, \ast, TI)\) is a monoidal category.

Other versions are available: e.g. closed, symmetric, cartesian...
Skew monoidal case ($\mathcal{C}$ skew monoidal, $\overline{T}$ strong)

Classify left-linear maps

Construct an action $\mathcal{T} \times \mathcal{C} \to \mathcal{T}$

Extend to a skew monoidal structure on $\mathcal{T}$
Skew monoidal case (\( \mathcal{C} \) skew monoidal, \( \mathcal{T} \) strong)

Classify left-linear maps

Construct an action \( \mathcal{C}^\mathcal{T} \times \mathcal{C} \to \mathcal{C}^\mathcal{T} \)

Extend to a skew monoidal structure on \( \mathcal{C}^\mathcal{T} \)

Background assumption:
\( \mathcal{C} \) skew monoidal, \( \mathcal{T} \) strong (\( st : T(A) \otimes B \to T(A \otimes B) \))
Factoring the proof

\[ \mathcal{C} \text{ has reflexive coequalizers, which } T \text{ preserves} \]

\[ \mathcal{C}^\top \text{ has reflexive coequalizers} \]

\[ \mathcal{C} \text{ has a } (1, 2, 3)\text{-left linear classifier} \]

\[ \mathcal{C}^\top \text{ skew monoidal} \]

\[ \mathcal{C} \text{ acts on } \mathcal{C}^\top \]

\[ (\cdot) \otimes \mathcal{X} \text{ preserves reflexive coeqs.} \]

\[ \mathcal{C} \text{ closed or } \alpha \text{ invertible} \]

or

\[ (\cdot) \otimes \mathcal{X} \text{ preserves reflexive coeqs.} \]
Factoring the proof

\( \mathcal{C} \) has reflexive coequalizers, which \( T \) preserves

\( \mathcal{C} \) has a \((1, 2, 3)\)-left linear classifier

\( (\_ \otimes X) \) preserves reflexive coeqs.

\( \mathcal{C} \) acts on \( \mathcal{C}^T \)

\( \mathcal{C}^T \) skew monoidal
Left-linear maps

Definition (c.f. Kock (1971))

For \((A, a), (B, b) \in \mathcal{C}^\mathsf{T}\) and \(P \in \mathcal{C}\), a map \(h : A \otimes P \to C\) is left linear if

\[
\begin{array}{c}
\xrightarrow{T(A) \otimes P} \xrightarrow{\mathsf{st}_{A,B}} \xrightarrow{T(A \otimes P)} \xrightarrow{Th} \xrightarrow{TB}
\end{array}
\]

\[
\begin{array}{c}
\downarrow \mathsf{a} \otimes P \quad \downarrow h \\
A \otimes P \quad \quad \quad B
\end{array}
\]
Left-linear classifiers

Definition (c.f. Guitart ('80), Jacobs ('94), Seal ('13))

A left-linear classifier is a family of maps $\sigma_{A,P} : A \otimes P \to A \star P$ such that

1. $(A \star P, \tau_{A,P}) \in \mathcal{C}^{\mathbb{T}}$
2. $\sigma_{A,B}$ is left-linear,
3. Every left-linear map $A \otimes P \to B$ factors uniquely:

$$
\begin{array}{ccc}
A \otimes P & \xrightarrow{\sigma} & A \star P \\
\forall \text{ left-linear maps} & \downarrow & \exists! \text{ algebra map} \\
& B & \\
\end{array}
$$

Determines an isomorphism $\mathcal{C}^{\mathbb{T}}(A \star P, B) \cong \text{LeftLin}_C(A, P; B)$. 

Left-linear classifiers

Definition (c.f. Guitart ('80), Jacobs ('94), Seal ('13))

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$$
A \otimes P \xrightarrow{\sigma} A \ast P
$$

$\forall$ left-linear maps $\downarrow \exists!$ algebra map $\downarrow B$

Determines an isomorphism $\mathcal{C}^\top(A \ast P, B) \cong \text{LeftLin}_\mathcal{C}(A, P; B)$.

Need to build in a preservation property to guarantee coherence
**Definition**

For \((A, a), (B, b) \in \mathcal{C}^\square\) and \(P_1, \ldots, P_n \in \mathcal{C}\), a map

\[ h : \left( \cdots \left( (A \otimes P_1) \otimes P_2 \right) \cdots \otimes P_{n-1} \right) \otimes P_n \to B \]

is \(n\)-left linear if

\[
\begin{align*}
T(A) \otimes P_1 \otimes \cdots \otimes P_n & \xrightarrow{\text{st} \otimes^n} T(A \otimes P_1 \otimes \cdots \otimes P_n) & \xrightarrow{T h} & T B \\
A \otimes P_1 \otimes \cdots \otimes P_n & \xrightarrow{a \otimes P_1 \otimes \cdots \otimes P_n} & A \otimes P_1 \otimes \cdots \otimes P_n & \xrightarrow{h} & B \\
\end{align*}
\]

where \(\text{st}^{\otimes 1} := \text{st}\) and \(\text{st}^{\otimes (n+1)} := \text{st} \circ \text{st}^{\otimes n}\).
**n-left linear maps**

**Definition**

For \((A, a), (B, b) \in C^\uparrow\) and \(P_1, \ldots, P_n \in C\), a map

\[
h : \left( \cdots \left( (A \otimes P_1) \otimes P_2 \right) \cdots \otimes P_{n-1} \right) \otimes P_n \rightarrow B
\]

is \(n\)-left linear if

\[
T(A) \otimes P_1 \otimes \cdots \otimes P_n \xrightarrow{st \otimes^n} T(A \otimes P_1 \otimes \cdots \otimes P_n) \xrightarrow{Th} TB
\]

where \(st^{\otimes 1} := st\) and \(st^{\otimes(n+1)} := st \circ st^{\otimes n}\).

\(\leadsto\) An \(n\)-parameter version of left-linearity.
Definition

A \textit{n-left linear classifier} is a family of maps \( \sigma_{A,P_1} : A \otimes P_1 \to A \star P_1 \) such that

1. \((A \star P_1, \tau_{A,P_1}) \in \mathcal{C}^\uparrow\)

2. \(\sigma_{A,B}\) is left-linear,

3. Every \(n\)-left linear map \((\cdots ((A \otimes P_1) \otimes P_2) \cdots) \otimes P_n \to B\) factors uniquely:

\[
\begin{array}{c}
A \otimes P_1 \otimes \cdots \otimes P_n \\
\sigma \otimes P_2 \otimes \cdots \otimes P_n \\
\forall \text{ } n\text{-left linear}
\end{array}
\]

\[
\xrightarrow{\sigma \otimes P_2 \otimes \cdots \otimes P_n}
\]

\[
\begin{array}{c}
(A \star P_1) \otimes P_2 \otimes \cdots \otimes P_n \\
\exists! (n-1)\text{-left linear map}
\end{array}
\]

\[
\xrightarrow{\exists! (n-1)\text{-left linear map}}
\]

\[
B
\]

A \((1, \ldots, n)\)-left linear classifier\ is a 1-left linear classifier that is also an \(i\)-left linear classifier \((1 \leq i \leq n)\).
$n$-left linear classifiers

Lemma

If $h : (\cdots ((A \otimes P_1) \otimes P_2) \cdots) \otimes P_{n+1} \rightarrow B$ is $(n+1)$-left linear, then (if they exist)

1. The transpose $\tilde{h} : A \otimes P_1 \otimes \cdots \otimes P_n \rightarrow [P_{n+1}, B]$ is $n$-left linear,

2. $h \circ \alpha^{-1} : (A \otimes P_1 \cdots \otimes P_{n-1}) \otimes (P_n \otimes P_{n+1}) \rightarrow B$ is $n$-left linear

Lemma

If $C$ has an $n$-left linear classifier and satisfies either

- $C$ is closed, or
- $\alpha$ is invertible

Then $C$ has an $(n+1)$-left linear classifier.
Factoring the proof

\[ C \text{ has reflexive coequalizers, which } T \text{ preserves} \]

\[ C^\top \text{ has reflexive coequalizers} \]

\[ C \text{ has a } (1, 2, 3)\text{-left linear classifier} \]

\[ \alpha \text{ invertible or } \alpha \text{ invertible} \]

\[ \alpha \text{ invertible or } \alpha \text{ invertible} \]

\[ (-) \otimes X \text{ preserves reflexive coeqs.} \]

\[ C \text{ acts on } C^\top \]

\[ C^\top \text{ skew monoidal} \]
Proposition

If \( C \) has a \((1, 2, 3)\)-left linear classifier \( \sigma_{A,B} : A \otimes B \to A \star B \), then

1. \( \star : C^\top \times C \to C^\top \) is a skew action, and
2. The free-forgetful adjunction \( F : C \leftrightarrow C^\top : U \) is strong.
Proposition

If $C$ has a $(1, 2, 3)$-left linear classifier $\sigma_{A,B} : A \otimes B \to A \star B$, then

1. $\star : C^\top \times C \to C^\top$ is a skew action, and
2. The free-forgetful adjunction $F : C \leftrightarrow C^\top : U$ is strong.

Holds in particular if $C$ has a 1-left linear classifier and

- $C$ is closed, or
- $\alpha$ is invertible
Factoring the proof

$C$ has reflexive coequalizers, which $T$ preserves

$\mathcal{C}^\mathbb{T}$ has reflexive coequalizers

$C$ has a $(1, 2, 3)$-left linear classifier

$\mathcal{C}^\mathbb{T}$ skew monoidal

$C$ closed or $\alpha$ invertible

$(-) \otimes X$ preserves reflexive coeqs.
From action to skew monoidal structure

Proposition

Given

1. A skew monoidal category \((\mathcal{C}, \otimes, I)\),
2. A category \(\mathcal{A}\),
3. A skew action \(\star : \mathcal{A} \times \mathcal{C} \rightarrow \mathcal{A}\),
4. A strong adjunction \((U, \text{st}^U) : \mathcal{A} \rightleftarrows \mathcal{C} : (F, \text{st}^F)\)

Then, setting

\[ A \circledast B := A \star UB \]

makes \((\mathcal{A}, \star, FI)\) a skew monoidal category.
Proposition

If $\mathcal{C}$ has any of

1. A $(1, 2, 3)$-left linear classifier $A \otimes B \rightarrow A \star B$,
2. A 1-left linear classifier $A \otimes B \rightarrow A \star B$, and $\mathcal{C}$ is closed,
3. A 1-left linear classifier $A \otimes B \rightarrow A \star B$, and $\alpha$ is invertible

Then $(\mathcal{C}^\top, \star, TI)$ is skew monoidal.
Proposition

If $C$ has any of

1. A (1, 2, 3)-left linear classifier $A \otimes B \rightarrow A \star B$,
2. A 1-left linear classifier $A \otimes B \rightarrow A \star B$, and $C$ is closed,
3. A 1-left linear classifier $A \otimes B \rightarrow A \star B$, and $\alpha$ is invertible

Then $(C^\top, \star, TI)$ is skew monoidal.

Question: how do we construct a (1, 2, 3)-left linear classifier?
Constructing a left-linear classifier

Construction

Reflexive coequalizer in $\mathcal{C}^T$:

\[
T(T(A) \otimes P) \xrightarrow{T_{st}} T^2(A \otimes P) \xrightarrow{\mu} T(A \otimes P) \xrightarrow{\text{coeq.}} A \star P
\]

Then

1. $\mathcal{C}^T(A \star P, B) \cong \text{LeftLin}_C(A, P; B)$,

2. If $T(\ - \otimes X)$ preserves reflexive coequalizers, get a $(1, 2, 3)$-left linear classifier.
Proposition

If $C$ has all reflexive coequalizers, $T$ preserves reflexive coequalizers, and any of the following:

1. Every $(-) \otimes P$ preserves reflexive coequalizers,
2. $C$ is closed,
3. $\alpha$ is invertible

Then $C$ has a $(1,2,3)$-left linear classifier:

\[ A \otimes B \xrightarrow{\eta} T(A \otimes B) \xrightarrow{\text{coeq.}} A \star B \]
Putting it all together

Theorem

If \( C \) has all reflexive coequalizers, \( T \) preserves reflexive coequalizers, and any of the following:

1. Every \((-) \otimes P\) preserves reflexive coequalizers,
2. \( C \) is closed,
3. \( \alpha \) is invertible

Then \((C^\top, \star, TI)\) is skew monoidal.

Remark

Can also do the calculation directly — but it is much more intricate! (c.f. Seal (2013))
Factoring the proof

$\mathcal{C}$ has reflexive coequalizers, which $T$ preserves

$\mathcal{C}^T$ has reflexive coequalizers

$\mathcal{C}$ has a $(1, 2, 3)$-left linear classifier

$\mathcal{C}$ closed or $\alpha$ invertible

$\mathcal{C}$ closed or $\alpha$ invertible

$(-) \otimes X$ preserves reflexive coeqs.

$\mathcal{C}$ acts on $\mathcal{C}^T$

$\mathcal{C}^T$ skew monoidal

$\mathcal{C}^T$ skew monoidal

$\mathcal{C}^T$ has reflexive coequalizers

$\mathcal{C}$ has reflexive coequalizers, which $T$ preserves
Monoids in skew monoidal categories

**Definition**

A *monoid* in $\mathcal{C}$ is an object $M$ with $(I \xrightarrow{e} M \xleftarrow{m} M \otimes M)$ such that

\[
\begin{array}{c}
I \otimes M \xrightarrow{e \otimes M} M \otimes M \\
\downarrow \lambda \quad \downarrow m \\
M \quad M
\end{array}
\quad
\begin{array}{c}
M \xrightarrow{\rho} M \otimes I \\
\downarrow \quad \downarrow M \otimes e \\
M \otimes M
\end{array}
\]

\[
\begin{array}{c}
(M \otimes M) \otimes M \xrightarrow{m \otimes M} M \otimes M \\
\downarrow \alpha \\
M \otimes (M \otimes M)
\end{array}
\quad
\begin{array}{c}
M \otimes M \xrightarrow{m} m \\
\downarrow m \\
m
\end{array}
\]

**Question:** how do we construct free monoids?
Lemma (folklore)

Let \((\mathcal{C}, \boxtimes, I)\) be a monoidal category with finite coproducts \((0, +)\) and \(\omega\)-colimits, and \(X \in \mathcal{C}\) such that

1. Every \((-) \boxtimes P\) preserves coproducts and \(\omega\)-colimits, and
2. \(X \boxtimes (-)\) preserves \(\omega\)-colimits

Then the initial \((I + X \boxtimes -)\)-algebra is the free monoid on \(X\).
Lemma

Let \((\mathcal{C}, \otimes, I)\) be a skew monoidal category with finite coproducts \((0, +)\) and \(\omega\)-colimits, and \(X \in \mathcal{C}\) such that

1. Every \((-) \otimes P\) preserves coproducts and \(\omega\)-colimits, and
2. \(X \otimes (-)\) preserves \(\omega\)-colimits

Then the initial \((I + X \otimes -)\)-algebra is the free monoid on \(X\).
Free monoids as colimits: \((\mathcal{C}, \otimes, I)\) monoidal

Lemma (Dubuc (1974), Melliès (2008), Lack (2008))

There exists a monoidal category \(\mathcal{P}\) such that

\[
\text{MonCat}^{\text{strong}}(\mathcal{P}, \mathcal{C}) \simeq (I/\mathcal{C})
\]

Lemma (Dubuc (1974), Melliès (2008), Lack (2008))

For \((I \overset{x}{\to} X) \in (I/\mathcal{C})\), if

1. \(\mathcal{C}\) has \(\mathcal{P}\)-colimits, and
2. Every \((-) \otimes \mathcal{C}\) and \(\mathcal{C} \otimes (-)\) preserves \(\mathcal{P}\)-colimits

Then \(\text{colim } D_x\) is the free monoid on \((I \overset{x}{\to} X)\), for \(D_x : \mathcal{P} \to \mathcal{C}\) the monoidal functor corresponding to \((I \overset{x}{\to} X)\).
Free monoids as colimits: \((\mathcal{C}, \otimes, I)\) skew monoidal

Lemma

There exists a skew monoidal \(\mathcal{P}\) such that

\[
\text{SkMonCat}_{\text{strong}}(\mathcal{P}, \mathcal{C}) \simeq (I/\mathcal{C})
\]

Lemma

For \((I \xrightarrow{X} X) \in (I/\mathcal{C})\), if

1. \(\mathcal{C}\) has \(\mathcal{P}\)-colimits, and

2. Every \((-) \otimes \mathcal{C}\) and \(\mathcal{C} \otimes (-)\) preserves \(\mathcal{P}\)-colimits

Then \(\text{colim} D_x\) is the free monoid on \((I \xrightarrow{X} X)\), for \(D_x : \mathcal{P} \rightarrow \mathcal{C}\) the monoidal functor corresponding to \((I \xrightarrow{X} X)\).
Monoids in $(C^T, \star, TI)$ as $T$-monoids

Definition (c.f. Fiore et al. (1999))

For a strong monad $(\mathbb{T}, \text{st})$, a $T$-monoid is an object $M \in C$ with

1. A monoid structure $(M \otimes M \xrightarrow{m} M \xleftarrow{e} I)$,
2. An algebra structure $(M, \tau_M)$,

Such that the multiplication $m : M \otimes M \to M$ is a left-linear map.
Monoids in \((\mathcal{C}^\mathbb{T}, \star, TI)\) as \(T\)-monoids

**Definition (c.f. Fiore et al. (1999))**

For a strong monad \((\mathbb{T}, \text{st})\), a \(T\)-monoid is an object \(M \in \mathcal{C}\) with

1. A monoid structure \((M \otimes M \xrightarrow{m} M \leftarrow e I)\),
2. An algebra structure \((M, \tau_M)\),

Such that the multiplication \(m : M \otimes M \to M\) is a left-linear map.

**Example**

If \(\mathcal{C}\) has two monoidal structures \((\otimes, I)\) and \((\bullet, J)\) related by a *distributivity structure*, then for \(\mathbb{T}\) the free \(\bullet\)-monoid monad on \(\mathcal{C}\), a \(T\)-monoid in \((\mathcal{C}, \otimes, I)\) is a *near semiring object* (Fiore 2016, Fiore & S. 2017).
Monoids in \((\mathcal{C}^\top, \star, TI)\) as \(T\)-monoids

**Definition (c.f. Fiore et al. (1999))**

A **\(T\)-monoid** is an object \(M \in \mathcal{C}\) with

1. A monoid structure \((M \otimes M \xrightarrow{m} M \leftarrow I)\),
2. An algebra structure \((M, \tau_M)\),

Such that the multiplication \(m : M \otimes M \to M\) is a left-linear map.

**Proposition**

*If \(\mathcal{C}\) has a \((1, 2, 3)\)-left linear classifier \(\sigma_{A,B} : A \otimes B \to A \star B\), then*\[
T\text{-Mon}(\mathcal{C}, \otimes, I) \cong \text{Mon}(\mathcal{C}^\top, \star, TI)\]
Monoids in \((\mathcal{C}^{\mathbb{T}}, \star, TI)\) as \(T\)-monoids

Monoidal examples

1. If \(\mathcal{C}\) has finite coproducts,

\[ \mathcal{C}^{\mathbb{T}} \cong T-\text{Mon}((\mathcal{C}, +, 0)) \cong \text{Mon}(\mathcal{C}^{\mathbb{T}}) \]

2. For \(M \in \text{Mon}(\mathcal{C})\) and \(M\otimes := (M \otimes (-), m \otimes (-), e \otimes (-))\)

\[ (M/\text{Mon}(\mathcal{C})) \cong M\otimes-\text{Mon}(\mathcal{C}) \cong \text{Mon}(\mathcal{C}^{M\otimes}) \]

(Fiore & S. 2017).
Adapted classical construction of monoidal structure on $\mathcal{C}^\square$ to skew monoidal setting.
Adapted classical construction of monoidal structure on $C^\top$ to skew monoidal setting.

Proof simplified by focus on *n-left linear classifiers* and corresponding *skew monoidal actions*.
Adapted classical construction of monoidal structure on $C^\top$ to skew monoidal setting.

Proof simplified by focus on $n$-left linear classifiers and corresponding skew monoidal actions.

Construction of free monoids in skew setting is as for monoidal categories.
Adapted classical construction of monoidal structure on $C^\top$ to skew monoidal setting.

Proof simplified by focus on $n$-left linear classifiers and corresponding skew monoidal actions.

Construction of free monoids in skew setting is as for monoidal categories.

Monoids in $(C^\top, \star, TI)$ are $T$-monoids in $(C, \otimes, I)$. 
Adapted classical construction of monoidal structure on $\mathcal{C}^\mathbb{T}$ to skew monoidal setting.

Proof simplified by focus on *n-left linear classifiers* and corresponding *skew monoidal actions*.

Construction of free monoids in skew setting is as for monoidal categories.

Monoids in $(\mathcal{C}^\mathbb{T}, \star, Tl)$ are $T$-monoids in $(\mathcal{C}, \otimes, I)$.

Associated paper in preparation.