Proofs for "Integrating Dependent and Linear Types"

Neel Krishnaswami, Pierre Pradic, and Nick Benton

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1 Overview

The basic approach of this paper is to build a realizability model of dependent LNL in the style of Harper [4]. Essentially, we give an untyped operational semantics for the language, and then construct a PER for the syntactic types, and a function mapping each semantic type to a PER giving the equality relation for that type. For linear types, we give a map from semantic types to a map from monoid elements to PERs. This generalizes the pattern of L^3 [1] from unary to binary relations.

Below, the first occurence is the statement of the theorem, and the second is the proof. (The proofs all begin on page 19.)

List of Theorems

2 Untyped Syntax

See figures.

$$
\begin{array}{lcl}
e, t, X, A & ::= & \Pi x : X.\ Y \mid A \multimap B \mid \lambda x : C.\ e \mid e\ e' \mid \hat{\lambda} x.\ e \\
& \mid & 1 \mid \mathsf{I} \mid () \mid \mathsf{let}\ () = e\ \mathsf{in}\ e' \\
& \mid & \Sigma x : X.\ Y \mid A \otimes B \mid (e, e') \mid \pi_1\, e \mid \pi_2\, e \mid \mathsf{let}\ (x, y) = e\ \mathsf{in}\ e' \\
& \mid & \mathsf{G}\, e \mid \mathsf{G}^{-1}\, e \\
& \mid & \mathsf{F} x : X.\ A \mid \mathsf{F}\,(e, t) \mid \mathsf{let}\ \mathsf{F}\,(x, a) = t\ \mathsf{in}\ t' \\
& \mid & \top \mid A \,\&\, B \\
& \mid & \forall x : X.\ Y \mid \exists x : x.\ Y \\
& \mid & e =_X e' \mid \mathsf{refl} \\
& \mid & \mathbb{N} \mid 0 \mid \mathsf{s}(e) \mid \mathsf{iter}(e, 0 \to e_0, \mathsf{s}(x), y \to e_1) \\
& \mid & \mathsf{U}_i \mid \mathsf{L}_i \\
& \mid & x \mid \mathsf{fix}\ f\ x = e \\
& \mid & [A] \mid \mathsf{let}\ [x] = e\ \mathsf{in}\ e \mid * \\
& \mid & e \mapsto X \mid \mathsf{Loc} \mid \mathsf{new}_X\, e \mid \mathsf{free}\,(e, t) \mid l \\
& \mid & \mathsf{T}\, A \mid \mathsf{val}\, e \mid \mathsf{let\ val}\, x = e\ \mathsf{in}\ e' \\
\\
v & ::= & \lambda x : A.\ e \mid () \mid (e, e) \mid \mathsf{refl} \mid \mathsf{G}\, e \mid l \mid * \mid 0 \mid \mathsf{s}(v) \\
& \mid & \Pi x : X.\ Y \mid A \multimap B \mid \top \mid A \,\&\, B \\
& \mid & 1 \mid \mathsf{I} \mid \Sigma x : X.\ Y \mid A \otimes B \mid \mathsf{F} x : X.\ B \\
& \mid & e =_X e' \mid e \mapsto X \mid \mathsf{Loc} \mid \mathsf{U}_i \mid \mathsf{L}_i \\
\\
u & ::= & \lambda x.\ e \mid () \mid (e, e) \mid \mathsf{F}\,(e, e) \mid \hat{\lambda} x.\ e \mid (e, e') \\
& \mid & * \mid \mathsf{val}\, e \mid \mathsf{let\ val}\, x = e\ \mathsf{in}\ e \mid \mathsf{new}_X\, e \\
& \mid & \mathsf{let}\ (x, a) = \mathsf{get}(e, e')\ \mathsf{in}\ e'' \mid e :=_{e''} e' \\
\\
\sigma & ::= & \cdot \mid \sigma, l : v
\end{array}
$$

Figure 1: Terms e, t, X, A, values v, linear values u, stores σ

3 Operational Semantics

See figures.

4 CPPOs and Fixed Points

A *pointed partial order* is a triple $(X, \leqslant, \bot)$ such that X is a set, $\leqslant$ is a partial order on X, and $\bot$ is the least element of X. A subset $D \subseteq X$ is a *directed set* when every pair of elementsd $x, y \in D$ has an upper bound in D (i.e., there is a $z \in D$ such that $x \leqslant z$ and $y \leqslant z$). A pointed partial order is *complete* (i.e., forms a CPPO) when every directed set D has a supremum $\bigsqcup D$ in X.

The following lemma is in Harper '92, and is Theorem 8.22 in Davies and Priestley.

Lemma 1. *(Fixed Points on CPPOs) If* X *is a CPPO, and* $f : X \to X$ *is a monotone function on* X*, then* f *has a least fixed point.*

Proof. Construct the ordinal-indexed sequence x_α, where:

$$\begin{array}{lcl} x_0 & = & \bot \\ x_{\beta+1} & = & f(x_\beta) \\ x_\lambda & = & \bigsqcup_{\beta<\lambda} x_\beta \end{array}$$

Because f is monotone, we can show by transfinite induction that every initial segment is directed, which ensures the needed suprema exist and the sequence is well-defined.

Now, since we know there must be a stage λ such that $x_\lambda = x_{\lambda+1}$. If there were not, then we could construct a bijection between the ordinals and the strictly increasing chain of elements of the sequence x. However, the elements of the sequence x are all drawn of X. Since X is a set, it follows that the elements of x must themselves form a set. Since the ordinals do not form a set (they are a proper class), this leads to a contradiction. Hence, there must be a stage λ such that $x_\lambda = x_{\lambda+1}$. □

5 Partial Equivalence Relations and Semantic Type Systems

A *partial equivalence relation (*PER*)* is a symmetric, transitive relation on closed, terminating expressions. We further require that PERs be *closed under evaluation*. Given a PER R, we require that for all e, e', v, v' such that $e \Downarrow v$ and $e' \Downarrow v'$, we have that $(e, e') \in R$ if and only if $(v, v') \in R$. Given a PER P, we write P^* to close it up under evaluation.

A *partial evaluation relation on configurations (*CPER*)* is a symmetric, transitive relation on terminating machine configurations $\langle\sigma; e\rangle$. We further require that they be *closed under evaluation*. Given a CPER M, we require that for all $\langle\sigma_1; e_1\rangle$ such that $\langle\sigma_1; e_1\rangle \Downarrow \langle\sigma_1'; u_1\rangle$ and $\langle\sigma_2; e_2\rangle$ such that $\langle\sigma_2; e_2\rangle \Downarrow \langle\sigma_2'; u_2\rangle$, we have $(\langle\sigma_1; e_1\rangle, \langle\sigma_2; e_2\rangle) \in M$ if and only if $(\langle\sigma_1'; u_1\rangle, \langle\sigma_2'; u_2\rangle) \in M$.

Note that since evaluation (both ordinary and linear) is deterministic, an evaluation-closed PER is determined by its sub-PER on values (or value configurations).

A *semantic linear/non-linear type system* is a four-tuple $(I \in \text{PER}, L \in \text{PER}, \phi : I \to \text{PER}, \psi : L \to \text{CPER})$ such that ϕ respects I and ψ respects L. We say that I are the *semantic intuitionistic types*, L are the *semantic linear types*, and ϕ and ψ are *the type interpretation functions*.

The set of type systems forms a CPPO. The least element is the type system $(\emptyset, \emptyset, !_{\text{PER}}, !_{\text{CPER}})$ with an empty set of intuitionistic and linear types. The ordering $(I, L, \phi, \psi) \leqslant (I', L', \phi', \psi')$ is given by set inclusion on $I \subseteq I'$ and $L \subseteq L'$, when there is agreement between ϕ and ϕ' on the common part of their domains, and likewise for ψ and ψ' (which we write $\phi \sqsubseteq \phi'$ and $\psi \sqsubseteq \psi'$). Given a directed set, the join is given by taking unions pointwise (treating the functions ϕ and ψ as graphs).

We define the following constructions on PERs in Figure **??**.

$$\boxed{e \Downarrow v}$$

$$\frac{}{v \Downarrow v} \qquad \frac{e_1 \Downarrow \lambda x : A.\ e \qquad [e_2/x]e \Downarrow v}{e_1\ e_2 \Downarrow v}$$

$$\frac{e \Downarrow (e_1, e_2) \qquad e_1 \Downarrow v}{\pi_1\, e \Downarrow v} \qquad \frac{e \Downarrow (e_1, e_2) \qquad e_2 \Downarrow v}{\pi_2\, e \Downarrow v}$$

$$\frac{e \Downarrow v}{\mathsf{s}(e) \Downarrow \mathsf{s}(v)} \qquad \frac{e \Downarrow 0 \qquad e_0 \Downarrow v}{\mathsf{iter}(e, 0 \to e_0, \mathsf{s}(x), y \to e_1) \Downarrow v}$$

$$\frac{e \Downarrow \mathsf{s}(n) \qquad \mathsf{iter}(v, 0 \to e_0, \mathsf{s}(x), y \to e_1) \Downarrow v \qquad [n/x, v/y]e_1 \Downarrow v'}{\mathsf{iter}(e, 0 \to e_0, \mathsf{s}(x), y \to e_1) \Downarrow v}$$

$$\frac{e \Downarrow \mathsf{fix}\ f\ x = e_0 \qquad [(\mathsf{fix}\ f\ x = e_0)/f, e'/x]e_0 \Downarrow v}{e\ e' \Downarrow v}$$

$$\boxed{\langle \sigma; e \rangle \Downarrow \langle \sigma'; u \rangle}$$

$$\frac{}{\langle \sigma; u \rangle \Downarrow \langle \sigma; u \rangle}\ \textsc{LVal} \qquad \frac{\langle \sigma; e_1 \rangle \Downarrow \langle \sigma'; \lambda x.\ e_1' \rangle \qquad \langle \sigma'; [e_2/x]e_1' \rangle \Downarrow \langle \sigma''; u'' \rangle}{\langle \sigma; e_1\ e_2 \rangle \Downarrow \langle \sigma''; u'' \rangle}\ \textsc{LApp}$$

$$\frac{\langle \sigma; e_1 \rangle \Downarrow \langle \sigma'; \hat{\lambda} x.\ e \rangle \qquad \langle \sigma'; [e_2/x]e \rangle \Downarrow \langle \sigma''; u'' \rangle}{\langle \sigma; e_1\ e_2 \rangle \Downarrow \langle \sigma''; u'' \rangle}\ \textsc{LPiApp} \qquad \frac{\langle \sigma; e \rangle \Downarrow \langle \sigma'; (e_1, e_2) \rangle \qquad \langle \sigma'; e_1 \rangle \Downarrow \langle \sigma''; u'' \rangle}{\langle \sigma; \pi_1\, e \rangle \Downarrow \langle \sigma''; u'' \rangle}\ \textsc{LFst}$$

$$\frac{\langle \sigma; e \rangle \Downarrow \langle \sigma'; (e_1, e_2) \rangle \qquad \langle \sigma'; e_2 \rangle \Downarrow \langle \sigma''; u'' \rangle}{\langle \sigma; \pi_2\, e \rangle \Downarrow \langle \sigma''; u'' \rangle}\ \textsc{LSnd} \qquad \frac{\langle \sigma; e \rangle \Downarrow \langle \sigma'; () \rangle \qquad \langle \sigma'; e' \rangle \Downarrow \langle \sigma''; u \rangle}{\langle \sigma; \mathsf{let}\ () = e\ \mathsf{in}\ e' \rangle \Downarrow \langle \sigma''; u \rangle}\ \textsc{LUnit}$$

$$\frac{\langle \sigma; e \rangle \Downarrow \langle \sigma'; (e_1, e_2) \rangle \qquad \langle \sigma'; [e_1/a, e_2/b]e' \rangle \Downarrow \langle \sigma''; u \rangle}{\langle \sigma; \mathsf{let}\ (a, b) = e\ \mathsf{in}\ e' \rangle \Downarrow \langle \sigma''; u \rangle}\ \textsc{LPair}$$

$$\frac{\langle \sigma; e \rangle \Downarrow \langle \sigma'; \mathsf{F}\,(e_1, e_2) \rangle \qquad \langle \sigma'; [e_1/x, e_2/a]e' \rangle \Downarrow \langle \sigma''; u \rangle}{\langle \sigma; \mathsf{let}\ \mathsf{F}\,(x, a) = e\ \mathsf{in}\ e' \rangle \Downarrow \langle \sigma''; u \rangle}\ \textsc{LF} \qquad \frac{e \Downarrow \mathsf{G}\, e' \qquad \langle \sigma; e' \rangle \Downarrow \langle \sigma'; u \rangle}{\langle \sigma; \mathsf{G}^{-1}\, e \rangle \Downarrow \langle \sigma'; u \rangle}\ \textsc{LRunG}$$

$$\frac{\langle \sigma; e \rangle \Downarrow \langle \sigma'; u' \rangle}{\langle \sigma; \mathsf{let}\ [] = e_0\ \mathsf{in}\ e \rangle \Downarrow \langle \sigma'; u' \rangle}\ \textsc{LIrrDrop} \qquad \frac{\langle \sigma; [*/a, */b]e \rangle \Downarrow \langle \sigma'; u' \rangle}{\langle \sigma; \mathsf{let}\ [a, b] = e_0\ \mathsf{in}\ e \rangle \Downarrow \langle \sigma'; u' \rangle}\ \textsc{LIrrSplit}$$

$$\frac{e \Downarrow l \qquad \langle \sigma; e' \rangle \Downarrow \langle \sigma', l : v; * \rangle \qquad \langle \sigma', l : v; [v/x, */c]e'' \rangle \Downarrow \langle \sigma''; u \rangle}{\langle \sigma; \mathsf{let}\ (x, c) = \mathsf{get}(e, e')\ \mathsf{in}\ e'' \rangle \Downarrow \langle \sigma''; u \rangle}\ \textsc{LDeref}$$

$$\boxed{\langle \sigma; e \rangle \rightsquigarrow \langle \sigma'; \mathsf{val}\, v \rangle}$$

$$\frac{}{\langle \sigma; \mathsf{val}\, e \rangle \rightsquigarrow \langle \sigma; \mathsf{val}\, e \rangle} \qquad \frac{\langle \sigma; e \rangle \Downarrow \langle \sigma'; u \rangle \qquad e \neq \mathsf{val}\, u_0 \qquad \langle \sigma; u \rangle \rightsquigarrow \langle \sigma''; \mathsf{val}\, u' \rangle}{\langle \sigma; e \rangle \rightsquigarrow \langle \sigma''; \mathsf{val}\, u' \rangle}$$

$$\frac{\langle \sigma; e \rangle \rightsquigarrow \langle \sigma_1; \mathsf{val}\, e_1 \rangle \qquad \langle \sigma_1; [e_1/x]e' \rangle \rightsquigarrow \langle \sigma'; \mathsf{val}\, v \rangle}{\langle \sigma; \mathsf{let\ val}\, x = e\ \mathsf{in}\ e' \rangle \rightsquigarrow \langle \sigma'; \mathsf{val}\, v \rangle} \qquad \frac{e \Downarrow v \qquad l \notin \mathrm{dom}(\sigma)}{\langle \sigma; \mathsf{new}_X\, e \rangle \rightsquigarrow \langle \sigma, l : v; \mathsf{val}\, \mathsf{F}\,(l, *) \rangle}$$

$$\frac{e \Downarrow l \qquad e' \Downarrow v \qquad \langle \sigma; e'' \rangle \Downarrow \langle \sigma', l : v'; * \rangle}{\langle \sigma, l : v'; e :=_{e''} e' \rangle \rightsquigarrow \langle \sigma', l : v; \mathsf{val}\, * \rangle}\ \textsc{LAssign} \qquad \frac{e \Downarrow l \qquad \langle \sigma; t \rangle \Downarrow \langle \sigma', l : v; * \rangle}{\langle \sigma; \mathsf{free}\,(e, t) \rangle \rightsquigarrow \langle \sigma'; \mathsf{val}\, () \rangle}\ \textsc{LDelete}$$

Figure 2: Operational Semantics

$$
\begin{array}{lcl}
\mathit{Loc} & = & \{(l,l) \mid l \in \mathsf{Loc}\} \\
\hat{\mathrm{I}} & = & \{((\,),(\,))\} \\
\mathit{Id}(a,b,E) & = & \{(\mathsf{refl},\mathsf{refl}) \mid (a,b) \in E\} \\
\Pi(E,\Phi) & = & \{(v,v') \mid \forall (a,a') \in E.\ (va, v'a') \in \Phi(a)\} \\
\Sigma(E,\Phi) & = & \{((a,b),(a',b')) \mid (a,a') \in E \wedge (b,b') \in \Phi(a)\} \\
\mathrm{G}(C) & = & \{(\mathsf{G}\,e, \mathsf{G}\,e') \mid (\langle \cdot; e\rangle, \langle \cdot; e'\rangle) \in C\} \\
\hat{\forall}(E,\Phi) & = & \{(v,v') \mid \forall (e,e') \in E.\ (v,v') \in \Phi(e)\} \\
\hat{\exists}(E,\Phi) & = & \{(v,v') \mid \exists (e,e') \in E.\ (v,v') \in \Phi(e)\}^{\dagger} \\
\hat{\mathbb{N}} & = & \left\{(\mathsf{s}^k(0), \mathsf{s}^k(0)) \;\middle|\; k \text{ is a natural number}\right\} \\
\hat{\top}_{\mathrm{I}} & = & \{(v,v') \mid v \in \mathrm{Val} \wedge v' \in \mathrm{Val}\}
\end{array}
$$

Figure 3: Intuitionistic PER constructions

$$
\begin{array}{lcl}
\hat{\top} & = & \{(\langle \sigma; (\,)\rangle, \langle \sigma'; (\,)\rangle) \mid \sigma, \sigma' \in \mathrm{Store}\} \\
A \mathbin{\hat{\&}} B & = & \left\{ \begin{array}{l} (\langle \sigma; (a,b)\rangle, \\ \quad \langle \sigma'; (a',b')\rangle) \end{array} \;\middle|\; \begin{array}{l} (\langle \sigma; a\rangle, \langle \sigma'; a'\rangle) \in A \wedge \\ (\langle \sigma; b\rangle, \langle \sigma'; b'\rangle) \in B \end{array} \right\} \\
\hat{\mathrm{I}} & = & \{(\langle \cdot; (\,)\rangle, \langle \cdot; (\,)\rangle)\} \\
(C \mathbin{\hat{\otimes}} D) & = & \left\{ \begin{array}{l} (\langle \sigma; (c,d)\rangle, \\ \quad \langle \sigma'; (c',d')\rangle) \end{array} \;\middle|\; \begin{array}{l} \exists \sigma_C, \sigma_D, \sigma_{C'}, \sigma_{D'}. \\ \quad \sigma = \sigma_C \cdot \sigma_D \wedge \\ \quad \sigma' = \sigma'_C \cdot \sigma'_D \wedge \\ \quad (\langle \sigma_C; c\rangle, \langle \sigma'_C; c'\rangle) \in C \wedge \\ \quad (\langle \sigma_D; d\rangle, \langle \sigma'_D; d'\rangle) \in D \end{array} \right\} \\
(C \mathbin{\hat{\multimap}} D) & = & \left\{ \begin{array}{l} (\langle \sigma; u\rangle, \\ \quad \langle \sigma'; u'\rangle) \end{array} \;\middle|\; \begin{array}{l} \forall \sigma_0 \# \sigma, \sigma'_0 \# \sigma', c, c'. \\ \quad \text{if } (\langle \sigma_0; c\rangle, \langle \sigma'_0; c'\rangle) \in C \\ \quad \text{then } \left(\begin{array}{l} \langle \sigma \cdot \sigma_0; uc\rangle, \\ \langle \sigma' \cdot \sigma'_0; u'c'\rangle \end{array} \right) \in D \end{array} \right\} \\
F(E,\Psi) & = & \left\{ \begin{array}{l} (\langle \sigma; \mathsf{F}\,(a,b)\rangle, \\ \quad \langle \sigma'; \mathsf{F}\,(a',b')\rangle) \end{array} \;\middle|\; \begin{array}{l} E(a,a') \wedge \\ (\langle \sigma; b\rangle, \langle \sigma'; b'\rangle) \in \Psi(a) \end{array} \right\} \\
\Pi_L(E,\Psi) & = & \left\{ \begin{array}{l} (\langle \sigma; u\rangle, \\ \quad \langle \sigma'; u'\rangle) \end{array} \;\middle|\; \begin{array}{l} \forall (e,e') \in E. \\ \quad (\langle \sigma; u\ e\rangle, \langle \sigma'; u'\ e'\rangle) \in \Psi(e) \end{array} \right\} \\
\hat{\forall}_L(E,\Psi) & = & \left\{ \begin{array}{l} (\langle \sigma; u\rangle, \\ \quad \langle \sigma'; u'\rangle) \end{array} \;\middle|\; \begin{array}{l} \forall (e,e') \in E. \\ \quad (\langle \sigma; u\rangle, \langle \sigma'; u'\rangle) \in \Psi(e) \end{array} \right\} \\
\hat{\exists}_L(E,\Psi) & = & \left\{ \begin{array}{l} (\langle \sigma; u\rangle, \\ \quad \langle \sigma'; u'\rangle) \end{array} \;\middle|\; \begin{array}{l} \exists (e,e') \in E. \\ \quad (\langle \sigma; u\rangle, \langle \sigma'; u'\rangle) \in \Psi(e) \end{array} \right\}^{\dagger} \\
\hat{\mathrm{T}}(A) & = & \left\{ \begin{array}{l} (\langle \sigma_1; e_1\rangle, \\ \quad \langle \sigma_2; e_2\rangle) \end{array} \;\middle|\; \begin{array}{l} \forall \sigma_f \# \sigma_1, \sigma_g \# \sigma_2.\ \exists \sigma'_1, \sigma'_2, u_1, u_2. \\ \langle \sigma_1 \cdot \sigma_f; e_1\rangle \leadsto \langle \sigma'_1 \cdot \sigma_f; \mathsf{val}\, u_1\rangle \wedge \\ \langle \sigma_2 \cdot \sigma_g; e_2\rangle \leadsto \langle \sigma'_2 \cdot \sigma_g; \mathsf{val}\, u_2\rangle \wedge \\ (\langle \sigma'_1; u_1\rangle, \langle \sigma'_2; u_2\rangle) \in A \end{array} \right\} \\
\mathit{Ptr}(e,E) & = & \left\{ (\langle \sigma_1; *\rangle, \langle \sigma_2; *\rangle) \;\middle|\; \begin{array}{l} \sigma_1 = [l : v_1] \wedge \sigma_2 = [l : v_2] \wedge \\ (e,l) \in \mathit{Loc} \wedge (v_1,v_2) \in E \end{array} \right\} \\
\mathit{Irr}(A) & = & \left\{ (\langle \sigma; *\rangle, \langle \sigma'; *\rangle) \;\middle|\; \exists a, a'.\ (\langle \sigma; a\rangle, \langle \sigma; a'\rangle) \in A \right\}
\end{array}
$$

Figure 4: Linear PER Constructions

$$
\begin{array}{lcl}
\phi'(\mathsf{Loc}) & = & \mathit{Loc} \\
\phi'(\mathbb{N}) & = & \hat{\mathbb{N}} \\
\phi'(\top_I) & = & \hat{\top}_I \\
\phi'(e_1 =_X e_2) & = & \mathit{Id}(e_1, e_2, \phi(X)) \\
\phi'(\Pi x : X.\ Y[x]) & = & \Pi(\phi(X), \lambda v.\ \phi(Y[v])) \\
\phi'(\Sigma x : X.\ Y[x]) & = & \Sigma(\phi(X), \lambda v.\ \phi(Y[v])) \\
\phi'(\forall x : X.\ Y[x]) & = & \hat{\forall}(\phi(X), \lambda v.\ \phi(Y[v])) \\
\phi'(\exists x : X.\ Y[x]) & = & \hat{\exists}(\phi(X), \lambda v.\ \phi(Y[v])) \\
\phi'(\mathsf{G}\, A) & = & G(\psi(A)) \\
\phi'(U_i) & = & \mathsf{let}\ (\hat{U}, \hat{\phi}, \hat{L}, \hat{\psi}) = \mathsf{fix}(T_i)\ \mathsf{in}\ \hat{U} \\
\phi'(L_i) & = & \mathsf{let}\ (\hat{U}, \hat{\phi}, \hat{L}, \hat{\psi}) = \mathsf{fix}(T_i)\ \mathsf{in}\ \hat{L} \\
\\
\psi'(\mathsf{I}) & = & \hat{I} \\
\psi'(A \otimes B) & = & \psi(A)\ \hat{\otimes}\ \psi(B) \\
\psi'(\top) & = & \hat{\top} \\
\psi'(A \,\&\, B) & = & \psi(A)\ \hat{\&}\ \psi(B) \\
\psi'(A \multimap B) & = & \psi(A)\ \hat{\multimap}\ \psi(B) \\
\psi'(\mathsf{F} x : X.\ A[x]) & = & F(\phi(X), \lambda v.\ \psi(A[v])) \\
\psi'(\Pi x : X.\ A[x]) & = & \Pi_L(\phi(X), \lambda v.\ \psi(A[v])) \\
\phi'(\forall x : X.\ A[x]) & = & \hat{\forall}_L(\phi(X), \lambda v.\ \psi(A[v])) \\
\phi'(\exists x : X.\ A[x]) & = & \hat{\exists}_L(\phi(X), \lambda v.\ \psi(A[v])) \\
\psi'(\mathsf{T}\, A) & = & T(\psi(A)) \\
\psi'(e \mapsto X) & = & \mathit{Ptr}(e, \phi(X))
\end{array}
$$

Figure 5: Definition of T_k, interpretation part

We can now define an operator T_k on type systems:

$$T_k(I, L, \phi, \psi) = (I'^*, L'^*, \phi', \psi')$$

where I' and L' are defined in Figure 6 and ϕ' and ψ' are defined in Figure 5.

Lemma 2 (T_k is a type system operator)**.** *We have that T_k is a monotone function on type systems.*

Lemma 3 (Expansion)**.** *If $i \leqslant k$ and τ is a type system then $T_i(\tau) \leqslant T_k(\tau)$.*

Lemma 4 (Universe Cumulativity)**.** *If $i \leqslant k$ then $\mathcal{T}_i \leqslant \mathcal{T}_k$.*

The interpretation of the i-th universe is the least fixed point of T_i.

The T_k are monotone operators, and so they have least fixed points. Furthermore, our definition of T_k refers to the fixed point itself, but only for smaller stages $i < k$.

Define $\mathcal{T}_i$ as the least fixed point of T_i. Notice $\mathcal{T}_i \sqsubseteq \mathcal{T}_{i+1}$. We shall thus consider the following type system in the sequel:

$$\mathcal{T}_\omega := \bigsqcup_{i \in \mathbb{N}} \mathcal{T}_i$$

$$
\begin{aligned}
I' = \;& \{(\mathsf{Loc}, \mathsf{Loc})\} \cup \\
& \{(\mathbb{N}, \mathbb{N})\} \cup \\
& \{(\top_I, \top_I)\} \cup \\
& \{(e_1 =_X e_2, t_1 =_Y t_2) \mid I(X, Y) \wedge \Phi(X)(e_1, t_1) \wedge \Phi(X)(e_2, t_2)\} \cup \\
& \left\{ \begin{array}{l} (\Pi x : X.\ Y[x], \\ \quad \Pi x : X'.\ Y'[x]) \end{array} \;\middle|\; \begin{array}{l} I(X, X') \wedge \\ \forall (v, v') \in \Phi(X).\ I(Y[v], Y'[v']) \end{array} \right\} \cup \\
& \left\{ \begin{array}{l} (\Sigma x : X.\ Y[x] \\ \quad \Sigma x : X'.\ Y'[x]) \end{array} \;\middle|\; \begin{array}{l} I(X, X') \wedge \\ \forall (v, v') \in \Phi(X).\ I(Y[v], Y'[v']) \end{array} \right\} \cup \\
& \left\{ \begin{array}{l} (\forall x : X.\ Y[x] \\ \quad \forall x : X'.\ Y'[x]) \end{array} \;\middle|\; \begin{array}{l} I(X, X') \wedge \\ \forall (v, v') \in \Phi(X).\ I(Y[v], Y'[v']) \end{array} \right\} \cup \\
& \left\{ \begin{array}{l} (\exists x : X.\ Y[x] \\ \quad \exists x : X'.\ Y'[x]) \end{array} \;\middle|\; \begin{array}{l} I(X, X') \wedge \\ \forall (v, v') \in \Phi(X).\ I(Y[v], Y'[v']) \end{array} \right\} \cup \\
& \{(\mathsf{G}\, A, \mathsf{G}\, A') \mid L(A, A')\} \cup \\
& \{(\mathsf{U}_i, \mathsf{U}_i) \mid i < k\} \cup \\
& \{(\mathsf{L}_i, \mathsf{L}_i) \mid i < k\}
\end{aligned}
$$

$$
\begin{aligned}
L' = \;& \{(\mathsf{I}, \mathsf{I})\} \cup \\
& \{(A \otimes B, A' \otimes B') \mid L(A, A') \wedge L(B, B')\} \cup \\
& \{(A \multimap B, A' \multimap B') \mid L(A, A') \wedge L(B, B')\} \cup \\
& \left\{ \begin{array}{l} (\mathsf{F} x : X.\ A[x], \\ \quad \mathsf{F} x : X'.\ A'[x]) \end{array} \;\middle|\; \begin{array}{l} I(X, X') \wedge \\ \forall (v, v') \in \Phi(X).\ L(A[v], A'[v']) \end{array} \right\} \cup \\
& \left\{ \begin{array}{l} (\Pi x : X.\ A[x], \\ \quad \Pi x : X'.\ A'[x]) \end{array} \;\middle|\; \begin{array}{l} I(X, X') \wedge \\ \forall (v, v') \in \Phi(X).\ L(A[v], A'[v']) \end{array} \right\} \cup \\
& \left\{ \begin{array}{l} (\forall x : X.\ A[x], \\ \quad \forall x : X'.\ A'[x]) \end{array} \;\middle|\; \begin{array}{l} I(X, X') \wedge \\ \forall (v, v') \in \Phi(X).\ L(A[v], A'[v']) \end{array} \right\} \cup \\
& \left\{ \begin{array}{l} (\exists x : X.\ A[x], \\ \quad \exists x : X'.\ A'[x]) \end{array} \;\middle|\; \begin{array}{l} I(X, X') \wedge \\ \forall (v, v') \in \Phi(X).\ L(A[v], A'[v']) \end{array} \right\} \cup \\
& \{(\top, \top)\} \cup \\
& \{(A \,\&\, B, A' \,\&\, B') \mid L(A, A') \wedge L(B, B')\} \cup \\
& \{(\mathsf{T}\, A, \mathsf{T}\, A') \mid (A, A') \in L\} \cup \\
& \{(e \mapsto X, e' \mapsto X') \mid (e, e') \in \mathsf{Loc} \wedge (X, X') \in U\}
\end{aligned}
$$

Figure 6: Definition of type part of T_k

6 Environments

6.1 Semantic Environments

6.1.1 Intuitionistic

$$
\begin{array}{lcl}
[\![\cdot]\!] & = & \{\langle\rangle\} \\
[\![\Gamma, x : X]\!] & = & \{(\gamma, (e_1, e_2)/x) \mid \gamma \in [\![\Gamma]\!] \wedge (\gamma_1(X), \gamma_2(X)) \in U_i \wedge (e_1, e_2) \in \phi_i(\gamma(X))\}
\end{array}
$$

6.1.2 Linear

$$
\begin{array}{lcl}
[\![\cdot]\!] & = & \{(\langle \epsilon; \langle\rangle \rangle, \langle \epsilon; \langle\rangle \rangle)\} \\
[\![\Delta_1, \Delta_2]\!] & = & \left\{ (\langle \sigma; \delta_1, \delta_2 \rangle, \langle \sigma'; \delta_1', \delta_2' \rangle) \;\middle|\; \begin{array}{l} \exists \sigma_1, \sigma_2, \sigma_1', \sigma_2'. \\ \quad \sigma = \sigma_1 \cdot \sigma_2 \wedge \sigma' = \sigma_1' \cdot \sigma_2' \wedge \\ \quad (\langle \sigma_1; \delta_1 \rangle, \langle \sigma_1'; \delta_1' \rangle) \in [\![\Delta_1]\!] \wedge \\ \quad (\langle \sigma_2; \delta_2 \rangle, \langle \sigma_2'; \delta_2' \rangle) \in [\![\Delta_2]\!] \end{array} \right\} \\
[\![a : A]\!] & = & \{(\langle \sigma; e/a \rangle, \langle \sigma'; e'/a \rangle) \mid (A, A) \in L_i \wedge (\langle \sigma; e \rangle, \langle \sigma; e' \rangle) \in \psi(A)\}
\end{array}
$$

7 Typing Rules

The judgements are:

- Γ ok
- $\Gamma \vdash \Delta$ ok
- $\Gamma \vdash X$ type
- $\Gamma \vdash A$ linear
- $\Gamma \vdash X \equiv Y$ type
- $\Gamma \vdash A \equiv B$ linear
- $\Gamma \vdash e : X$
- $\Gamma; \Delta \vdash e : A$
- $\Gamma \vdash e \equiv e' : X$
- $\Gamma; \Delta \vdash e \equiv e' : A$

We maintain the following implicit premises in all of the rules:

- Every rule of the form $\Gamma \vdash e : X$ has $\Gamma \vdash X$ type as a premise.
- Every rule of the form $\Gamma \vdash e \equiv e' : X$ has $\Gamma \vdash e : X$, and $\Gamma \vdash e' : X$ and $\Gamma \vdash X$ type as premises.
- Every rule of the form $\Gamma; \Delta \vdash e : A$ has $\Gamma \vdash A$ linear as a premise.
- Every rule of the form $\Gamma; \Delta \vdash e \equiv e' : A$ has $\Gamma; \Delta \vdash e : A$, and $\Gamma; \Delta \vdash e' : A$ and $\Gamma \vdash A$ linear as premises.

In the figures, we suppress these premises for readability.

8 Fundamental Property

Theorem 1 (Fundamental Property).
Assuming that Γ ok *and* $\gamma \in [\![\Gamma]\!]$ *and* $\Gamma \vdash \Delta$ ok *and* $(\sigma, \delta) \in [\![\gamma_1(\Delta)]\!]$*, we have that:*

1. *If* $\Gamma \vdash X$ type *then* $\gamma(X) \in U(\gamma_1(X))$.
2. *If* $\Gamma \vdash X \equiv Y$ type *then* $(\gamma_1(X), \gamma_2(Y)) \in U(\gamma_1(X))$.
3. *If* $\Gamma \vdash e : X$ *then* $\gamma(e) \in \phi(\gamma_1(X))$.
4. *If* $\Gamma \vdash e_1 \equiv e_2 : X$ *then* $(\gamma_1(e_1), \gamma_2(e_2)) \in \phi(\gamma_1(X))$.
5. *If* $\Gamma \vdash A$ linear *then* $\gamma(A) \in L(\gamma_1(X))$.
6. *If* $\Gamma \vdash A \equiv B$ linear *then* $(\gamma_1(A), \gamma_2(B)) \in L(\gamma_1(X))$.
7. *If* $\Gamma; \Delta \vdash e : A$ *then* $((\sigma_1, \gamma_1(\delta_1(e))), (\sigma_2, \gamma_2(\delta_2(e)))) \in \psi(\gamma_1(X))$.
8. *If* $\Gamma; \Delta \vdash e_1 \equiv e_2 : A$ *then* $((\sigma_1, \gamma_1(\delta_1(e_1))), (\sigma_2, \gamma_2(\delta_2(e_2)))) \in \psi(\gamma_1(X))$.
9. *If* $\Gamma; \Delta \vdash e \div A$ *then there exists* t *and* t' *such that for every* $\gamma \in [\![\Gamma]\!]$ *and every* $(\sigma, \delta) \in [\![\gamma_1(\Delta)]\!]$, $((\sigma_1, \delta_1(\gamma_1(t))), (\sigma_2, \delta_2(\gamma_2(t'$ $\psi(\gamma_1(A))$.

9 Technical Lemmas

Lemma 5 (Context Shrinking).
If Γ, Γ' ok *then* Γ ok.

Lemma 6 (Linear Context Shrinking).
If $\Gamma \vdash \Delta, \Delta'$ ok *then* $\Gamma \vdash \Delta$ ok *and* $\Gamma \vdash \Delta'$ ok.

Lemma 7 (Substitution Shrinking).
If $\gamma \in [\![\Gamma_0, \Gamma_1]\!]$ *then there are* γ_0, γ_1 *such that* $\gamma = \gamma_0, \gamma_1$ *and* $\gamma_0 \in [\![\Gamma_0]\!]$.

Lemma 8 (Free Variables of Linear Contexts).
If $\Gamma \vdash \Delta$ ok *then* $\mathrm{FV}(\Delta) \subseteq \mathrm{dom}(\Gamma)$.

Lemma 9 (Linear Heap Preservation).
If $\langle \sigma; e \rangle \Downarrow \langle \sigma'; u \rangle$ *then* $\sigma = \sigma'$.

Lemma 10 (Linear Evaluation Frame Property).
If $\langle \sigma; e \rangle \Downarrow \langle \sigma'; u \rangle$ *and* $\sigma_f \# \sigma$ *then* $\sigma' \# \sigma_f$ *and* $\langle \sigma \cdot \sigma_f; e \rangle \Downarrow \langle \sigma' \cdot \sigma_f; u \rangle$.

$$\boxed{\Gamma\ \mathsf{ok}} \qquad \boxed{\Gamma \vdash \Delta\ \mathsf{ok}}$$

$$\frac{}{\cdot\ \mathsf{ok}}\ \textsc{CtxNil} \qquad \frac{\Gamma\ \mathsf{ok} \qquad \Gamma \vdash X\ \mathsf{type}}{\Gamma, x : X\ \mathsf{ok}}\ \textsc{CtxCons}$$

$$\frac{}{\Gamma \vdash \cdot\ \mathsf{ok}}\ \textsc{LCtxNil} \qquad \frac{\Gamma \vdash \Delta\ \mathsf{ok} \qquad \Gamma \vdash A\ \mathsf{linear}}{\Gamma \vdash \Delta, a : A\ \mathsf{ok}}\ \textsc{LCtxCons}$$

$$\boxed{\Gamma \vdash X\ \mathsf{type}} \qquad \boxed{\Gamma \vdash A\ \mathsf{linear}}$$

$$\frac{\Gamma \vdash X : \mathsf{U}_i}{\Gamma \vdash X\ \mathsf{type}}\ \textsc{Tp} \qquad \frac{\Gamma \vdash A : \mathsf{L}_i}{\Gamma \vdash A\ \mathsf{linear}}\ \textsc{LTp}$$

$$\boxed{\Gamma \vdash X \equiv Y\ \mathsf{type}} \qquad \boxed{\Gamma \vdash A \equiv B\ \mathsf{linear}}$$

$$\frac{\Gamma \vdash X \equiv Y : \mathsf{U}_i}{\Gamma \vdash X \equiv Y\ \mathsf{type}}\ \textsc{TpEq} \qquad \frac{\Gamma \vdash A \equiv B : \mathsf{L}_i}{\Gamma \vdash A \equiv B\ \mathsf{linear}}\ \textsc{LTpEq}$$

Figure 7: Structural judgements

$$\frac{}{\Gamma, x : X, \Gamma' \vdash x : X}\ \textsc{IHyp} \qquad \frac{\Gamma \vdash e : Y \qquad \Gamma \vdash X \equiv Y\ \mathsf{type}}{\Gamma \vdash e : X}\ \textsc{ITpEq} \qquad \frac{}{\Gamma \vdash () : 1}\ \textsc{IUnitI}$$

$$\frac{\Gamma \vdash e : X \qquad \Gamma \vdash e' : [e/x]Y}{\Gamma \vdash (e, e') : \Sigma x : X.\ Y}\ \textsc{IPairI} \qquad \frac{\Gamma \vdash e : \Sigma x : X.\ Y}{\Gamma \vdash \pi_1\ e : X}\ \textsc{IPairE1} \qquad \frac{\Gamma \vdash e : \Sigma x : X.\ Y}{\Gamma \vdash \pi_2\ e : [\pi_1\ e/x]Y}\ \textsc{IPairE2}$$

$$\frac{\Gamma \vdash \Pi x : X.\ Y\ \mathsf{type} \qquad \Gamma, x : X \vdash e : Y}{\Gamma \vdash \lambda x.\ e : \Pi x : X.\ Y}\ \textsc{IFunI} \qquad \frac{\Gamma \vdash e : \Pi x : X.\ Y \qquad \Gamma \vdash e' : X}{\Gamma \vdash e\ e' : [e'/x]Y}\ \textsc{IFunE}$$

$$\frac{\Gamma \vdash e \equiv e' : X}{\Gamma \vdash \mathsf{refl} : e =_X e'}\ \textsc{IEqI} \qquad \frac{\Gamma; \cdot \vdash e : A}{\Gamma \vdash \mathsf{G}\,e : \mathsf{G}\,A}\ \textsc{IGI}$$

$$\frac{}{\Gamma \vdash 0 : \mathbb{N}}\ \textsc{INIzero} \qquad \frac{\Gamma \vdash e : \mathbb{N}}{\Gamma \vdash \mathsf{s}(e) : \mathbb{N}}\ \textsc{INIsucc}$$

$$\frac{\Gamma \vdash C : \mathbb{N} \to \mathsf{U} \qquad \Gamma \vdash e : \mathbb{N} \qquad \Gamma \vdash e_0 : C\ 0 \qquad \Gamma, x, y : C\ x \vdash e_1 : C(\mathsf{s}(x))}{\Gamma \vdash \mathsf{iter}(e, 0 \to e_0, \mathsf{s}(x), y \to e_1) : C\ e}\ \textsc{INE}$$

Figure 8: Intuitionistic Typing

$$\frac{}{\Gamma \vdash \mathsf{U}_i : \mathsf{U}_{i+1}}\ \textsc{IU} \qquad \frac{}{\Gamma \vdash \mathsf{L}_i : \mathsf{U}_{i+1}}\ \textsc{IL}$$

$$\frac{\Gamma \vdash X : \mathsf{U}_i \qquad \Gamma, x : X \vdash Y : \mathsf{U}_i}{\Gamma \vdash \Pi x : X.\ Y : \mathsf{U}_i}\ \textsc{IPi} \qquad \frac{\Gamma \vdash X : \mathsf{U}_i \qquad \Gamma, x : X \vdash Y : \mathsf{U}_i}{\Gamma \vdash \Sigma x : X.\ Y : \mathsf{U}_i}\ \textsc{ISigma} \qquad \frac{}{\Gamma \vdash 1 : \mathsf{U}_i}\ \textsc{IUnit}$$

$$\frac{}{\Gamma \vdash \mathbb{N} : \mathsf{U}_i}\ \textsc{INat} \qquad \frac{\Gamma \vdash A : \mathsf{L}_i}{\Gamma \vdash \mathsf{G}\,A : \mathsf{U}_i}\ \textsc{IG}$$

$$\frac{\Gamma \vdash X : \mathsf{U}_i \qquad \Gamma \vdash e : X \qquad \Gamma \vdash e' : X}{\Gamma \vdash e =_X e' : \mathsf{U}_i}\ \textsc{IEq}$$

$$\frac{}{\Gamma \vdash \mathsf{I} : \mathsf{L}_i}\ \textsc{IOne} \qquad \frac{\Gamma \vdash A : \mathsf{L}_i \qquad \Gamma \vdash B : \mathsf{L}_i}{\Gamma \vdash A \otimes B : \mathsf{L}_i}\ \textsc{ITensor} \qquad \frac{\Gamma \vdash A : \mathsf{L}_i \qquad \Gamma \vdash B : \mathsf{L}_i}{\Gamma \vdash A \multimap B : \mathsf{L}_i}\ \textsc{ILolli}$$

$$\frac{\Gamma \vdash X : \mathsf{U}_i \qquad \Gamma, x : X \vdash A : \mathsf{L}_i}{\Gamma \vdash \Pi x : X.\ A : \mathsf{L}_i}\ \textsc{ILPi} \qquad \frac{\Gamma \vdash X : \mathsf{U}_i \qquad \Gamma, x : X \vdash A : \mathsf{L}_i}{\Gamma \vdash \mathsf{F} x : X.\ A : \mathsf{L}_i}\ \textsc{IF} \qquad \frac{}{\Gamma \vdash \top : \mathsf{L}_i}\ \textsc{ITop}$$

$$\frac{\Gamma \vdash A : \mathsf{L}_i \qquad \Gamma \vdash B : \mathsf{L}_i}{\Gamma \vdash A \,\&\, B : \mathsf{L}_i}\ \textsc{IWith}$$

Figure 9: Type Well-formedness

$$\frac{}{\Gamma \vdash \mathsf{Loc} : \mathsf{U}_i}\ \textsc{ILoc} \qquad \frac{\Gamma \vdash e : \mathsf{Loc} \qquad \Gamma \vdash X : \mathsf{U}_i}{\Gamma \vdash e \mapsto X : \mathsf{L}_i}\ \textsc{IPtr} \qquad \frac{\Gamma \vdash A : \mathsf{L}_i}{\Gamma \vdash \mathsf{T}\,A : \mathsf{L}_i}\ \textsc{IT}$$

$$\frac{\Gamma \vdash A : \mathsf{L}_i}{\Gamma \vdash [A] : \mathsf{L}_i}\ \textsc{IIrr}$$

$$\frac{\Gamma \vdash X : \mathsf{U}_i \qquad \Gamma, x : X \vdash Y : \mathsf{U}_i}{\Gamma \vdash \forall x : X.\ Y : \mathsf{U}_i} \qquad \frac{\Gamma \vdash X : \mathsf{U}_i \qquad \Gamma, x : X \vdash Y : \mathsf{L}_i}{\Gamma \vdash \forall x : X.\ Y : \mathsf{L}_i}$$

$$\frac{\Gamma \vdash X : \mathsf{U}_i \qquad \Gamma, x : X \vdash Y : \mathsf{U}_i}{\Gamma \vdash \exists x : X.\ Y : \mathsf{U}_i} \qquad \frac{\Gamma \vdash X : \mathsf{U}_i \qquad \Gamma, x : X \vdash Y : \mathsf{L}_i}{\Gamma \vdash \exists x : X.\ Y : \mathsf{L}_i}$$

$$\frac{}{\Gamma \vdash \top_I : \mathsf{U}_i}\ \textsc{IAny}$$

Figure 10: Well-formedness of extensions

$$\frac{}{\Gamma; a : A \vdash a : A}\ \textsc{LHyp} \qquad \frac{\Gamma; \Delta \vdash e : B \qquad \Gamma \vdash A \equiv B\ \mathsf{linear}}{\Gamma; \Delta \vdash e : A}\ \textsc{LEq}$$

$$\frac{}{\Gamma; \cdot \vdash () : \mathsf{I}}\ \textsc{LOneI} \qquad \frac{\Gamma; \Delta \vdash e : \mathsf{I} \qquad \Gamma; \Delta' \vdash e' : C}{\Gamma; \Delta, \Delta' \vdash \mathsf{let}\ () = e\ \mathsf{in}\ e' : C}\ \textsc{LOneE}$$

$$\frac{\Gamma; \Delta \vdash e : A \qquad \Gamma; \Delta' \vdash e' : B}{\Gamma; \Delta, \Delta' \vdash (e, e') : A \otimes B}\ \textsc{LTensorI} \qquad \frac{\Gamma; \Delta \vdash e : A \otimes B \qquad \Gamma; \Delta', a : A, b : B \vdash e' : C}{\Gamma; \Delta, \Delta' \vdash \mathsf{let}\ (a, b) = e\ \mathsf{in}\ e' : C}\ \textsc{LTensorE}$$

$$\frac{\Gamma; \Delta, a : A \vdash e : B}{\Gamma; \Delta \vdash \lambda a.\ e : A \multimap B}\ \textsc{LFunI} \qquad \frac{\Gamma; \Delta \vdash e : A \multimap B \qquad \Gamma; \Delta' \vdash e' : A}{\Gamma; \Delta, \Delta' \vdash e\ e' : B}\ \textsc{LFunE}$$

$$\frac{\Gamma, x : X; \Delta \vdash e : A}{\Gamma; \Delta \vdash \hat{\lambda} x.\ e : \Pi x : X.\ A}\ \textsc{LPiI} \qquad \frac{\Gamma; \Delta \vdash e : \Pi x : X.\ A \qquad \Gamma \vdash e' : X}{\Gamma; \Delta \vdash e\ e' : [e'/x]A}\ \textsc{LPiE}$$

$$\frac{}{\Gamma; \Delta \vdash () : \top}\ \textsc{LTopI}$$

$$\frac{\Gamma; \Delta \vdash e_1 : A_1 \qquad \Gamma; \Delta \vdash e_2 : A_2}{\Gamma; \Delta \vdash (e_1, e_2) : A_1 \,\&\, A_2}\ \textsc{LWithI} \qquad \frac{\Gamma; \Delta \vdash e : A \,\&\, B}{\Gamma; \Delta \vdash \pi_1\ e : A}\ \textsc{LWithEFst} \qquad \frac{\Gamma; \Delta \vdash e : A \,\&\, B}{\Gamma; \Delta \vdash \pi_2\ e : B}\ \textsc{LWithESndI}$$

$$\frac{\Gamma \vdash e : X \qquad \Gamma; \Delta \vdash t : [e/x]A}{\Gamma; \Delta \vdash \mathsf{F}\,(e, t) : \mathsf{F}x : X.\ A}\ \textsc{LFI} \qquad \frac{\Gamma \vdash C\ \mathsf{linear} \qquad \Gamma; \Delta \vdash e : \mathsf{F}x : X.\ A \qquad \Gamma, x : X; \Delta', a : A \vdash e' : C}{\Gamma; \Delta, \Delta' \vdash \mathsf{let}\ \mathsf{F}\,(x, a) = e\ \mathsf{in}\ e' : C}\ \textsc{LFE}$$

$$\frac{\Gamma \vdash e : \mathsf{G}\,A}{\Gamma; \cdot \vdash \mathsf{G}^{-1}\,e : A}\ \textsc{LGE}$$

Figure 11: Linear Typing

$$\frac{\Gamma, x : X \vdash e : Y \qquad x \notin FV(e)}{\Gamma \vdash e : \forall x : X.\ Y} \qquad \frac{\Gamma \vdash e : \forall x : X.\ Y \qquad \Gamma \vdash e' : X}{\Gamma \vdash e : [e'/x]Y}$$

$$\frac{\Gamma, x : X, y : Y \vdash e : Z \qquad x \notin FV(e)}{\Gamma, y : \exists x : X.\ Y \vdash e : Z} \qquad \frac{\Gamma, x : X \vdash Y\ \mathsf{type} \qquad \Gamma \vdash e' : X \qquad \Gamma \vdash e : [e'/x]Y}{\Gamma \vdash e : \exists x : X.\ Y}$$

$$\frac{\Gamma, x : X; \Delta \vdash e : A \qquad x \notin FV(e)}{\Gamma; \Delta \vdash e : \forall x : X.\ A} \qquad \frac{\Gamma; \Delta \vdash e : \forall x : X.\ A \qquad \Gamma \vdash e' : X}{\Gamma; \Delta \vdash e : [e'/x]A}$$

$$\frac{\Gamma, x : X; \Delta, a : A \vdash e : C \qquad x \notin FV(e)}{\Gamma; \Delta, a : \exists x : X.\ A \vdash e : C} \qquad \frac{\Gamma, x : X \vdash Y\ \mathsf{linear} \qquad \Gamma \vdash e' : X \qquad \Gamma; \Delta \vdash e : [e'/x]Y}{\Gamma; \Delta \vdash e : \exists x : X.\ Y}$$

$$\frac{\begin{array}{c}\Gamma, n : \mathbb{N} \vdash \Pi x : X[n].\ Y[n]\ \mathsf{type} \\ \Gamma, f : \top_I, x : X(0) \vdash e : Y(0) \qquad \Gamma, n : \mathbb{N}, f : \Pi x : X[n].\ Y[n], x : X[\mathsf{s}(n)] \vdash e : Y[\mathsf{s}(n)] \qquad n \notin FV(\mathsf{fix}\ f\ x = e)\end{array}}{\Gamma \vdash \mathsf{fix}\ f\ x = e : \forall n : \mathbb{N}.\ \Pi x : X[n].\ Y[n]}$$

Figure 12: Intersection and Union Types

$$\frac{\Gamma \vdash e : \mathsf{Loc} \qquad \Gamma; \Delta \vdash t : [e \mapsto X] \qquad \Gamma, x : X; \Delta', a : [e \mapsto X] \vdash t' : C}{\Gamma; \Delta, \Delta' \vdash \mathsf{let}\ (x, a) = \mathsf{get}(e, t)\ \mathsf{in}\ t' : C}\ \textsc{LGet}$$

$$\frac{\Gamma; \Delta \vdash e : A}{\Gamma; \Delta \vdash \mathsf{val}\, e : \mathsf{T}\, A}\ \textsc{LTI} \qquad \frac{\Gamma; \Delta \vdash e : \mathsf{T}\, A \qquad \Gamma; \Delta', a : A \vdash e' : \mathsf{T}\, C}{\Gamma; \Delta, \Delta' \vdash \mathsf{let\ val}\ a = e\ \mathsf{in}\ e' : \mathsf{T}\, C}\ \textsc{LTLet}$$

$$\frac{\Gamma \vdash e : X}{\Gamma; \cdot \vdash \mathsf{new}_X\, e : \mathsf{T}\,(\mathsf{F}x : \mathsf{Loc}.\ [x \mapsto X])}\ \textsc{LNew} \qquad \frac{\Gamma \vdash e : \mathsf{Loc} \qquad \Gamma; \Delta \vdash t : [e \mapsto X]}{\Gamma; \Delta \vdash \mathsf{free}\,(e, t) : \mathsf{T}\, \mathsf{I}}\ \textsc{LFree}$$

$$\frac{\Gamma \vdash e : \mathsf{Loc} \qquad \Gamma; \Delta \vdash t : [e \mapsto X] \qquad \Gamma \vdash e' : Y}{\Gamma; \Delta \vdash e :=_t e' : \mathsf{T}\,[e \mapsto Y]}\ \textsc{LSet}$$

$$\frac{\Gamma; \Delta \vdash e \div A}{\Gamma; \Delta \vdash * : [A]}\ \textsc{LIrr} \qquad \frac{\Gamma; \Delta \vdash e : A}{\Gamma; \Delta \vdash e \div A} \qquad \frac{\Gamma; \Delta \vdash e : [A] \qquad \Gamma; \Delta', x : A \vdash e' \div C}{\Gamma; \Delta, \Delta' \vdash \mathsf{let}\ [x] = e\ \mathsf{in}\ e' \div C}$$

$$\mathrm{o}\ \frac{\Gamma; \Delta \vdash e : [\mathsf{I}] \qquad \Gamma; \Delta' \vdash e' : C}{\Gamma; \Delta, \Delta' \vdash \mathsf{let}\ [] = e\ \mathsf{in}\ e' : C}\ \textsc{LIrrUnit} \qquad \frac{\Gamma; \Delta \vdash e : [A \otimes B] \qquad \Gamma; \Delta', a : [A], b : [B] \vdash e' : C}{\Gamma; \Delta, \Delta' \vdash \mathsf{let}\ [a, b] = e\ \mathsf{in}\ e' : C}\ \textsc{LIrrPair}$$

Figure 13: Typing of Imperative Programs

$$\boxed{\Gamma \vdash e \equiv e' : X} \qquad \boxed{\Gamma; \Delta \vdash t \equiv t' : A}$$

$$\frac{}{\Gamma \vdash (\lambda x.\ e)\ e' \equiv [e'/x]e : Z}\ \textsc{IFunBeta} \qquad \frac{}{\Gamma \vdash e \equiv \lambda x.\ e\ x : \Pi x : X.\ Y}\ \textsc{IFunEta}$$

$$\frac{}{\Gamma \vdash \pi_1\,(e, e') \equiv e : Z}\ \textsc{IPairBetaFst} \qquad \frac{}{\Gamma \vdash \pi_2\,(e, e') \equiv e' : Z}\ \textsc{IPairBetaSnd}$$

$$\frac{}{\Gamma \vdash e \equiv (\pi_1\,e, \pi_2\,e) : \Sigma x : X.\ Y}\ \textsc{IPairEta}$$

$$\frac{}{\Gamma \vdash e \equiv e' : 1}\ \textsc{IUnitEta}$$

$$\frac{}{\Gamma \vdash \mathsf{G}\,(\mathsf{G}^{-1}\,e) \equiv e : \mathsf{G}\,A}\ \textsc{IGEta} \qquad \frac{}{\Gamma; \cdot \vdash \mathsf{G}^{-1}\,(\mathsf{G}\,t) \equiv t : A}\ \textsc{IGBeta}$$

$$\frac{}{\Gamma \vdash \mathsf{iter}(0, 0 \to e_0, \mathsf{s}(x), y \to e_1) \equiv e_0 : X}\ \textsc{INZBeta}$$

$$\frac{}{\Gamma \vdash \mathsf{iter}(\mathsf{s}(e), 0 \to e_0, \mathsf{s}(x), y \to e_1) \equiv [e/x, \mathsf{iter}(e, 0 \to e_0, \mathsf{s}(x), y \to e_1)/y]e_1 : X}\ \textsc{INSBeta}$$

$$\frac{}{\Gamma; \Delta \vdash (\lambda x.\ e)\ e' \equiv [e'/x]e : C}\ \textsc{LFunBeta} \qquad \frac{}{\Gamma; \Delta \vdash e \equiv \lambda x.\ e\ x : A \multimap B}\ \textsc{LFunEta}$$

$$\frac{}{\Gamma; \Delta \vdash (\hat{\lambda} x.\ e)\ e' \equiv [e'/x]e : C}\ \textsc{LPiBeta} \qquad \frac{}{\Gamma; \Delta \vdash e \equiv \hat{\lambda} x.\ e\ x : \Pi x : X.\ A}\ \textsc{LPiEta}$$

$$\frac{}{\Gamma; \Delta \vdash e \equiv e' : \top}\ \textsc{LTopEta}$$

$$\frac{}{\Gamma; \Delta \vdash \pi_1\,(e, e') \equiv e : A}\ \textsc{LWithBetaFst} \qquad \frac{}{\Gamma; \Delta \vdash \pi_1\,(e, e') \equiv e' : B}\ \textsc{LWithBetaSnd}$$

$$\frac{}{\Gamma; \Delta \vdash e \equiv (\pi_1\,e, \pi_2\,e) : A \,\&\, B}\ \textsc{LWithEta}$$

$$\frac{}{\Gamma; \Delta \vdash \mathsf{let}\ () = ()\ \mathsf{in}\ e \equiv e : C}\ \textsc{LOneBeta} \qquad \frac{}{\Gamma; \Delta \vdash \mathsf{let}\ () = t\ \mathsf{in}\ [()\,/x]t' \equiv [t/x]t' : C}\ \textsc{LOneEta}$$

$$\frac{}{\Gamma; \Delta \vdash \mathsf{let}\ (a, b) = (t_1, t_2)\ \mathsf{in}\ t' \equiv [t_1/a, t_2/b]t' : C}\ \textsc{LTensorBeta}$$

$$\frac{}{\Gamma; \Delta \vdash \mathsf{let}\ (a, b) = t\ \mathsf{in}\ [(a, b)\,/x]t' \equiv [t/x]t' : C}\ \textsc{LTensorEta}$$

$$\frac{}{\Gamma; \Delta \vdash \mathsf{let}\ \mathsf{F}\,(x, a) = \mathsf{F}\,(e, t)\ \mathsf{in}\ t' \equiv [e/x, t/a]t' : C}\ \textsc{LFBeta}$$

$$\frac{}{\Gamma; \Delta \vdash \mathsf{let}\ \mathsf{F}\,(x, a) = t\ \mathsf{in}\ [\mathsf{F}\,(x, a)\,/y]t' \equiv [t/y]t' : C}\ \textsc{LFEta}$$

Figure 14: βη-Equality

$$\frac{\Gamma, x : X \vdash e \equiv e' : Y}{\Gamma \vdash e \equiv e' : \forall x : X.\ Y}\ \textsc{IAllEta} \qquad \frac{\Gamma \vdash e \equiv e' : \forall x : X.\ Y \qquad \Gamma \vdash t : X}{\Gamma \vdash e \equiv e' : [t/x]Y}\ \textsc{IAllBeta}$$

$$\frac{\Gamma \vdash e \equiv e' : [t/x]Y \qquad \Gamma \vdash t : X}{\Gamma \vdash e \equiv e' : \exists x : X.\ Y}\ \textsc{IExBeta} \qquad \frac{\Gamma, x : X, y : Y \vdash e \equiv e' : Z \qquad x \notin FV(e, e', Z)}{\Gamma, y : \exists x : X.\ Y \vdash e \equiv e' : Z}\ \textsc{IExEta}$$

$$\frac{}{\Gamma; \Delta \vdash \mathsf{let\ val}\, x = \mathsf{val}\, t\ \mathsf{in}\ t' \equiv [t/x]t' : \mathsf{T}\, C}\ \textsc{LTBeta} \qquad \frac{}{\Gamma; \Delta \vdash \mathsf{let\ val}\, x = t\ \mathsf{in\ val}\, x \equiv t : \mathsf{T}\, C}\ \textsc{LTEta}$$

$$\frac{x \notin FV(t_2)}{\Gamma; \Delta \vdash \mathsf{let\ val}\, y = (\mathsf{let\ val}\, x = t_1\ \mathsf{in}\ t_2)\ \mathsf{in}\ t_3 \equiv \mathsf{let\ val}\, x = t_1\ \mathsf{in\ let\ val}\, y = t_2\ \mathsf{in}\ t_3 : \mathsf{T}\, C}\ \textsc{LTAssoc}$$

$$\frac{}{\Gamma; \Delta \vdash e \equiv e' : [A]}\ \textsc{LIrrEq}$$

$$\frac{\Gamma, x : X; \Delta \vdash e \equiv e' : A}{\Gamma; \Delta \vdash e \equiv e' : \forall x : X.\ Y}\ \textsc{LAllEta} \qquad \frac{\Gamma; \Delta \vdash e \equiv e' : \forall x : X.\ A \qquad \Gamma \vdash t : X}{\Gamma; \Delta \vdash e \equiv e' : [t/x]A}\ \textsc{LAllBeta}$$

$$\frac{\Gamma; \Delta \vdash e \equiv e' : [t/x]A \qquad \Gamma \vdash t : X}{\Gamma; \Delta \vdash e \equiv e' : \exists x : X.\ A}\ \textsc{LExBeta} \qquad \frac{\Gamma, x : X; \Delta, a : A \vdash e \equiv e' : C \qquad x \notin FV(e, e', C)}{\Gamma \vdash \Delta, a : \exists x : X.\ A \equiv e : e'C}\ \textsc{LExEta}$$

$$\frac{}{\Gamma \vdash (\mathsf{fix}\ f\ x = e)\ e' \equiv [(\mathsf{fix}\ f\ x = e)/f, e'/x]e : Z}\ \textsc{IFixBeta}$$

Figure 15: Imperative Equality

$$\frac{\Gamma \vdash p : e =_X e'}{\Gamma \vdash e \equiv e' : X}\ \textsc{IReflect} \qquad \frac{\Gamma \vdash p : e =_X e \qquad \Gamma \vdash q : e =_X e}{\Gamma \vdash p \equiv q : e =_X e}\ \textsc{K}$$

$$\frac{\Gamma \vdash e : X}{\Gamma \vdash e \equiv e : X}\ \textsc{IReflex} \qquad \frac{\Gamma; \Delta \vdash t : A}{\Gamma; \Delta \vdash t \equiv t : A}\ \textsc{LReflex}$$

$$\frac{\Gamma \vdash e \equiv e' : X \qquad \Gamma \vdash e' \equiv e'' : X}{\Gamma \vdash e \equiv e'' : X}\ \textsc{ITrans} \qquad \frac{\Gamma; \Delta \vdash t \equiv t' : A \qquad \Gamma; \Delta \vdash t' \equiv t'' : A}{\Gamma; \Delta \vdash t \equiv t'' : A}\ \textsc{LTrans}$$

$$\frac{\Gamma \vdash A \equiv A' : \mathsf{L}_i \qquad \Gamma \vdash B \equiv B' : \mathsf{L}_i}{\Gamma \vdash A \multimap B \equiv A' \multimap B' : \mathsf{L}_i}\ \textsc{ILolliCong} \qquad \frac{\Gamma \vdash A \equiv A' : \mathsf{L}_i \qquad \Gamma \vdash B \equiv B' : \mathsf{L}_i}{\Gamma \vdash A \otimes B \equiv A' \otimes B' : \mathsf{L}_i}\ \textsc{ITensorCong}$$

$$\frac{\Gamma \vdash A \equiv A' : \mathsf{L}_i \qquad \Gamma \vdash B \equiv B' : \mathsf{L}_i}{\Gamma \vdash A \,\&\, B \equiv A' \,\&\, B' : \mathsf{L}_i}\ \textsc{IWithCong}$$

$$\frac{\Gamma \vdash A \equiv A' : \mathsf{L}_i}{\Gamma \vdash \mathsf{T}\,A \equiv \mathsf{T}\,A' : \mathsf{L}_i}\ \textsc{ITCong}$$

$$\frac{\Gamma \vdash X \equiv X' : \mathsf{U}_i \qquad \Gamma, x : X \vdash Y \equiv Y' : \mathsf{U}_i}{\Gamma \vdash \Pi x : X.\ Y \equiv \Pi x : X'.\ Y' : \mathsf{U}_i}\ \textsc{IPiCong} \qquad \frac{\Gamma \vdash X \equiv X' : \mathsf{U}_i \qquad \Gamma, x : X \vdash Y \equiv Y' : \mathsf{U}_i}{\Gamma \vdash \Sigma x : X.\ Y \equiv \Sigma x : X'.\ Y' : \mathsf{U}_i}\ \textsc{ISigmaCong}$$

$$\frac{\Gamma \vdash X \equiv X' : \mathsf{U}_i \qquad \Gamma, x : X \vdash Y \equiv Y' : \mathsf{U}_i}{\Gamma \vdash \forall x : X.\ Y \equiv \forall x : X'.\ Y' : \mathsf{U}_i}\ \textsc{IAllCong} \qquad \frac{\Gamma \vdash X \equiv X' : \mathsf{U}_i \qquad \Gamma, x : X \vdash Y \equiv Y' : \mathsf{U}_i}{\Gamma \vdash \exists x : X.\ Y \equiv \exists x : X'.\ Y' : \mathsf{U}_i}\ \textsc{IExCong}$$

$$\frac{\Gamma \vdash X \equiv X' : \mathsf{U}_i \qquad \Gamma, x : X \vdash A \equiv A' : \mathsf{L}_i}{\Gamma \vdash \forall x : X.\ A \equiv \forall x : X'.\ A' : \mathsf{L}_i}\ \textsc{LAllCong} \qquad \frac{\Gamma \vdash X \equiv X' : \mathsf{U}_i \qquad \Gamma, x : X \vdash A \equiv A' : \mathsf{L}_i}{\Gamma \vdash \exists x : X.\ A \equiv \exists x : A'.\ Y' : \mathsf{L}_i}\ \textsc{LExCong}$$

$$\frac{\Gamma \vdash X \equiv X' : \mathsf{U}_i \qquad \Gamma, x : X \vdash A \equiv A' : \mathsf{L}_i}{\Gamma \vdash \mathsf{F} x : X.\ A \equiv \mathsf{F} x : X'.\ A' : \mathsf{U}_i}\ \textsc{IFCong} \qquad \frac{\Gamma \vdash X \equiv X' : \mathsf{U}_i \qquad \Gamma, x : X \vdash A \equiv A' : \mathsf{L}_i}{\Gamma \vdash \Pi x : X.\ A \equiv \Pi x : X'.\ A' : \mathsf{U}_i}\ \textsc{ILPiCong}$$

$$\frac{\Gamma \vdash e \equiv e' : \mathsf{Loc} \qquad \Gamma \vdash X \equiv X' : \mathsf{U}_i}{\Gamma \vdash e \mapsto X \equiv e' \mapsto X' : \mathsf{U}_i}\ \textsc{IPtrCong}$$

$$\frac{\Gamma \vdash X \equiv X' : \mathsf{U}_i \qquad \Gamma \vdash e_1 \equiv e_2 : X \qquad \Gamma \vdash e_1' \equiv e_2' : X'}{\Gamma \vdash e_1 =_X e_2 \equiv e_1' =_{X'} e_2' : \mathsf{U}_i}\ \textsc{IEqCong}$$

Figure 16: Congruence rules, part 1

$$\frac{\Gamma, x : X \vdash e \equiv e' : Y}{\Gamma \vdash \lambda x : X.\ e \equiv \lambda x : X.\ e' : \Pi x : X.\ Y}\ \textsc{IFunCong} \qquad \frac{\Gamma \vdash e_1 \equiv e_1' : \Pi x : X.\ Y \qquad \Gamma \vdash e_2 \equiv e_2' : X}{\Gamma \vdash e_1\ e_2 \equiv e_1'\ e_2' : Y[e_2/x]}\ \textsc{IAppCong}$$

$$\frac{\Gamma \vdash e_1 \equiv e_1' : X \qquad \Gamma \vdash e_2 \equiv e_2' : Y[e_1/x]}{\Gamma \vdash (e_1, e_2) \equiv (e_1', e_2') : \Sigma x : X.\ Y}\ \textsc{IPairCong}$$

$$\frac{\Gamma \vdash e \equiv e' : \Sigma x : X.\ Y}{\Gamma \vdash \pi_1\ e \equiv \pi_1\ e' : X}\ \textsc{IFstCong} \qquad \frac{\Gamma \vdash e \equiv e' : \Sigma x : X.\ Y}{\Gamma \vdash \pi_2\ e \equiv \pi_2\ e' : Y[\pi_1\ e/x]}\ \textsc{ISndCong}$$

$$\frac{\Gamma, x : X \vdash e \equiv e' : Y \qquad x \notin FV(e, e')}{\Gamma \vdash e \equiv e' : \forall x : X.\ Y}\ \textsc{IAllCongTm}$$

$$\frac{\Gamma \vdash e \equiv e' : [e''/x]Y \qquad \Gamma \vdash e'' : X \qquad \Gamma, x : X \vdash Y\ \mathsf{type}}{\Gamma \vdash e \equiv e' : \exists x : X.\ Y}\ \textsc{IExCongTm}$$

$$\frac{\Gamma \vdash e \equiv e' : \mathbb{N}}{\Gamma \vdash \mathsf{s}(e) \equiv \mathsf{s}(e') : \mathbb{N}}\ \textsc{INSCong}$$

$$\frac{\Gamma \vdash e \equiv e' : \mathbb{N} \qquad \Gamma \vdash e_0 \equiv e_0' : C\ z \qquad \Gamma, x : \mathbb{N}, y : C\ x \vdash e_1 \equiv e_1' : C(\mathsf{s}(x)) \qquad \Gamma \vdash C\ \mathsf{type}\mathbb{N} \to \mathsf{U}_i}{\Gamma \vdash \mathsf{iter}(e, 0 \to e_0, \mathsf{s}(x), y \to e_1) \equiv \mathsf{iter}(e', 0 \to e_0', \mathsf{s}(x), y \to e_1') : C\ e}\ \textsc{INIterCong}$$

$$\frac{\Gamma; \Delta' \vdash t_1 \equiv t_1' : \mathsf{I} \qquad \Gamma; \Delta \vdash t_2 \equiv t_2' : B}{\Gamma; \Delta, \Delta' \vdash \mathsf{let}\ () = t_1\ \mathsf{in}\ t_2 \equiv \mathsf{let}\ () = t_1'\ \mathsf{in}\ t_2' : B}\ \textsc{LUnitECong}$$

$$\frac{\Gamma; \Delta \vdash t_1 \equiv t_1' : A \qquad \Gamma; \Delta' \vdash t_2 \equiv t_2' : A}{\Gamma; \Delta, \Delta' \vdash (t_1, t_2) \equiv (t_1', t_2') : A \otimes B}\ \textsc{LTensorCong}$$

$$\frac{\Gamma; \Delta \vdash t_1 \equiv t_1' : A \otimes B \qquad \Gamma; \Delta', x : A, y : B \vdash t_2 \equiv t_2' : C}{\Gamma; \Delta, \Delta' \vdash \mathsf{let}\ (x, y) = t_1\ \mathsf{in}\ t_2 \equiv \mathsf{let}\ (x, y) = t_1'\ \mathsf{in}\ t_2' : C}\ \textsc{LTensorECong}$$

$$\frac{\Gamma; \Delta, x : A \vdash t \equiv t' : B}{\Gamma; \Delta \vdash \lambda x : A.\ t \equiv \lambda x : A.\ t' : A \multimap B}\ \textsc{LFunCong} \qquad \frac{\Gamma; \Delta \vdash t_1 \equiv t_1' : A \multimap B \qquad \Gamma; \Delta' \vdash t_2 \equiv t_2' : A}{\Gamma; \Delta, \Delta' \vdash t_1\ t_2 \equiv t_1'\ t_2' : B}\ \textsc{LAppCong}$$

$$\frac{\Gamma; \Delta \vdash e_1 \equiv e_1' : A \qquad \Gamma; \Delta' \vdash e_2 \equiv e_2' : B}{\Gamma; \Delta, \Delta' \vdash (e_1, e_2) \equiv (e_1', e_2') : A \,\&\, B}\ \textsc{LPairCong} \qquad \frac{\Gamma; \Delta \vdash e \equiv e' : A \,\&\, B}{\Gamma; \Delta \vdash \pi_1\ e \equiv \pi_1\ e' : A}\ \textsc{LFstCong}$$

$$\frac{\Gamma; \Delta \vdash e \equiv e' : A \,\&\, B}{\Gamma; \Delta \vdash \pi_2\ e \equiv \pi_2\ e' : B}\ \textsc{LSndCong}$$

$$\frac{\Gamma, x : X \vdash \Delta \equiv e : e'A \qquad x \notin FV(e, e')}{\Gamma \vdash \Delta \equiv e : e'\forall x : X.\ A}\ \textsc{LAllCongTm}$$

$$\frac{\Gamma; \Delta \vdash e \equiv e' : [e''/x]A \qquad \Gamma \vdash e'' : X \qquad \Gamma, x : X \vdash Y\ \mathsf{type}}{\Gamma; \Delta \vdash e \equiv e' : \exists x : X.\ A}\ \textsc{LExCongTm}$$

$$\frac{\Gamma, x : X; \Delta \vdash e \equiv e' : A}{\Gamma; \Delta \vdash \hat{\lambda} x.\ e \equiv \hat{\lambda} x.\ e' : \Pi x : X.\ A}\ \textsc{LPiFunCong} \qquad \frac{\Gamma; \Delta \vdash e_1 \equiv e_1' : \Pi x : X.\ A \qquad \Gamma \vdash \Delta \equiv e_2 : e_2'X}{\Gamma; \Delta, \Delta' \vdash e_1\ e_2 \equiv e_1'\ e_2' : [e_1/x]A}\ \textsc{LPiAppCong}$$

$$\frac{\Gamma \vdash e \equiv e' : X \qquad \Gamma; \Delta \vdash t[e/x] \equiv t'[e/x] : A[e/x]}{\Gamma; \Delta \vdash \mathsf{F}\,(e, t) \equiv \mathsf{F}\,(e', t') : \mathsf{F} x : X.\ A}\ \textsc{LFICong}$$

$$\frac{\Gamma; \Delta \vdash t_1 \equiv t_1' : \mathsf{F} x : X.\ A \qquad \Gamma, x : X; \Delta', a : A \vdash t_2 \equiv t_2' : B}{\Gamma; \Delta, \Delta' \vdash \mathsf{let}\ \mathsf{F}\,(x, a) = t_1\ \mathsf{in}\ t_2 \equiv \mathsf{let}\ \mathsf{F}\,(x, a) = t_1'\ \mathsf{in}\ t_2' : B}\ \textsc{LFECong}$$

Figure 17: Congruence rules, part 2

$$\frac{\Gamma;\Delta \vdash e \equiv e' : A}{\Gamma;\Delta \vdash \mathsf{val}\, e \equiv \mathsf{val}\, e' : \mathsf{T}\, A}\ \textsc{LValCong} \qquad \frac{\Gamma;\Delta \vdash e_1 \equiv e_1' : \mathsf{T}\, A \qquad \Gamma;\Delta', a : A \vdash e_2 \equiv e_2' : \mathsf{T}\, C}{\Gamma;\Delta,\Delta' \vdash \mathsf{let\ val}\ a = e_1\ \mathsf{in}\ e_2 \equiv \mathsf{let\ val}\ a = e_1'\ \mathsf{in}\ e_2' : \mathsf{T}\, C}\ \textsc{LLetCong}$$

$$\frac{\Gamma \vdash e \equiv e' : X}{\Gamma;\cdot \vdash \mathsf{new}_X\, e \equiv \mathsf{new}_X\, e' : \mathsf{T}\,(\mathsf{F}x : \mathsf{Loc}.\ x \mapsto e)}\ \textsc{LNewCong}$$

$$\frac{\Gamma \vdash e \equiv e' : \mathsf{Loc} \qquad \Gamma;\Delta \vdash t \equiv t' : e \mapsto e_0}{\Gamma;\Delta \vdash \mathsf{free}\,(e, t) \equiv \mathsf{free}\,(e', t') : \mathsf{T}\, \mathsf{I}}\ \textsc{LFreeCong}$$

$$\frac{\Gamma \vdash e \equiv e' : \mathsf{Loc} \qquad \Gamma;\Delta \vdash t_1 \equiv t_1' : e \mapsto X \qquad \Gamma, x : X;\Delta', a : e \mapsto X \vdash t_2 \equiv t_2' : C}{\Gamma;\Delta,\Delta' \vdash \mathsf{let}\ (x, a) = \mathsf{get}(e, t_1)\ \mathsf{in}\ t_2 \equiv \mathsf{let}\ (x, a) = \mathsf{get}(e', t_1')\ \mathsf{in}\ t_2' : C}\ \textsc{LGetCong}$$

$$\frac{\Gamma \vdash e_1 \equiv e_1' : \mathsf{Loc} \qquad \Gamma;\Delta \vdash t_1 \equiv t_1' : e \mapsto X \qquad \Gamma \vdash e_2 \equiv e_2' : Y}{\Gamma;\Delta \vdash e_1 :=_t e_2 \equiv e_1' :=_{t'} e_2' : \mathsf{T}\,(e \mapsto Y)}\ \textsc{LAssignCong}$$

Figure 18: Congruence rules, part 3

10 Proofs

Lemma 2 (T_k is a type system operator)**.** *We have that* T_k *is a monotone function on type systems.*

Proof. We proceed by strong induction on k:

1. First, we check that $T_0(I, L, \phi, \psi)$ is a type system.

   To check this, we check that ϕ' and ψ' respect equivalence on I' and L'. Since PERs are determined by their values, we need only consider the value cases of $(X, X') \in I'$ and $(A, A') \in L'$.

   - Case $(\mathsf{Loc}, \mathsf{Loc}) \in I'$:
     Immediate.
   - Case $(\mathbb{N}, \mathbb{N}) \in I'$:
     Immediate.
   - Case $(\top_I, \top_I) \in I'$:
     Immediate.
   - Case $(1, 1) \in I'$:
     Immediate.
   - Case $(\Pi x : X.\ Y, \Pi x : X'.\ Y') \in I'$:
     We know that $(X, X') \in I$.
     We know that $\forall (v, v') \in \phi(X).\ ([v/x]Y, [v'/x]Y') \in I$.

     Since ϕ respects I, we know that $\phi(X) = \phi(X')$.

     Assume that we have $(v, v) \in \phi(X) = \phi(X')$.
     Then we know that $([v/x]Y, [v/x]Y') \in I$.
     Since ϕ respects I, we know that $\phi([v/x]Y) = \phi([v/x]Y')$.
     Therefore $\lambda v \in \phi(X).\ \phi([v/x]Y) = \lambda v \in \phi(X').\ \phi([v/x]Y')$.

     Therefore $\Pi(\phi(X), \lambda v.\ \phi([v/x]Y)) = \Pi(\phi(X'), \lambda v.\ \phi[v/x]Y')$.
     Therefore $\phi'(\Pi x : X.\ Y) = \phi'(\Pi x : X'.\ Y')$.
     Therefore ϕ' respects I'.
   - Case $(\Sigma x : X.\ Y, \Sigma x : X'.\ Y') \in I'$:
     Similar to $\Pi x : X.\ Y$ case.
   - Case $(\forall x : X.\ Y, \forall x : X'.\ Y') \in I'$:
     Similar to $\Pi x : X.\ Y$ case.
   - Case $(\exists x : X.\ Y, \exists x : X'.\ Y') \in I'$:
     Similar to $\Pi x : X.\ Y$ case.
   - Case $(\mathsf{G}\,A, \mathsf{G}\,A') \in I'$:
     We know that $(A, A') \in L$.
     Since $(A, A') \in L$, we know that $\psi(A)$ and $\psi(A')$ are CPER's.
     Hence $G(\psi(A))$ and $G(\psi(A'))$ are PER's.

     We know that ψ respects L and that $(A, A') \in L$.
     Therefore $\psi(A) = \psi(A')$.
     Therefore $G(\psi(A)) = G(\psi(A'))$.
     By definition of ϕ', $\phi'(\mathsf{G}\,A) = \phi'(\mathsf{G}\,A')$.

     So ϕ' respects L'.

- Case $(e_1 =_X e_2, e'_1 =_{X'} e'_2) \in I'$:
  We know $(X, X') \in I$.
  We know $(e_1, e'_1) \in \phi(X)$.
  We know $(e_2, e'_2) \in \phi(X)$.

  We want to show $\phi'(e_1 =_X e_2) = \phi'(e'_1 =_{X'} e'_2)$.
  By definition, it suffices to show $\mathrm{Id}(e_1, e_2, \phi(X)) = \mathrm{Id}(e'_1, e'_2, \phi(X'))$.
  Since ϕ respects I, we know $\phi(X) = \phi(X')$.
  So it suffices to show $\mathrm{Id}(e_1, e_2, \phi(X)) = \mathrm{Id}(e'_1, e'_2, \phi(X))$.

  We know $\mathrm{Id}(e_1, e_2, \phi(X)) = \{(\mathsf{refl}, \mathsf{refl}) \mid (e_1, e_2) \in \phi(X)\}$.
  We know $\mathrm{Id}(e'_1, e'_2, \phi(X')) = \{(\mathsf{refl}, \mathsf{refl}) \mid (e'_1, e'_2) \in \phi(X')\}$.

  So we want to show that $(e_1, e_2) \in \phi(X)$ iff $(e'_1, e'_2) \in \phi(X)$.
  Assume $(e_1, e_2) \in \phi(X)$.
  We know $(e'_1, e_1) \in \phi(X)$.
  We know $(e_2, e'_2) \in \phi(X)$.
  By transitivity of $\phi(X)$, $(e'_1, e'_2) \in \phi(X)$.
  Therefore $(e_1, e_2) \in \phi(X)$ implies $(e'_1, e'_2) \in \phi(X)$.

  Similarly, $(e'_1, e'_2) \in \phi(X)$ implies $(e_1, e_2) \in \phi(X)$.

  Therefore $(e_1, e_2) \in \phi(X)$ iff $(e'_1, e'_2) \in \phi(X)$.
  Therefore $\mathrm{Id}(e_1, e_2, \phi(X)) = \mathrm{Id}(e'_1, e'_2, \phi(X))$.
  Therefore $\mathrm{Id}(e_1, e_2, \phi(X)) = \mathrm{Id}(e'_1, e'_2, \phi(X'))$.
  Therefore $\phi'(e_1 =_X e_2) = \phi'(e'_1 =_{X'} e'_2)$.
- Case $(\mathsf{U}_i, \mathsf{U}_i) \in I$ where $i < k$:
  Since $i < k$, by induction we can assume that T_i is a monotone function on type systems.
  Hence the fixed point $\mathsf{fix}(T_i)$ exists, and T_k is well-defined at this case.
  Then it is immediate that $\phi'(\mathsf{U}_i) = \phi'(\mathsf{U}_i)$.
- Case $(\mathsf{I}, \mathsf{I}) \in L'$:
  Similar to previous case.
- Case $(A \otimes B, A' \otimes B') \in L'$:
  We know that $(A, A') \in L$ and $(B, B') \in L'$.
  Since ψ respects L, $\psi(A) = \psi(A')$ and $\psi(B) = \psi(B')$.
  Therefore $\psi(A) \hat{\otimes} \psi(B) = \psi(A') \hat{\otimes} \psi(B')$.
  Therefore $\phi'(A \otimes B) = \phi'(A' \otimes B')$.

- Case $(A \multimap B, A' \multimap B') \in L'$:
  We know that $(A, A') \in L$ and $(B, B') \in L'$.
  Since ψ respects L, $\psi(A) = \psi(A')$ and $\psi(B) = \psi(B')$.
  Therefore $\psi(A) \hat{\multimap} \psi(B) = \psi(A') \hat{\multimap} \psi(B')$.
  Therefore $\phi'(A \multimap B) = \phi'(A' \multimap B')$.

- Case $(\mathsf{F}x : X.\ A, \mathsf{F}x : X'.\ A') \in L'$:
  We know that $(X, X') \in I$.
  We know that for all $(v, v') \in \phi(X)$, we have $([v/x]A, [v'/x]A') \in L$.

Assume $(v, v') \in \phi(X)$.
Then $([v/x]A, [v'/x]A') \in L$.
Since ψ respects L, we know $\psi([v/x]A) = \psi([v/x]A')$.
By extensionality, $\lambda v \in \phi(X).\ \psi([v/x]A) = \lambda v \in \phi(X').\ \psi([v/x]A')$.

Hence $F(\phi(X), \lambda v \in \phi(X).\ \psi([v/x]A)) = F(\phi(X'), \lambda v \in \phi(X').\ \psi([v/x]A'))$.
By definition, $\psi'(\mathsf{F}x : X.\ A) = \psi'(\mathsf{F}x : X'.\ A')$.

- Case $(\top, \top) \in L'$:
  Similar to (I, I) case.
- Case $(A \,\&\, B, A' \,\&\, B') \in L'$:
  We know that $(A, A') \in L$ and $(B, B') \in L'$.
  Since ψ respects L, $\psi(A) = \psi(A')$ and $\psi(B) = \psi(B')$.
  Therefore $\psi(A) \,\hat{\&}\, \psi(B) = \psi(A') \,\hat{\&}\, \psi(B')$.
  Therefore $\phi'(A \,\&\, B) = \phi'(A' \,\&\, B')$.

- Case $(\Pi x : X.\ A, \Pi x : X'.\ A') \in L'$:
  We know that $(X, X') \in I$.
  We know that for all $(v, v') \in \phi(X)$, we have $([v/x]A, [v'/x]A') \in L$.

  Assume $(v, v') \in \phi(X)$.
  Then $([v/x]A, [v'/x]A') \in L$.
  Since ψ respects L, we know $\psi([v/x]A) = \psi([v/x]A')$.
  By extensionality, $\lambda v \in \phi(X).\ \psi([v/x]A) = \lambda v \in \phi(X').\ \psi([v/x]A')$.

  Hence $\Pi_L(\phi(X), \lambda v \in \phi(X).\ \psi([v/x]A)) = \Pi_L(\phi(X'), \lambda v \in \phi(X').\ \psi([v/x]A'))$.
  By definition, $\psi'(\Pi x : X.\ A) = \psi'(\Pi x : X'.\ A')$.
- Case $(\forall x : X.\ A, \forall x : X'.\ A') \in L'$:
  Similar to $\Pi x : X.\ A$ case.
- Case $(\exists x : X.\ A, \exists x : X'.\ A') \in L'$:
  Similar to $\Pi x : X.\ A$ case.
- Case $(\mathsf{T}\,A, \mathsf{T}\,A') \in L'$:
  We know $(A, A') \in L$.
  Since ψ respects L, $\psi(A) = \psi(A')$.
  Therefore $\hat{\mathsf{T}}(\psi(A)) = \hat{\mathsf{T}}(\psi(A'))$.
  By definition, $\psi'(\mathsf{T}\,A) = \psi'(\mathsf{T}\,A')$.
- Case $(e \mapsto X, e' \mapsto X') \in L'$:
  We know that $(e, e') \in Loc$.
  We know that $(X, X') \in I$.

  We want to show that $\psi'(e \mapsto X) = \psi'(e' \mapsto X')$.
  This is equivalent to showing $Ptr(e, \phi(X)) = Ptr(e', \phi(X))$.
  It suffices to show that $(\langle \sigma; \bullet \rangle, \langle \sigma'; \bullet \rangle) \in Ptr(e, \phi(X))$ iff $(\langle \sigma; \bullet \rangle, \langle \sigma'; \bullet \rangle) \in Ptr(e', \phi(X'))$.

  $\Rightarrow$: Assume $(\langle \sigma; \bullet \rangle, \langle \sigma'; \bullet \rangle) \in Ptr(e, \phi(X))$.
  Therefore $\sigma = [l : v]$ and $\sigma' = [l : v']$
  where $(e, l) \in Loc$ and $(v, v') \in \phi(X)$.
  By symmetry, $(e', e) \in Loc$, and by transivity $(e', l) \in Loc$.
  We know that since ϕ respects I, $\phi(X) = \phi(X')$.

Therefore $(v, v') \in \phi(X')$.
Therefore $(\langle\sigma; \bullet\rangle, \langle\sigma'; \bullet\rangle) \in \mathit{Ptr}(e', \phi(X'))$.

$\Leftarrow$: The other direction is similar.

2. Next, we will show that that if $(I_1, L_1, \phi_1, \psi_1) \leqslant (I_2, L_2, \phi_2, \psi_2)$ then $T_k(I_1, L_1, \phi_1, \psi_1) \leqslant T_k(I_2, L_2, \phi_2, \psi_2)$. Let $(I_1', L_1', \phi_1', \psi_1') = T_k(I_1, L_1, \phi_1, \psi_1)$ and $(I_2', L_2', \phi_2', \psi_2') = T_k(I_2, L_2, \phi_2, \psi_2)$. We have four cases to show:

   (a) $I_1' \subseteq I_2'$: To show this, we want to show that if $(X, X') \in I_1'$, then $(X, X') \in I_2'$. Since PER's are closed under evaluation, it suffices to consider the value forms of (X, X'):

   - $(X, X') = (\mathsf{Loc}, \mathsf{Loc})$:
     By definition of T_k, $(\mathsf{Loc}, \mathsf{Loc}) \in I_2'$.

   - $(X, X') = (1, 1)$:
     Similar to previous case.

   - $(X, X') = (\mathbb{N}, \mathbb{N})$:
     Similar to previous case.

   - $(X, X') = (\top_I, \top_I)$:
     Similar to previous case.

   - $(X, X') = (e =_Y t, e' =_{Y'} t')$:
     By definition of T_k, we know that $(Y, Y') \in I_1$ and $(e, e') \in \phi_1(Y)$ and $(t, t') \in \phi_1(Y)$.
     Since $I_1 \subseteq I_2$, we know $(Y, Y') \in I_2$.
     By the definition of the preorder, $\phi_2(Y) = \phi_1(Y)$.
     Therefore $(e, e') \in \phi_2(Y)$ and $(t, t') \in \phi_2(Y)$.
     Hence $(X, X') \in I_2'$.

   - $(X, X') = (\Pi y : Y.\ Z[y], \Pi y : Y'.\ Z'[y])$:
     By definition of T_k, we know $(Y, Y') \in I_1$.
     By definition of T_k, we know $\forall (v, v') \in \phi_1(Y).\ (Z[v], Z'[v']) \in I_1$.
     Since $I_1 \subseteq I_2$, we know $(Y, Y') \in I_2$.

     Assume $(v, v') \in \phi_2(Y)$.
     Since $(Y, Y') \in I_1$, it follows that $\phi_1(Y) = \phi_2(Y)$.
     Hence $(v, v') \in \phi_1(Y)$.
     Therefore $(Z[v], Z'[v']) \in I_1$.
     Since $I_1 \subseteq I_2$, we know $(Z[v], Z'[v']) \in I_2$.
     Therefore $\forall (v, v') \in \phi_2(Y).\ (Z[v], Z'[v']) \in I_2$.

     Therefore $(X, X') \in I_2'$.

   - Case $(X, X') = (\Sigma y : Y.\ Z, \Sigma y : Y'.\ Z')$:
     Similar to the pi case.

   - Case $(X, X') = (\forall y : Y.\ Z, \forall y : Y'.\ Z')$:
     Similar to the pi case.

- Case $(X, X') = (\exists y : Y.\ Z, \exists y : Y'.\ Z')$:
  Similar to the pi case.

- Case $(X, X') = (\mathsf{G}\,A, \mathsf{G}\,A')$:
  By definition of T_k, we know $(A, A') \in L_1$.
  Since $L_1 \subseteq L_2$, we know $(A, A') \in L_2$.
  Hence $(\mathsf{G}\,A, \mathsf{G}\,A') \in I_2'$.

- $(X, X') = (\mathsf{U}_i, \mathsf{U}_i)$:
  By the definition of T_k, $i < k$.
  Hence $(\mathsf{U}_i, \mathsf{U}_i) \in I_2'$.

- $(X, X') = (\mathsf{L}_i, \mathsf{L}_i)$:
  Similar to the previous case.

(b) $L_1' \subseteq L_2'$: To show this, we want to show that if $(C, C') \in L_1'$, then $(C, C') \in L_2'$. Since PER's are closed under evaluation, it suffices to consider the value forms of (X, X'):

- Case $(C, C') = (\mathsf{I}, \mathsf{I})$:
  By definition of T_k, $(\mathsf{I}, \mathsf{I}) \in L_2'$.

- Case $(C, C') = (A \otimes B, A' \otimes B')$:
  By definition of T_k, we know $(A, A') \in L_1$.
  By definition of T_k, we know $(B, B') \in L_1$.
  Since $L_1 \subseteq L_2$, we know $(A, A') \in L_2$.
  Since $L_1 \subseteq L_2$, we know $(B, B') \in L_2$.
  By definition of T_k, we have $(A \otimes B, A' \otimes B') \in L_2'$.

- Case $(C, C') = (A \multimap B, A' \multimap B')$:
  Similar to the previous case.

- Case $(C, C') = (A \mathbin{\&} B, A' \mathbin{\&} B')$:
  Similar to the previous case.

- Case $(C, C') = (\top, \top)$:
  By definition of T_k, $(\top, \top) \in L_2'$.

- Case $(C, C') = (\mathsf{T}\,A, \mathsf{T}\,A')$:
  By definition of T_k, we know $(A, A') \in L_1$.
  Since $L_1 \subseteq L_2$, we know $(A, A') \in L_2$.
  By definition of T_k, we have $(\mathsf{T}\,A, \mathsf{T}\,A') \in L_2'$.

- Case $(C, C') = (\mathsf{F}x : X.\ A[x], \mathsf{F}x : X'.\ A'[x])$:
  By definition of T_k, we know $(X, X') \in I_1$.
  By definition of T_k, we know $\forall (v, v') \in \phi_1(X).\ (A[v], A'[v']) \in L_1$.
  Since $I_1 \subseteq I_2$, we know $(X, X') \in I_2$.

  Assume $(v, v') \in \phi_2(X)$.
  Since $(X, X') \in I_1$, by properties of extension, $\phi_2(X) = \phi_1(X)$.
  Hence $(v, v') \in \phi_1(X)$.
  Hence $(A[v], A'[v']) \in L_1$.

Since $L_1 \subseteq L_2$, we have $(A[v], A'[v']) \in L_2$.
Therefore $\forall (v, v') \in \phi_2(X).\ (A[v], A'[v']) \in L_2$.

Therefore $(\mathsf{F}x : X.\ A[x], \mathsf{F}x : X'.\ A'[x]) \in L_2'$.

- Case $(C, C') = (\Pi x : X.\ A[x], \Pi x : X'.\ A'[x])$:
  Similar to previous case.

- Case $(C, C') = (\forall x : X.\ A[x], \forall x : X'.\ A'[x])$:
  Similar to previous case.

- Case $(C, C') = (\exists x : X.\ A[x], \exists x : X'.\ A'[x])$:
  Similar to previous case.

- Case $(C, C') = (e \mapsto X, e' \mapsto X')$:
  By definition of T_k, we know $(e, e') \in \text{Loc}$.
  By definition of T_k, we know $(X, X') \in I_1$.
  Since $I_1 \subseteq I_2$, we have $(X, X') \in I_2$.
  Therefore $(C, C') \in L_2'$.

(c) Next, we want to show that if $(X, X') \in I_1'$, then $\phi_1'(X) = \phi_2'(X)$. Since PERs are determined by values, we proceed by cases on the value part of $(X, X') \in I_1'$.

- Case $(X, X') = (\mathsf{Loc}, \mathsf{Loc})$:
  By definition of T_k, we see that $\phi_1'(\mathsf{Loc}) = \phi_2'(\mathsf{Loc}) = \text{Loc}$.

- Case $(X, X') = (1, 1)$:
  Similar to previous case.

- Case $(X, X') = (\mathbb{N}, \mathbb{N})$:
  Similar to previous case.

- Case $(X, X') = (\top_I, \top_I)$:
  Similar to previous case.

- Case $(X, X') = (\mathsf{G}\,A, \mathsf{G}\,A')$:
  By definition of T_k, we know that $(A, A') \in L_1$.
  Since $L_1 \subseteq L_2$, we know that $(A, A') \in L_2$.
  Since $\psi_1 \sqsubseteq \psi_2$, we know $\psi_1(A) = \psi_2(A)$.
  Therefore $G(\psi_1(A)) = G(\psi_2(A))$.
  Therefore $\phi_1'(\mathsf{G}\,A) = \phi_2'(\mathsf{G}\,A)$.

- Case $(X, X') = (\Pi y : Y.\ Z[y], \Pi y : Y'.\ Z'[y'])$:
  By definition of T_k, we know that $(Y, Y') \in I_1$.
  By definition of T_k, we know that $\forall (v, v') \in \phi_1(Y).\ (Y[v], Y'[v'] \in I_1$.

  Since $I_1 \subseteq I_2$, we know $(Y, Y') \in I_2$.
  Since $\phi_1 \sqsubseteq \phi_2$, we know $\phi_1(Y) = \phi_2(Y)$.
  Assume $(v, v) \in \phi_2(Y)$.
  Then we know $(v, v) \in \phi_1(Y)$.
  Hence $(Z[v], Z'[v]) \in I_1$.

Since $I_1 \subseteq I_2$, we know $(Z[v], Z'[v]) \in I_2$, too.
Since $\phi_1 \sqsubseteq \phi_2$, we know $\phi_1(Z[v]) = \phi_2(Z[v])$.
Therefore for all $(v, v) \in \phi_2(Y)$, we have $\phi_1(Z[v]) = \phi_2(Z[v])$.
By extensionality, $\lambda v.\ \phi_1(Z[v]) = \lambda v.\ \phi_2(Z[v])$.

Therefore $\Pi(\phi_1(Y), \lambda v.\ \phi_1(Z[v])) = \Pi(\phi_2(Y), \lambda v.\ \phi_2(Z[v]))$.

By definition of T_k, we have $\phi'_1(\Pi y : Y.\ Z[y]) = \phi'_2(\Pi y : Y'.\ Z'[y'])$.

- Case $(X, X') = (\Sigma y : Y.\ Z[y], \Sigma y : Y'.\ Z'[y'])$:
  Similar to the previous case.

- Case $(X, X') = (\forall y : Y.\ Z[y], \forall y : Y'.\ Z'[y'])$:
  Similar to the previous case.

- Case $(X, X') = (\exists y : Y.\ Z[y], \exists y : Y'.\ Z'[y'])$:
  Similar to the previous case.

- Case $(X, X') = (e =_Y t, e' =_{Y'} t')$:
  By definition of T_k, we know $(Y, Y') \in I_1$.
  Since $I_1 \subseteq I_2$, we get $(Y, Y') \in I_2$.
  Since $\phi_1 \sqsubseteq \phi_2$, $\phi_1(Y) = \phi_2(Y)$.
  Therefore $Id(\phi_1(Y), e, t) = Id(\phi_2(Y), e, t)$.
  Therefore $\phi'_1(e =_Y t) = \phi'_2(e =_Y t)$.

- $(X, X') = (\mathsf{U}_i, \mathsf{U}_i)$:
  By definition of T_k, $\phi'_1(\mathsf{U}_i) = \phi'_2(\mathsf{U}_i) = \mathsf{let}\ (\mathsf{U}, \_, \_, \_) = \mathsf{fix}(T_i)\ \mathsf{in}\ \mathsf{U}$.

- $(X, X') = (\mathsf{L}_i, \mathsf{L}_i)$:
  Similar to previous case.

(d) Finally, we must show that if $(C, C') \in L'_1$, then $\psi'_1(C) = \psi'_2(C)$. Since PERs are determined by value configurations, we proceed by cases on the value part of $(C, C') \in L'_1$.

- Case $(C, C') = (\mathsf{I}, \mathsf{I})$:
  By definition of T_k, $\psi'_1(\mathsf{I}) = \psi'_2(\mathsf{I}) = \hat{I}$.

- Case $(C, C') = (A \otimes B, A' \otimes B')$:
  By definition of T_k, we know $(A, A') \in L_1$.
  Since $L_1 \subseteq L_2$, we get $(A, A') \in L_2$.
  Since $\psi_1 \sqsubseteq \psi_2$, we get $\psi_1(A) = \psi_2(A)$.

  By definition of T_k, we know $(B, B') \in L_1$.
  Since $L_1 \subseteq L_2$, we get $(B, B') \in L_2$.
  Since $\psi_1 \sqsubseteq \psi_2$, we get $\psi_1(B) = \psi_2(B)$.

  Therefore $\psi_1(A) \hat{\otimes} \psi_1(B) = \psi_2(A) \hat{\otimes} \psi_2(B)$.
  By definition of T_k, we have $\psi'_1(A \otimes B) = \psi'_2(A \otimes B)$.

- Case $(C, C') = (A \multimap B, A' \multimap B')$:

Similar to the previous case.

- Case $(C, C') = (A \,\&\, B, A' \,\&\, B')$:
  Similar to the previous case.

- Case $(C, C') = (\top, \top)$:
  By definition of T_k, $\psi_1'(\top) = \psi_2'(\top) = \hat{\top}$.

- Case $(C, C') = (\mathsf{F}x : X.\ A[x], \mathsf{F}x : X'.\ A'[x])$:
  By definition of T_k, we know $(X, X') \in I_1$.
  By definition of T_k, we know $\forall(v, v') \in \phi_1(X).\ (A[v], A'[v']) \in L_1$.

  Since $I_1 \subseteq I_2$, we get $(X, X') \in I_2$.
  Since $\phi_1 \sqsubseteq \phi_2$, we have $\phi_1(X) = \phi_2(X)$.

  Assume $(v, v') \in \phi_2(X)$.
  Then $(v, v') \in \phi_1(X)$.
  Therefore $(A[v], A'[v']) \in L_1$.
  Since $L_1 \subseteq L_2$, we get $(A[v], A'[v']) \in L_2$.
  Since $\psi_1 \sqsubseteq \psi_2$, we have $\psi_1(A[v]) = \psi_2(A[v])$.
  Therefore $\forall(v, v') \in \phi_2(X).\ \psi_1(A[v]) = \psi_2(A[v])$.
  By extensionality, $\lambda v \in \phi_1(X).\ \psi_1(A[v]) = \lambda v \in \phi_2(X).\ \psi_2(A[v])$.

  Therefore $F(\phi_1(X), \lambda v \in \phi_1(X).\ \psi_1(A[v])) = F(\phi_2(X), \lambda v \in \phi_2(X).\ \psi_2(A[v]))$.
  By definition of T_k, we have $\psi_1'(\mathsf{F}x : X.\ A[x]) = \psi_2'(\mathsf{F}x : X.\ A[x])$.

- Case $(C, C') = (\Pi x : X.\ A[x], \Pi x : X'.\ A'[x])$:
  Similar to previous case.

- Case $(C, C') = (\forall x : X.\ A[x], \forall x : X'.\ A'[x])$:
  Similar to previous case.

- Case $(C, C') = (\exists x : X.\ A[x], \exists x : X'.\ A'[x])$:
  Similar to previous case.

- Case $(C, C') = (\mathsf{T}\,A, \mathsf{T}\,A')$:
  By definition of T_k, we know $(A, A') \in L_1$.
  Since $L_1 \subseteq L_2$, we get $(A, A') \in L_2$.
  Since $\psi_1 \sqsubseteq \psi_2$, we get $\psi_1(A) = \psi_2(A)$.

  Therefore $\hat{T}(\psi_1(A)) = \hat{T}(\psi_2(A))$.
  By definition of T_k, we have $\psi_1'(\mathsf{T}\,A) = \psi_2'(\mathsf{T}\,A)$.

- Case $(C, C') = (e \mapsto X, e' \mapsto X')$:
  By definition of T_k, we know $(X, X') \in I_1$.
  Since $I_1 \subseteq I_2$, we have $(X, X') \in I_2$.
  Hence $\phi_1(X) = \phi_2(X)$.
  Hence $Ptr(e, \phi_1(X)) = Ptr(e, \phi_2(X))$.
  By definition of T_k, we have $\phi_1'(e \mapsto X) = \phi_2'(e \mapsto X)$.

□

Lemma 3 (Expansion). *If* $i \leqslant k$ *and* τ *is a type system then* $T_i(\tau) \leqslant T_k(\tau)$.

Proof. Immediate, since the definition of T_k only adds those universes U_j such that $i \leqslant j < k$. □

Lemma 4 (Universe Cumulativity). *If* $i \leqslant k$ *then* $\mathcal{T}_i \leqslant \mathcal{T}_k$.

Proof. First, note that by monotonicity, $T_i(\tau) \leqslant T_i(\tau')$ for any $\tau \leqslant \tau'$.
Then by expansion, we know that $T_i(\tau') \leqslant T_k(\tau')$.
Hence by transivity, $T_i(\tau) \leqslant T_k(\tau')$.
Next, consider the ordinal-indexed sequences:

- $s_0 = \emptyset$
- $s_{\beta+1} = T_i(s_\beta)$
- $s_\lambda = \bigsqcup_{\beta<\lambda} s_\beta$

- $t_0 = \emptyset$
- $t_{\beta+1} = T_k(t_\beta)$
- $t_\lambda = \bigsqcup_{\beta<\lambda} t_\beta$

Now observe that for every ordinal α, $s_\alpha \leqslant t_\alpha$.
Since both of these sequences reach fixed points, it follows that $\mathcal{T}_i \leqslant \mathcal{T}_k$. □

Lemma 5 (Context Shrinking).
If Γ, Γ' ok *then* Γ ok.

Proof. The proof is by induction on the structure of Γ':

- Case $\Gamma' = \cdot$:

  We have Γ, Γ' ok.
  Then $\Gamma, \Gamma' = \Gamma$, and so we already have Γ ok.

- Case $\Gamma' = \Gamma'', x : X$:

  We have $\Gamma, \Gamma'', x : X$ ok.
  By inversion, we get Γ, Γ'' ok.
  By induction, we get Γ ok.

□

Lemma 6 (Linear Context Shrinking).
If $\Gamma \vdash \Delta, \Delta'$ ok *then* $\Gamma \vdash \Delta$ ok *and* $\Gamma \vdash \Delta'$ ok.

Proof. The proof is by induction on the structure of Δ':

- Case $\Delta' = \cdot$:

  We have $\Gamma \vdash \Delta, \Delta'$ ok.
  ☞ Then $\Delta, \Delta' = \Delta$, and so we already have $\Gamma \vdash \Delta$ ok.
  By LCtxNil, we have $\Gamma \vdash \cdot$ ok.
  ☞ Therefore $\Gamma \vdash \Delta'$ ok.

- Case $\Delta' = \Delta'', a : A$:

  We have $\Gamma \vdash \Delta, \Delta'', a : A$ ok.
  By inversion, we get $\Gamma \vdash A$ linear.
  By inversion, we get $\Gamma \vdash \Delta, \Delta''$ ok.
  ☞ By induction, we get $\Gamma \vdash \Delta$ ok.
  By induction, we get $\Gamma \vdash \Delta''$ ok.
  By LCTXCONS, we get $\Gamma \vdash \Delta'', a : A$ ok.
  ☞ Therefore $\Gamma \vdash \Delta'$ ok.

□

Lemma 7 (Substitution Shrinking)**.**
If $\gamma \in [\![\Gamma_0, \Gamma_1]\!]$ *then there are* γ_0, γ_1 *such that* $\gamma = \gamma_0, \gamma_1$ *and* $\gamma_0 \in [\![\Gamma_0]\!]$*.*

Proof. We proceed by induction on Γ_1:

- Case $\Gamma_1 = \cdot$:

  Then $\Gamma_0, \Gamma_1 = \Gamma_0$.
  So $\gamma \in [\![\Gamma_0]\!]$.
  Take $\gamma_0 = \gamma$.
  Take $\gamma_1 = \langle\rangle$.
  ☞ So $\gamma_0 \in [\![\Gamma_0]\!]$.
  ☞ Note $\gamma = \gamma, \langle\rangle = \gamma_0, \gamma_1$.

- Case $\Gamma_1 = \Gamma_1', x : X$:

  We have $\gamma \in [\![\Gamma_0, \Gamma_1', x : X]\!]$.
  By definition of $[\![-]\!]$, we know $\gamma = (\gamma, (e, e')/x)$.
  By definition of $[\![-]\!]$, we know $\gamma'(X) \in I$.
  By definition of $[\![-]\!]$, we know $(e, e') \in \phi(\gamma_1'(X))$.
  By definition of $[\![-]\!]$, we know $\gamma' \in [\![\Gamma_0, \Gamma_1']\!]$.
  By induction, we have γ_0, γ_1' such that $\gamma' = \gamma_0, \gamma_1'$.
  ☞ By induction, we have $\gamma_0 \in [\![\Gamma_0]\!]$.
  Take $\gamma_1 = \gamma_1', (e, e')/x$.
  ☞ Note that $\gamma = (\gamma', (e, e')/x) = (\gamma_0, \gamma_1', (e, e')/x) = (\gamma_0, \gamma_1)$

□

Lemma 8 (Free Variables of Linear Contexts)**.**
If $\Gamma \vdash \Delta$ ok *then* $\mathrm{FV}(\Delta) \subseteq \mathrm{dom}(\Gamma)$.

Proof. We proceed by induction on Δ:

- Case $\Delta = \cdot$:

  Then $\mathrm{FV}(\Delta) = \emptyset$.
  Immediately, $\mathrm{FV}(\Delta) \subseteq \mathrm{dom}(\Gamma)$.

- Case $\Delta = \Delta', a : A$:

  By inversion on $\Gamma \vdash \Delta$ ok, we get $\Gamma \vdash \Delta'$ ok.
  By inversion on $\Gamma \vdash \Delta$ ok, we get $\Gamma \vdash A$ linear.
  By induction, $\mathrm{FV}(\Delta') \subseteq \mathrm{dom}(\Gamma)$.
  By properties of typing, $\mathrm{FV}(A) \subseteq \mathrm{dom}(\Gamma)$.
  Hence $\mathrm{FV}(\Delta') \cup \mathrm{FV}(A) \subseteq \mathrm{dom}(\Gamma)$.

Hence $FV(\Delta', a : A) \subseteq dom(\Gamma)$.
By equality, $FV(\Delta) \subseteq dom(\Gamma)$.

□

Lemma 9 (Linear Heap Preservation)**.**
If $\langle \sigma; e \rangle \Downarrow \langle \sigma'; u \rangle$ *then* $\sigma = \sigma'$*.*

Proof. Routine induction on derivations. The only interesting case is dereference:

- **Case** LDEREF: $\dfrac{e \Downarrow l \qquad \langle \sigma; e' \rangle \Downarrow \langle \sigma', l : v; * \rangle \qquad \langle \sigma', l : v; [v/x, */c]e'' \rangle \Downarrow \langle \sigma''; u \rangle}{\langle \sigma; \mathsf{let}\ (x, c) = \mathsf{get}(e, e')\ \mathsf{in}\ e'' \rangle \Downarrow \langle \sigma''; u \rangle}$

  By induction, we know that $\sigma = (\sigma', l : v)$.
  By induction, we know that $(\sigma', l : v) = \sigma''$.
  By transitivity, $\sigma = \sigma''$.

□

Lemma 10 (Linear Evaluation Frame Property)**.**
If $\langle \sigma; e \rangle \Downarrow \langle \sigma'; u \rangle$ *and* $\sigma_f \# \sigma$ *then* $\sigma' \# \sigma_f$ *and* $\langle \sigma \cdot \sigma_f; e \rangle \Downarrow \langle \sigma' \cdot \sigma_f; u \rangle$*.*

Proof. We proceed by induction on the derivation of $\langle \sigma; e \rangle \Downarrow \langle \sigma'; u \rangle$.

- **Case** LVAL: $\overline{\langle \sigma; u \rangle \Downarrow \langle \sigma; u \rangle}$

  ☞ By assumption, $\sigma \# \sigma_f$.
  Since $\sigma \# \sigma_f$, we know $\sigma \cdot \sigma_f$ is defined.
  ☞ Hence by rule LVAL, $\langle \sigma \cdot \sigma''; u \rangle \Downarrow \langle \sigma \cdot \sigma''; u \rangle$.

- **Case** LAPP: $\dfrac{\langle \sigma; e_1 \rangle \Downarrow \langle \sigma'; \lambda x.\ e_1' \rangle \qquad \langle \sigma'; [e_2/x]e_1' \rangle \Downarrow \langle \sigma''; u'' \rangle}{\langle \sigma; e_1\ e_2 \rangle \Downarrow \langle \sigma''; u'' \rangle}$

  By assumption, we have $\sigma \# \sigma_f$.
  By inversion, we have $\langle \sigma; e_1 \rangle \Downarrow \langle \sigma'; \lambda x.\ e_1' \rangle$.
  By inversion, we have $\langle \sigma'; [e_2/x]e_1' \rangle \Downarrow \langle \sigma''; u'' \rangle$.

  By induction, we get $\langle \sigma \cdot \sigma_f; e_1 \rangle \Downarrow \langle \sigma' \cdot \sigma_f; \lambda x.\ e_1' \rangle$. (a)
  We also get $\sigma' \# \sigma_f$.
  By induction, we get $\langle \sigma' \cdot \sigma_f; [e_2/x]e_1' \rangle \Downarrow \langle \sigma'' \cdot \sigma_f; u'' \rangle$. (b)
  ☞ We also get $\sigma'' \# \sigma_f$.
  ☞ By rule LAPP on (a) and (b), we get $\langle \sigma \cdot \sigma_f; e_1\ e_2 \rangle \Downarrow \langle \sigma'' \cdot \sigma_f; u'' \rangle$.

- **Case** LUNIT: $\dfrac{\langle \sigma; e \rangle \Downarrow \langle \sigma'; () \rangle \qquad \langle \sigma'; e' \rangle \Downarrow \langle \sigma''; u \rangle}{\langle \sigma; \mathsf{let}\ () = e\ \mathsf{in}\ e' \rangle \Downarrow \langle \sigma''; u \rangle}$

  By assumption, we have $\sigma \# \sigma_f$.
  By inversion, we have $\langle \sigma; e \rangle \Downarrow \langle \sigma'; () \rangle$.
  By inversion, we have $\langle \sigma'; e' \rangle \Downarrow \langle \sigma''; u \rangle$.

  By induction, we get $\langle \sigma \cdot \sigma_f; e \rangle \Downarrow \langle \sigma' \cdot \sigma_f; () \rangle$. (a)
  We also get $\sigma' \# \sigma_f$.
  By induction, we get $\langle \sigma' \cdot \sigma_f; e' \rangle \Downarrow \langle \sigma'' \cdot \sigma_f; u \rangle$. (b)
  ☞ We also get $\sigma'' \# \sigma_f$.
  ☞ By rule LUNIT on (a) and (b), we get $\langle \sigma \cdot \sigma_f; \mathsf{let}\ () = e\ \mathsf{in}\ e' \rangle \Downarrow \langle \sigma'' \cdot \sigma_f; u \rangle$.

- **Case** LPAIR: $\dfrac{\langle \sigma; e\rangle \Downarrow \langle \sigma'; (e_1, e_2)\rangle \qquad \langle \sigma'; [e_1/a, e_2/b]e'\rangle \Downarrow \langle \sigma''; u\rangle}{\langle \sigma; \mathsf{let}\ (a, b) = e\ \mathsf{in}\ e'\rangle \Downarrow \langle \sigma''; u\rangle}$

  By assumption, we have $\sigma \# \sigma_f$.
  By inversion, we have $\langle \sigma; e\rangle \Downarrow \langle \sigma'; (e_1, e_2)\rangle$.
  By inversion, we have $\langle \sigma'; [e_1/a, e_2/b]e'\rangle \Downarrow \langle \sigma''; u\rangle$.

  By induction, we get $\langle \sigma \cdot \sigma_f; e\rangle \Downarrow \langle \sigma' \cdot \sigma_f; (e_1, e_2)\rangle$. (a)
  We also get $\sigma' \# \sigma_f$.
  By induction, we get $\langle \sigma' \cdot \sigma_f; [e_1/a, e_2/b]e'\rangle \Downarrow \langle \sigma'' \cdot \sigma_f; u\rangle$. (b)
  ☞ We also get $\sigma'' \# \sigma_f$.
  ☞ By rule LPAIR on (a) and (b), we get $\langle \sigma \cdot \sigma_f; \mathsf{let}\ (a, b) = e\ \mathsf{in}\ [e_1/a, e_2/b]e'\rangle \Downarrow \langle \sigma'' \cdot \sigma_f; u\rangle$.

- **Case** LPIAPP: $\dfrac{\langle \sigma; e_1\rangle \Downarrow \langle \sigma'; \hat{\lambda} x.\ e\rangle \qquad \langle \sigma'; [e_2/x]e\rangle \Downarrow \langle \sigma''; u''\rangle}{\langle \sigma; e_1\ e_2\rangle \Downarrow \langle \sigma''; u''\rangle}$

  By assumption, we have $\sigma \# \sigma_f$.
  By inversion, $\langle \sigma; e_1\rangle \Downarrow \langle \sigma'; \hat{\lambda} x.\ e\rangle$.
  By inversion, $\langle \sigma'; [e_2/x]e\rangle \Downarrow \langle \sigma''; u''\rangle$.

  (a) By induction, $\langle \sigma \cdot \sigma_f; e_1\rangle \Downarrow \langle \sigma' \cdot \sigma_f; \hat{\lambda} x.\ e\rangle$.
  We also get $\sigma' \# \sigma_f$.
  By (b) induction, $\langle \sigma' \cdot \sigma_f; [e_2/x]e\rangle \Downarrow \langle \sigma'' \cdot \sigma_f; u''\rangle$.
  ☞ We also get $\sigma'' \# \sigma_f$.
  ☞ By rule LPIAPP on (a) and (b), we get $\langle \sigma \cdot \sigma_f; e_1\ e_2\rangle \Downarrow \langle \sigma'' \cdot \sigma_f; u''\rangle$.

- **Case** LFST: $\dfrac{\langle \sigma; e\rangle \Downarrow \langle \sigma'; (e_1, e_2)\rangle \qquad \langle \sigma'; e_1\rangle \Downarrow \langle \sigma''; u''\rangle}{\langle \sigma; \pi_1\ e\rangle \Downarrow \langle \sigma''; u''\rangle}$

  By assumption, we have $\sigma \# \sigma_f$.
  By inversion, $\langle \sigma; e\rangle \Downarrow \langle \sigma'; (e_1, e_2)\rangle$.
  By inversion, $\langle \sigma'; e_1\rangle \Downarrow \langle \sigma''; u''\rangle$.

  By induction, $\langle \sigma \cdot \sigma_f; e\rangle \Downarrow \langle \sigma' \cdot \sigma_f; (e_1, e_2)\rangle$.
  We also have $\sigma' \# \sigma_f$.

  By induction, $\langle \sigma' \cdot \sigma_f; e_1\rangle \Downarrow \langle \sigma'' \cdot \sigma_f; u''\rangle$.
  ☞ We also have $\sigma'' \# \sigma_f$.
  ☞ By rule LFST on (a) and (b), we get $\langle \sigma \cdot \sigma_f; \pi_1\ e\rangle \Downarrow \langle \sigma'' \cdot \sigma_f; u''\rangle$.

- **Case** LSND: $\dfrac{\langle \sigma; e\rangle \Downarrow \langle \sigma'; (e_1, e_2)\rangle \qquad \langle \sigma'; e_2\rangle \Downarrow \langle \sigma''; u''\rangle}{\langle \sigma; \pi_2\ e\rangle \Downarrow \langle \sigma''; u''\rangle}$

  Similar to previous case.

- **Case** LF: $\dfrac{\langle \sigma; e\rangle \Downarrow \langle \sigma'; \mathsf{F}\,(e_1, e_2)\rangle \qquad \langle \sigma'; [e_1/x, e_2/a]e'\rangle \Downarrow \langle \sigma''; u\rangle}{\langle \sigma; \mathsf{let}\ \mathsf{F}\,(x, a) = e\ \mathsf{in}\ e'\rangle \Downarrow \langle \sigma''; u\rangle}$

  By assumption, we have $\sigma \# \sigma_f$.
  By inversion, we have $\langle \sigma; e\rangle \Downarrow \langle \sigma'; \mathsf{F}\,(e_1, e_2)\rangle$.
  By inversion, we have $\langle \sigma'; [e_1/x, e_2/b]e'\rangle \Downarrow \langle \sigma''; u\rangle$.

By induction, we get $\langle \sigma \cdot \sigma_f; e \rangle \Downarrow \langle \sigma' \cdot \sigma_f; \mathsf{F}\,(e_1, e_2) \rangle$. (a)
We also get $\sigma' \# \sigma_f$.
By induction, we get $\langle \sigma' \cdot \sigma_f; [e_1/a, e_2/b]e' \rangle \Downarrow \langle \sigma'' \cdot \sigma_f; u \rangle$. (b)
☞ We also get $\sigma'' \# \sigma_f$.
☞ By rule LF on (a) and (b), we get $\langle \sigma \cdot \sigma_f; \mathsf{let}\ (a, b) = e\ \mathsf{in}\ [e_1/a, e_2/b]e' \rangle \Downarrow \langle \sigma'' \cdot \sigma_f; u \rangle$.

- **Case** LRUNG: $\dfrac{e \Downarrow \mathsf{G}\, e' \qquad \langle \sigma; e' \rangle \Downarrow \langle \sigma'; u \rangle}{\left\langle \sigma; \mathsf{G}^{-1}\, e \right\rangle \Downarrow \langle \sigma'; u \rangle}$

  By assumption, we have $\sigma \# \sigma_f$.
  (a) By inversion, we get $e \Downarrow \mathsf{G}\, e'$.
  By inversion, we get $\langle \sigma; e' \rangle \Downarrow \langle \sigma'; u \rangle$.

  (b) By induction, we get $\langle \sigma \cdot \sigma_f; e' \rangle \Downarrow \langle \sigma' \cdot \sigma_f; u \rangle$.
  ☞ We also get $\sigma' \# \sigma_f$.
  By rule LRUNGon (a) and (b), $\left\langle \sigma \cdot \sigma_f; \mathsf{G}^{-1}\, e \right\rangle \Downarrow \langle \sigma' \cdot \sigma_f; u \rangle$.

- **Case** LDEREF: $\dfrac{e \Downarrow l \qquad \langle \sigma; e' \rangle \Downarrow \langle \sigma', l : v; \bullet \rangle \qquad \langle \sigma', l : v; [v/x, \bullet/c]e'' \rangle \Downarrow \langle \sigma''; u \rangle}{\langle \sigma; \mathsf{let}\ (x, c) = \mathsf{get}(e, e')\ \mathsf{in}\ e'' \rangle \Downarrow \langle \sigma''; u \rangle}$

  By assumption, we have $\sigma \# \sigma_f$.
  (a) By inversion, $e \Downarrow l$.
  By inversion, $\langle \sigma; e' \rangle \Downarrow \langle \sigma', l : v; \bullet \rangle$.
  By inversion, $\langle \sigma; [v/x, \bullet/c]e'' \rangle \Downarrow \langle \sigma', l : v; u \rangle$.
  (b) By induction, $\langle \sigma, l : v, \sigma_f; [v/x, \bullet/c]e'' \rangle \Downarrow \langle \sigma' \cdot \sigma_f; u \rangle$.
  We also get $\sigma', l : v \# \sigma_f$.
  (c)By induction, $\langle (\sigma', l : v) \cdot \sigma_f; [v/x, \bullet/c]e'' \rangle \Downarrow \langle \sigma'' \cdot \sigma_f; u \rangle$.
  ☞ We also get $\sigma'' \# \sigma_f$.
  By rule LDEREF on (a), (b) and (c), we get

  $$\langle \sigma \cdot \sigma_f; \mathsf{let}\ (x, c) = \mathsf{get}(e, e')\ \mathsf{in}\ e'' \rangle \Downarrow \langle \sigma'' \cdot \sigma_f; u \rangle$$

□

Theorem 1 (Fundamental Property)**.**
Assuming that Γ ok *and* $\gamma \in [\![\Gamma]\!]$ *and* $\Gamma \vdash \Delta$ ok *and* $(\sigma, \delta) \in [\![\gamma_1(\Delta)]\!]$*, we have that:*

1. *If* $\Gamma \vdash X$ type *then* $\gamma(X) \in U(\gamma_1(X))$.
2. *If* $\Gamma \vdash X \equiv Y$ type *then* $(\gamma_1(X), \gamma_2(Y)) \in U(\gamma_1(X))$.
3. *If* $\Gamma \vdash e : X$ *then* $\gamma(e) \in \phi(\gamma_1(X))$.
4. *If* $\Gamma \vdash e_1 \equiv e_2 : X$ *then* $(\gamma_1(e_1), \gamma_2(e_2)) \in \phi(\gamma_1(X))$.
5. *If* $\Gamma \vdash A$ linear *then* $\gamma(A) \in L(\gamma_1(X))$.
6. *If* $\Gamma \vdash A \equiv B$ linear *then* $(\gamma_1(A), \gamma_2(B)) \in L(\gamma_1(X))$.
7. *If* $\Gamma; \Delta \vdash e : A$ *then* $((\sigma_1, \gamma_1(\delta_1(e))), (\sigma_2, \gamma_2(\delta_2(e)))) \in \psi(\gamma_1(X))$.
8. *If* $\Gamma; \Delta \vdash e_1 \equiv e_2 : A$ *then* $((\sigma_1, \gamma_1(\delta_1(e_1))), (\sigma_2, \gamma_2(\delta_2(e_2)))) \in \psi(\gamma_1(X))$.
9. *If* $\Gamma; \Delta \vdash e \div A$ *then there exists* t *and* t' *such that for every* $\gamma \in [\![\Gamma]\!]$ *and every* $(\sigma, \delta) \in [\![\gamma_1(\Delta)]\!]$, $((\sigma_1, \delta_1(\gamma_1(t))), (\sigma_2, \delta_2(\gamma_2(t'$
   $\psi(\gamma_1(A))$.

Proof. Assume that Γ ok and $\gamma \in [\![\Gamma]\!]$.
This proof has 9 main cases, all mutually inductive:

1. If $\Gamma \vdash X$ type then $\gamma(X) \in \mathcal{U}_k(\gamma_1(X))$ for some k.

   We case analyze the derivation of $\Gamma \vdash X$ type.

   - **Case** TP: $\dfrac{\Gamma \vdash X : U_i}{\Gamma \vdash X \text{ type}}$

     By induction, we know that $(\gamma_1(X), \gamma_2(X)) \in \phi(U_i)$.
     Thus $(\gamma_1(X), \gamma_2(X)) \in I$ at $\mathcal{T}_i$.

2. If $\Gamma \vdash X \equiv Y$ type then $(\gamma_1(X), \gamma_2(Y)) \in \mathcal{U}(\gamma_1(X))$.

   We case analyze the derivation of $\Gamma \vdash X \equiv Y$ type.

   - **Case** TPEQ: $\dfrac{\Gamma \vdash X \equiv Y : U_i}{\Gamma \vdash X \equiv Y \text{ type}}$

     By induction, we know that $(\gamma_1(X), \gamma_2(X)) \in \phi(U_i))$.
     Thus $(\gamma_1(X), \gamma_2(X))$ is in the I of type system $\mathcal{T}_i$.

3. If $\Gamma \vdash e : X$ then $\gamma(e) \in \phi(\gamma_1(X))$.

   We case analyze the derivation of $\Gamma \vdash e : X$:

   - **Case** IU: $\dfrac{}{\Gamma \vdash U_i : U_{i+1}}$

     Notice that $\gamma(U_i, U_i) = (U_i, U_i) \in \phi(U_{i+1})$ in $\mathcal{T}_{i+2}$ since it is in I in $\mathcal{T}_{i+1}$.

   - **Case** IL: $\dfrac{}{\Gamma \vdash L_i : U_{i+1}}$

     The same remark applies if one substitutes U_i by L_i.

   - **Case** IPI: $\dfrac{\Gamma \vdash X : U_i \qquad \Gamma, x : X \vdash Y : U_i}{\Gamma \vdash \Pi x : X.\ Y : U_i}$

     By induction and $\Gamma, x : X$ ok, we have

     – $(\gamma_1(X), \gamma_2(X)) \in \phi(U_i)$
     – $\forall (e_1, e_2) \in \phi(\gamma_1(X)), ((\gamma_1, e_1/x)(Y), (\gamma_2, e_2/x)(Y)) \in \phi(U_i)$

     which is exactly the requirement needed for $\Pi x : X.\ Y$ to be in I in $T_i(\mathcal{T}_i) = \mathcal{T}_i$ and thus in $\phi(U_i)$ in $\mathcal{T}_\omega$.

   - **Case** ISIGMA: $\dfrac{\Gamma \vdash X : U_i \qquad \Gamma, x : X \vdash Y : U_i}{\Gamma \vdash \Sigma x : X.\ Y : U_i}$

     The argument is the same as in the case IPI.

   - **Case** ILPI: $\dfrac{\Gamma \vdash X : U_i \qquad \Gamma, x : X \vdash A : L_i}{\Gamma \vdash \Pi x : X.\ A : L_i}$

     The argument is similar to the IF case.

   - **Case** : $\dfrac{\Gamma \vdash X : U_i \qquad \Gamma, x : X \vdash Y : U_i}{\Gamma \vdash \forall x : X.\ Y : U_i}$

     The argument is similar to the IF case.

   - **Case** : $\dfrac{\Gamma \vdash X : U_i \qquad \Gamma, x : X \vdash Y : L_i}{\Gamma \vdash \forall x : X.\ Y : L_i}$

     The argument is similar to the IF case.

- **Case** : $\dfrac{\Gamma \vdash X : \mathsf{U}_i \qquad \Gamma, x : X \vdash Y : \mathsf{U}_i}{\Gamma \vdash \exists x : X.\ Y : \mathsf{U}_i}$
  The argument is similar to the IF case.
- **Case** : $\dfrac{\Gamma \vdash X : \mathsf{U}_i \qquad \Gamma, x : X \vdash Y : \mathsf{L}_i}{\Gamma \vdash \exists x : X.\ Y : \mathsf{L}_i}$
  The argument is similar to the IF case.
- **Case** IWITH: $\dfrac{\Gamma \vdash A : \mathsf{L}_i \qquad \Gamma \vdash B : \mathsf{L}_i}{\Gamma \vdash A \,\&\, B : \mathsf{L}_i}$
  Same argument as the ITENSOR case.
- **Case** IUNIT: $\overline{\Gamma \vdash 1 : \mathsf{U}_i}$
  Clearly, $\gamma(1,1) = (1,1) \in I$ at $\mathcal{T}_i$.
- **Case** ILOC: $\overline{\Gamma \vdash \mathsf{Loc} : \mathsf{U}_i}$
  Same argument as IUNIT.
- **Case** INAT: $\overline{\Gamma \vdash \mathbb{N} : \mathsf{U}_i}$
  Same argument as IUNIT.
- **Case** IG: $\dfrac{\Gamma \vdash A : \mathsf{L}_i}{\Gamma \vdash \mathsf{G}\,A : \mathsf{U}_i}$
  By induction, $(\gamma_1(A), \gamma_2(A)) \in \phi(\gamma_1(\mathsf{L}_i))$, thus in I at $\mathcal{T}_i$.
  Since it is a fixpoint of T_i, we have also $(\mathsf{G}\,\gamma_1(A), \mathsf{G}\,\gamma_2(A)) \in \phi(\mathsf{U}_i)$.
- **Case** IEQ: $\dfrac{\Gamma \vdash X : \mathsf{U}_i \qquad \Gamma \vdash e : X \qquad \Gamma \vdash e' : X}{\Gamma \vdash e =_X e' : \mathsf{U}_i}$
  Similar to IPTR.
- **Case** IONE: $\overline{\Gamma \vdash \mathsf{I} : \mathsf{L}_i}$
  Same argument as IUNIT.
- **Case** ITENSOR: $\dfrac{\Gamma \vdash A : \mathsf{L}_i \qquad \Gamma \vdash B : \mathsf{L}_i}{\Gamma \vdash A \otimes B : \mathsf{L}_i}$
  Same argument as IG.
- **Case** ILOLLI: $\dfrac{\Gamma \vdash A : \mathsf{L}_i \qquad \Gamma \vdash B : \mathsf{L}_i}{\Gamma \vdash A \multimap B : \mathsf{L}_i}$
  Same argument as IG.
- **Case** IF: $\dfrac{\Gamma \vdash X : \mathsf{U}_i \qquad \Gamma, x : X \vdash A : \mathsf{L}_i}{\Gamma \vdash \mathsf{F}x : X.\ A : \mathsf{L}_i}$
  By induction and $\Gamma, x : X$ ok,
  $$(\gamma_1(X), \gamma_2(X)) \in \mathsf{U}$$
  $$\forall (e_1, e_2) \in \mathsf{U},\ ((\gamma_1, e_1/x)(A), (\gamma_2, e_2/x)(A)) \in L$$
  Thus by definition, $(\gamma_1(\mathsf{F}x : X.\ A), \gamma_2(\mathsf{F}x : X.\ A)) \in L$ since we're in a fixpoint of T_i. Thus, we have the expected result.
- **Case** IPTR: $\dfrac{\Gamma \vdash e : \mathsf{Loc} \qquad \Gamma \vdash X : \mathsf{U}_i \qquad \Gamma \vdash e' : X}{\Gamma \vdash e \mapsto X : \mathsf{L}_i}$
  By induction
  $$(\gamma_1(e), \gamma_2(e)) \in \phi(\mathsf{Loc})$$

$$(\gamma_1(X), \gamma_2(X)) \in \phi(\mathsf{U}_i)$$

Thus by definition, $(\gamma_1(e \mapsto X), \gamma_2(e \mapsto X)) \in \mathsf{L}$ since $\mathcal{T}_i$ is a fixpoint of T_i. Thus, We have the expected result.

- **Case** IT: $\dfrac{\Gamma \vdash A : \mathsf{L}_i}{\Gamma \vdash \mathsf{T}\, A : \mathsf{L}_i}$
  Same argument as IG.
- **Case** IHYP: $\overline{\Gamma, x : X, \Gamma' \vdash x : X}$
  By hypothesis, $\gamma \in [\![\Gamma, x : X, \Gamma']\!]$.
  We can therefore get a restriction γ' of γ belonging to $[\![\Gamma, x : X]\!]$ such that γ and γ' agree on $\Gamma, x : X$. Therefore, since all free variables in X appear in Γ, we have $\gamma(X) = \gamma'(X)$ and $\gamma(x) = \gamma'(x)$. By definition of $\Gamma, x : X$ ok, we have $(\gamma_1'(x), \gamma_2'(x)) \in \phi(\gamma_1'(X))$.
- **Case** IUNITI: $\overline{\Gamma \vdash () : 1}$
  $(\gamma_1(()), \gamma_2(())) = ((), ()) \in \phi(1)$
- **Case** ITPEQ: $\dfrac{\Gamma \vdash e : Y \qquad \Gamma \vdash X \equiv Y \text{ type}}{\Gamma \vdash e : X}$
  By induction, we have:
  - $(\gamma_1(X), \gamma_2(Y)) \in \mathsf{U}$
  - $(\gamma_1(e), \gamma_2(e)) \in \phi(\gamma_1(Y))$
  - $(\gamma_1(X), \gamma_2(X)) \in \mathsf{U}$ since X is a type

  Thus we have $\phi(\gamma_1(X)) = \phi(\gamma_2(X)) = \phi(\gamma_1(Y))$. Then $(\gamma_1(e), \gamma_2(e)) \in \phi(\gamma_1(X))$.
- **Case** IPAIRI: $\dfrac{\Gamma \vdash e : X \qquad \Gamma \vdash e' : [e/x]Y}{\Gamma \vdash (e, e') : \Sigma x : X.\ Y}$
  By induction, we have:
  - $(\gamma_1(e), \gamma_2(e)) \in \phi(\gamma_1(X))$
  - $((\gamma_1, \gamma_1(e)/x)(e'), (\gamma_2, \gamma_2(e)/x)(e')) \in \phi((\gamma_1, \gamma_1(e)/x)(Y))$

  Which gives us the result.
- **Case** IPAIRE1: $\dfrac{\Gamma \vdash e : \Sigma x : X.\ Y}{\Gamma \vdash \pi_1\, e : X}$
  By induction, we know that $(\gamma_1(e), \gamma_2(e)) \in \phi(\gamma_1(\Sigma x : X.\ Y)) = \phi(\Sigma x : \gamma_1(X).\ \gamma_1(Y))$.
  Thus we have some $((e_1', e_1''), (e_2', e_2''))$ such that

  $$\gamma_1(e) \Downarrow (e_1', e_1'')$$
  $$\gamma_2(e) \Downarrow (e_2', e_2'')$$
  $$(e_1', e_2') \in \phi(\gamma_1(X))$$

  It means in particular that

  $$\pi_1\,(\gamma_1(e)) \Downarrow e_1'$$
  $$\pi_1\,(\gamma_2(e)) \Downarrow e_2'$$

  Since our PERs are closed under evalutation, $(\gamma_1(\pi_1\, e), \gamma_2(\pi_2\,(e))) \in \phi(\gamma_1(X))$.

- **Case** IPAIRE2: $\dfrac{\Gamma \vdash e : \Sigma x : X.\ Y}{\Gamma \vdash \pi_2\ e : [\pi_1\ e/x]Y}$
  By induction, we know that
  $$(\gamma_1(e), \gamma_2(e)) \in \phi(\Sigma x : \gamma_1(X).\ \gamma_1(Y))$$
  Thus we have some $((e_1', e_1''), (e_2', e_2''))$ such that
  $$\gamma_1(e) \Downarrow (e_1', e_1'')$$
  $$\gamma_2(e) \Downarrow (e_2', e_2'')$$
  $$(e_1', e_2') \in \phi(\gamma_1(X))$$
  $$(e_1'', e_2'') \in \phi((\gamma_1, e_1'/x)(Y))$$
  It means in particular that
  $$\pi_1\ (\gamma_1(e)) \Downarrow e_1' \wedge \pi_2\ (\gamma_1(e)) \Downarrow e_1''$$
  $$\pi_1\ (\gamma_2(e)) \Downarrow e_2' \wedge \pi_2\ (\gamma_2(e)) \Downarrow e_2''$$
  Since $(\pi_1\ (\gamma_1(e_1)), e_1') \in \phi(\gamma_1(X))$, $\phi([e_1'/x]\gamma_1(Y)) = \phi([\pi_1\ (\gamma_1(e))/x]\gamma_1(Y))$.
  Then, by closure under evaluation, the PER structure and the previous equality, we have $(\pi_2\gamma_1(e), \pi_2\gamma_2(e)) \in \phi(\gamma_1([\pi_1\ e/x](Y)))$.

- **Case** INIZERO: $\dfrac{}{\Gamma \vdash 0 : \mathbb{N}}$
  Note that $\gamma(\mathbb{N}) = (\mathbb{N}, \mathbb{N})$ and $\gamma(0) = (0, 0)$.
  We know that $\phi(\mathbb{N}) = \hat{\mathbb{N}}$.
  By definition of $\hat{\mathbb{N}}$, we have $(0, 0) \in \hat{\mathbb{N}}$.

- **Case** INISUCC: $\dfrac{\Gamma \vdash e : \mathbb{N}}{\Gamma \vdash \mathsf{s}(e) : \mathbb{N}}$
  Note that $\gamma(\mathbb{N}) = (\mathbb{N}, \mathbb{N})$, and $\phi(\mathbb{N}) = \hat{\mathbb{N}}$.
  By induction $(\gamma_1(e), \gamma_2(e)) \in \hat{\mathbb{N}}$.
  Hence $\gamma_1(e) \Downarrow v$ and $\gamma_2(e) \Downarrow v'$ such that $(v, v') \in \hat{\mathbb{N}}$.
  Hence $v = v' = \mathsf{s}^k(0)$ for some k.
  By evaluation rules, $\mathsf{s}(e) \Downarrow \mathsf{s}^{k+1}(0)$ and $\mathsf{s}(e') \Downarrow \mathsf{s}^{k+1}(0)$.
  By definition $(\mathsf{s}^{k+1}(0), \mathsf{s}^{k+1}(0)) \in \hat{\mathbb{N}}$.

- **Case** INE: $\dfrac{\Gamma \vdash C : \mathbb{N} \to U \qquad \Gamma \vdash e : \mathbb{N} \qquad \Gamma \vdash e_0 : C\ 0 \qquad \Gamma, x, y : C\ x \vdash e_1 : C(\mathsf{s}(x))}{\Gamma \vdash \mathsf{iter}(e, 0 \to e_0, \mathsf{s}(x), y \to e_1) : C\ e}$
  By induction, $(\gamma_1(e), \gamma_2(e)) \in \phi(\gamma(\mathbb{N})) = \hat{\mathbb{N}}$.
  Hence $\gamma_1(e) \Downarrow v_1$ and $\gamma_2(e) \Downarrow v_2$ such that $(v_1, v_2) \in \phi(\mathbb{N}) = \hat{\mathbb{N}}$.
  Hence $v_1 = v_2 = \mathsf{s}^k(0)$.
  We proceed by nested induction on k:
  - Case $k = 0$:
    Then $v_1 = v_2 = 0$.
    By induction, $(\gamma_1(e_0), \gamma_2(e_0)) \in \phi(\gamma_1(C\ 0))$.
    Hence there are v_i' such that $\gamma_i(e_0) \Downarrow v_i'$ such that $(v_1', v_2') \in \phi(\gamma_1(C\ 0))$.
    We want to show that $\phi(\gamma_1(C\ z)) = \phi(\gamma_1(C\ e))$.
    By induction, we know $(\gamma_1(C), \gamma_2(C)) \in \phi(\mathbb{N} \to U)$.
    Hence $(\gamma_1(C), \gamma_1(C)) \in \phi(\mathbb{N} \to U)$.
    We know $(\gamma_1(e), 0) \in \hat{\mathbb{N}}$.

Hence $(\gamma_1(C)\ \gamma_1(e), \gamma_1(C)\ 0) \in I$.
By properties of substitution $(\gamma_1(C\ e), \gamma_1(C\ 0) \in I$.
Since ϕ respects I, we know $\phi(\gamma_1(C\ e)) = \phi\gamma_1(C\ 0)$.
By reduction relation, $\mathsf{iter}(\gamma_i(e), 0 \to \gamma_i(e_0), \mathsf{s}(x), y \to \gamma_i(e_1)) \Downarrow v_i'$.
Hence $\gamma(\mathsf{iter}(e_0, 0 \to x, \mathsf{s}(y), e_1 \to)) \in \phi(\gamma_1(C\ e))$.

- Case $k = j + 1$:
  Then $v_1 = v_2 = \mathsf{s}^{j+1}(0)$.
  By nested induction, $\gamma(\mathsf{iter}(\mathsf{s}^j(0), 0 \to e_0, \mathsf{s}(x), y \to e_1)) \in \phi(\gamma(C\ \mathsf{s}^j(0)))$.
  Note that $(\mathsf{s}^j(0), \mathsf{s}^j(0)) \in \hat{\mathbb{N}}$.
  Hence $(\gamma, (\mathsf{s}^j(0), \mathsf{s}^j(0))/x, \gamma(\mathsf{iter}(\mathsf{s}^j(0), 0 \to e_0, \mathsf{s}(x), y \to e_1))/y) \in [\![\Gamma, x : X, y : C\ x]\!]$.
  By induction, $(\gamma, (\mathsf{s}^j(0), \mathsf{s}^j(0))/x, \gamma(\mathsf{iter}(\mathsf{s}^j(0), 0 \to e_0, \mathsf{s}(x), y \to e_1))/y)e_1 \in \phi((\gamma, (\mathsf{s}^j(0), \mathsf{s}^j(0))/x, \gamma(\mathsf{iter}(\mathsf{s}^j$ $e_0, \mathsf{s}(x), y \to e_1))/y)(C(\mathsf{s}(x))))$.
  Simplifying, $[(\mathsf{s}^j(0), \mathsf{s}^j(0))/x, \gamma(\mathsf{iter}(\mathsf{s}^j(0), 0 \to e_0, \mathsf{s}(x), y \to e_1))/y]e_1 \in \phi(\gamma_1(C\ \mathsf{s}^{j+1}(0)))$.
  By a similar argument to the previous case, $\phi(\gamma_1(C\ \mathsf{s}^{j+1}(0))) = \phi(\gamma_1(C\ e))$.

- **Case** IFUNI: $\dfrac{\Gamma, x : X \vdash e : Y}{\Gamma \vdash \lambda x.\ e : \Pi x : X.\ Y}$
  By induction and $\Gamma, x : X$ ok, we have $\forall (e_1', e_2') \in \phi(\gamma_1(X)), ((\gamma_1, e_1'/x)(e), (\gamma_2, e_2'/x)(e) \in \phi([e_1'/x]\gamma_1(Y))$, which directly implies that
  $$\gamma_1(\lambda x.\ e) = \lambda x.\ \gamma_1(e) \in \phi(\Pi x : \gamma_1(X).\ \gamma_1(Y)) = \phi(\gamma_1(\Pi x : X.\ Y))$$
- **Case** IFUNE: $\dfrac{\Gamma \vdash e : \Pi x : X.\ Y \qquad \Gamma \vdash e' : X}{\Gamma \vdash e\ e' : [e'/x]Y}$
  By induction, we have:
  - $(\gamma_1(e'), \gamma_2(e')) \in \phi(\gamma_1(X))$
  - $(\gamma_1(e), \gamma_2(e)) \in \phi(\Pi x : \gamma_1(X).\ \gamma_1(Y))$

  This second hypothesis tells us that
  $$\forall (e_1'', e_2'') \in \phi(\gamma_1(X)), (\gamma_1(e)\ e_1'', \gamma_2(e)\ e_2'') \in \phi([e_1''/x]\gamma_1(Y))$$
  In particular, $(\gamma_1(e\ e'), \gamma_2(e\ e')) \in \phi([\gamma_1(e')/x]\gamma_1(Y))$.
- **Case** IEQI: $\dfrac{\Gamma \vdash e \equiv e' : X}{\Gamma \vdash \mathsf{refl} : e =_X e'}$
  Notice that $(\gamma_1, \gamma_1) \in [\![\Gamma]\!]$.
  By induction, $(\gamma_1(e), \gamma_1(e')) \in \phi(\gamma_1(X))$ at some $\mathcal{T}_i$.
  But then it means that $(\mathsf{refl}, \mathsf{refl}) \in \phi(\gamma_1(e) =_{\gamma_1(X)} \gamma_1(e'))$ at $T_i(\mathcal{T}_i) = \mathcal{T}_i$, which is what we want.
- **Case** IGI: $\dfrac{\Gamma; \cdot \vdash t : A}{\Gamma \vdash \mathsf{G}\,t : \mathsf{G}\,A}$
  By induction, $\gamma(t) \in \psi(\gamma_1(A))\epsilon$, which is what we need.
- **Case** : $\dfrac{\Gamma, n : \mathbb{N} \vdash \Pi x : X[n].\ Y[n]\ \mathsf{type} \quad \Gamma, f : \top_I, x : X(0) \vdash e : Y(0) \qquad \Gamma, n : \mathbb{N}, f : \Pi x : X[n].\ Y[n], x : X[\mathsf{s}(n)] \vdash e : Y[\mathsf{s}(n)]}{\Gamma \vdash \mathsf{fix}\ f\ x = e : \forall n : \mathbb{N}.\ \Pi x : X[n].\ Y[n]}$
  Assume $\gamma \in [\![\Gamma]\!]$.
  We want to show that $\gamma(\mathsf{fix}\ f\ x = e) \in \phi(\gamma_1(\forall n : \mathbb{N}.\ \Pi x : X[n].\ Y[n]))$.
  So it suffices to show that $\gamma(\mathsf{fix}\ f\ x = e) \in \phi(\forall n : \mathbb{N}.\ \gamma(\Pi x : X[n].\ Y[n]))$.
  To show this, assume $(e_0, e_0') \in \phi(\mathbb{N})$.
  Hence $e_0 \Downarrow \mathsf{s}^k(0)$ and $e_0' \Downarrow \mathsf{s}^k(0)$ and $(\mathsf{s}^k(0), \mathsf{s}^k(0)) \in \phi(\mathbb{N})$.
  Hence we want to show that $\gamma(\mathsf{fix}\ f\ x = e) \in \phi((\gamma_1, e_0/n)(\Pi x : X[n].\ Y))$. We proceed by nested induction on k, to show that $\gamma(\mathsf{fix}\ f\ x = e) \in \phi((\gamma_1, (\mathsf{s}^k(0), \mathsf{s}^k(0))/n)(\Pi x : X[n].\ Y))$.

– Case $k = 0$: We want to show $\gamma(\mathsf{fix}\ f\ x = e) \in \phi((\gamma_1, 0/n)(\Pi x : X[n].\ Y[n]))$.
By properties of substitution, it suffices to show $\gamma(\mathsf{fix}\ f\ x = e) \in \phi(\gamma_1(\Pi x : X[0].\ Y[0]))$.
So for all $(t_1, t_2) \in \phi(\gamma_1(X[0]))$, we want to show that $(\gamma_1(\mathsf{fix}\ f\ x = e)\ t_1, \gamma_2(\mathsf{fix}\ f\ x = e)\ t_2) \in \phi((\gamma_1, t_1/x)Y[0])$.
Note that $(\gamma, (t_1, t_2)/x) \in [\![\Gamma, x : X[0]]\!]$.
Note that $\gamma(\mathsf{fix}\ f\ x = e) \in \phi(\top_I)$.
Hence $(\gamma, \gamma(\mathsf{fix}\ f\ x = e)/f, (t_1, t_2)/x) \in [\![\Gamma, f : \top_I, x : X[0]]\!]$.
By induction, $(\gamma, \gamma(\mathsf{fix}\ f\ x = e)/f, (t_1, t_2)/x)e \in \phi((\gamma_1, t_1/x)Y[0])$.
So $(\gamma_1, \gamma_1(\mathsf{fix}\ f\ x = e)/f, t_1/x)e \Downarrow v_1$
and $(\gamma_2, \gamma_2(\mathsf{fix}\ f\ x = e)/f, t_2/x)e \Downarrow v_2$
such that $(v_1, v_2) \in \phi((\gamma_1, t_1/x)Y[0])$.
By properties of substitution, $(\gamma_i(\mathsf{fix}\ f\ x = e)/f, t_i/x)(\gamma_i(e)) \Downarrow v_i$.
By evaluation rules, $eval(\mathsf{fix}\ f\ x = \gamma_i(e))\ t_i v_i$.
Hence $\gamma(\mathsf{fix}\ f\ x = e) \in \phi((\gamma_1, 0/n)(\Pi x : X[n].\ Y[n]))$.

– Case $k = j + 1$: By induction, we know $\gamma(\mathsf{fix}\ f\ x = e) \in \phi((\gamma_1, \mathsf{s}^j(0)/n)(\Pi x : X[n].\ Y[n]))$.
We want to show that $\gamma(\mathsf{fix}\ f\ x = e) \in \phi((\gamma_1, \mathsf{s}^{j+1}(0)/n)(\Pi x : X[n].\ Y[n]))$.
So for all $(t_1, t_2) \in \phi(\gamma_1(X[\mathsf{s}^{j+1}(0)]))$, we want to show that $(\gamma_1(\mathsf{fix}\ f\ x = e)\ t_1, \gamma_2(\mathsf{fix}\ f\ x = e)\ t_2) \in \phi((\gamma_1, t_1/x)Y[\mathsf{s}^{j+1}(0)])$.
Note that $(\gamma, (t_1, t_2)/x) \in [\![\Gamma, x : X[\mathsf{s}^{j+1}(0)]]\!]$.
Hence $(\gamma, \gamma(\mathsf{fix}\ f\ x = e)/f, (t_1, t_2)/x) \in [\![\Gamma, f : \Pi x : X[n].\ Y[n], x : X[\mathsf{s}^{j+1}(0)]]\!]$.
By induction (and $n \notin FV(\mathsf{fix}\ f\ x = e)$), we have$(\gamma, \gamma(\mathsf{fix}\ f\ x = e)/f, (t_1, t_2)/x)e \in \phi((\gamma_1, t_1/x)Y[\mathsf{s}^{j+1}(0)])$.
So $(\gamma_1, \gamma_1(\mathsf{fix}\ f\ x = e)/f, t_1/x)e \Downarrow v_1$
and $(\gamma_2, \gamma_2(\mathsf{fix}\ f\ x = e)/f, t_2/x)e \Downarrow v_2$
such that $(v_1, v_2) \in \phi((\gamma_1, t_1/x)Y[\mathsf{s}^{j+1}(0)])$.
By properties of substitution, $(\gamma_i(\mathsf{fix}\ f\ x = e)/f, t_i/x)(\gamma_i(e)) \Downarrow v_i$.
By evaluation rules, $eval(\mathsf{fix}\ f\ x = \gamma_i(e))\ t_i v_i$.
Hence $\gamma(\mathsf{fix}\ f\ x = e) \in \phi((\gamma_1, \mathsf{s}^{j+1}(0)/n)(\Pi x : X[n].\ Y[n]))$.

Since $(e_0, \mathsf{s}^k(0)) \in \phi(\mathbb{N})$, by induction $(\gamma_1, (e_0, \mathsf{s}^k(0))/n)(\Pi x : X[n].\ Y[n]) \in I$.
Then by PER properties and the fact ϕ respects PERs,
we have $\phi((\gamma_1, \mathsf{s}^{j+1}(0)/n)(\Pi x : X[n].\ Y[n])) = \phi((\gamma_1, e_0/n)(\Pi x : X[n].\ Y[n]))$.
Hence $\gamma(\mathsf{fix}\ f\ x = e) \in \phi((\gamma_1, e_0 n)(\Pi x : X[n].\ Y[n]))$.

- **Case** : $\dfrac{\Gamma, x : X \vdash e : Y \qquad x \notin FV(e)}{\Gamma \vdash e : \forall x : X.\ Y}$
By induction and $\Gamma, x : X$ ok, we have $\forall (e_1', e_2') \in \phi(\gamma_1(X)), ((\gamma_1, e_1'/x)(e), (\gamma_2, e_2'/x)(e)) \in \phi([e_1'/x]\gamma_1(Y))$, so since x is free in e, $\forall (e_1', e_2') \in \psi(\gamma_1(X)), (\gamma_1(e), \gamma_2(e) \in \phi([e_1'/x]\gamma_1(Y))$. which directly implies that
$$\gamma_1(e) \in \phi(\forall x : \gamma_1(X).\ \gamma_1(Y)) = \phi(\gamma_1(\forall x : X.\ Y))$$

- **Case** : $\dfrac{\Gamma \vdash e : \forall x : X.\ Y \qquad \Gamma \vdash e' : X}{\Gamma \vdash e : [e'/x]Y}$
By induction, $(\gamma_1(e), \gamma_2(e)) \in \phi(\gamma_1(\forall x : X.\ Y))$, which means there exits $(e_1'', e_2'') \in \phi(\gamma_1(\forall x : X.\ Y))$ such that for every $(e_1''', e_2''') \in \phi(\gamma_1(X))$, we have
$$\gamma_1(e) \Downarrow e_1'' \wedge \gamma_2(e) \Downarrow e_2''$$
$$(e_1'', e_2'') \in \phi([e_1'''/x]\gamma_1(Y))$$
Thus by compatibility with reduction, it means
$$(\gamma_1(e), \gamma_2(e)) \in \phi([e_1'''/x]\gamma_1(Y))$$

Now we can take $(e_1''', e_2''') := (\gamma_1(e'), \gamma_2(e'))$ and conclude.

- **Case** : $\dfrac{\Gamma, x : X, y : Y \vdash e : Z \qquad x \notin FV(e)}{\Gamma, y : \exists x : X.\ Y \vdash e : Z}$
  Let $(\gamma_1, \gamma_2) \in [\![\Gamma]\!]$ and $(e_1', e_2') \in \phi(\gamma_1(\exists x : X.\ Y))$.
  That last fact tells us there exists some $(e_1'', e_2'') \in \phi(\gamma_1(X))$ such that $(e_1', e_2') \in \phi(\gamma_1([e_1''/x]Y))$ modulo a bit of reasoning with reductions.
  Thus, by noticing that $((\gamma_1, e_1''/x, e_1'/y), (\gamma_2, e_2''/x, e_1'/y)) \in [\![\Gamma, x : X, y : Y]\!]$, by induction (and $x \notin FV(e)$) we have
  $$(\gamma_1([e_1'/y]e), \gamma_2([e_2'/y]e)) \in \phi(\gamma_1(Z))$$
- **Case** : $\dfrac{\Gamma, x : X \vdash Y \text{ type} \qquad \Gamma \vdash e' : X \qquad \Gamma \vdash e : [e'/x]Y}{\Gamma \vdash e : \exists x : X.\ Y}$
  By induction, we have:
  - $(\gamma_1(e'), \gamma_2(e')) \in \phi(\gamma_1(X))$
  - $((\gamma_1, \gamma_1(e')/x)(e), (\gamma_2, \gamma_2(e')/x)(e)) \in \phi((\gamma_1, \gamma_1(e')/x)(Y))$

  Which gives us the result once we notice $x \notin FV(e)$ thanks to typing.

4. If $\Gamma \vdash e_1 \equiv e_2 : X$ then $(\gamma_1(e_1), \gamma_2(e_2)) \in \phi(\gamma_1(X))$.
   We case analyze the derivation of $\Gamma \vdash e_1 \equiv e_2 : X$:

   - **Case** IFUNBETA: $\overline{\Gamma \vdash (\lambda x.\ e)\ e' \equiv [e'/x]e : Z}$
     By induction:
     - $(\gamma_1([e'/x]e), \gamma_2([e'/x]e)) \in \phi(\gamma_1(Z))$
     - $(\gamma_1((\lambda x.\ e)\ e'), \gamma_2((\lambda x.\ e)\ e')) \in \phi(\gamma_1(Z))$

     This means that we have (e_1'', e_2'') such that
     $$\gamma_1([e'/x]e) \Downarrow e_1''$$
     $$\gamma_2([e'/x]e) \Downarrow e_2''$$
     We then build
     $$\frac{\lambda x.\ \gamma_1(e) \Downarrow \lambda x : A.\ \gamma_1(e) \qquad [\gamma_1(e')/x]\gamma_1(e) \Downarrow e_1''}{\gamma_1(\lambda x.\ e\ e') \Downarrow e_1''}$$
     Since our PER $\sim := \phi(\gamma_1(Z))$ is closed under evaluation, we have
     $$\gamma_1((\lambda x.\ e)\ e') \sim e_1'' \sim e_2'' \sim \gamma_2([e'/x]e)$$
   - **Case** IFUNETA: $\overline{\Gamma \vdash e \equiv \lambda x.\ e\ x : \Pi x : X.\ Y}$
     By induction
     $$(\gamma_1(e), \gamma_2(e)) \in \phi(\Pi x : \gamma_1(X).\ \gamma_1(Y))$$
     $$(\gamma_1(\lambda x.\ e), \gamma_2(\lambda x.\ e)) \in \phi(\Pi x : \gamma_1(X).\ \gamma_1(Y))$$
     Let $(e_1', e_2') \in \phi(\gamma_1(X))$.
     We know that there is $(v_1, v_2) \in \phi((\gamma_1, e_1')/x)(Y))$ such that
     $$\gamma_1(e)\ e_1' \Downarrow v_1$$
     $$\gamma_2(e)\ e_2' \Downarrow v_2$$

We can use this to build a derivation

$$\frac{\lambda x.\ \gamma_2(e)\ x \Downarrow \lambda x : A.\ \gamma_2(e)\ x \qquad \gamma_2(e)\ e_2' \Downarrow v_2}{\lambda x.\ \gamma_2(e)\ x\ e_2' \Downarrow v_2}$$

Then we have $(\gamma_1(e)\ e_1', \gamma_2(\lambda x.\ e)\ e_2') \in \phi((\gamma_1, e_1'/x)Y)$ which is what we need.

- **Case** IPAIRBETAFST: $\overline{\Gamma \vdash \pi_1\ (e, e') \equiv e : Z}$ By induction $(\gamma_1(e), \gamma_2(e)) \in \phi(\gamma_1(\Sigma x : X.\ Y))$, so

  $$\gamma_1(e) \Downarrow v_1 \wedge \gamma_2(e) \Downarrow v_2$$

  $$(v_1, v_2) \in \phi(\gamma_1(Z))$$

  Thus

  $$\frac{\gamma_1((e, e')) \Downarrow \gamma_1((e, e')) \qquad \gamma_1(e) \Downarrow v_1}{\gamma_1(\pi_1\ (e, e')) \Downarrow v_1}$$

  so $(\gamma_1(\pi_1\ (e, e')), \gamma_2(e)) \in \phi(\gamma_1(Z))$.

- **Case** IPAIRBETASND: $\overline{\Gamma \vdash \pi_2\ (e, e') \equiv e' : Z}$
  By induction $(\gamma_1(e), \gamma_2(e)) \in \phi(\gamma_1(\Sigma x : X.\ Y))$, so

  $$(\gamma_1(e'), \gamma_2(e')) \in \phi(\gamma_1(Z))$$

  $$\gamma_1(e') \Downarrow v_1' \wedge \gamma_2(e') \Downarrow v_2'$$

  Thus

  $$(v_1', v_2') \in \phi(\gamma_1(Z))$$

  $$\frac{\gamma_1((e, e')) \Downarrow \gamma_1((e, e')) \qquad \gamma_1(e') \Downarrow v_1'}{\gamma_1(\pi_2\ (e, e')) \Downarrow v_1'}$$

  so $(\gamma_1(\pi_2\ (e, e')), \gamma_2(e')) \in \phi(\gamma_1(Z))$.

- **Case** IPAIRETA: $\overline{\Gamma \vdash e \equiv (\pi_1\ e, \pi_2\ e) : \Sigma x : X.\ Y}$
  By induction $(\gamma_1(e), \gamma_2(e)) \in \phi(\gamma_1(\Sigma x : X.\ Y))$, so there exists e_1', e_2', e_1'', e_2'' such that

  $$\gamma_1(e) \Downarrow (e_1', e_1'')$$

  $$\gamma_2(e) \Downarrow (e_2', e_2'')$$

  $$(e_1', e_2') \in \phi(\gamma_1(X))$$

  $$(e_1'', e_2'') \in \phi((\gamma_1, e_1'/x)(Y))$$

  It suffices to show that $((e_1', e_1''), \gamma_2((\pi_1\ e, \pi_2\ e))) \in \phi(\gamma_1(\Sigma x : X.\ Y))$
  We can build the following derivations

  $$\frac{\gamma_2(e) \Downarrow (e_2', e_2'') \qquad e_2' \Downarrow e_2'}{\pi_1\ \gamma_2(e) \Downarrow e_2'} \qquad\qquad \frac{\gamma_2(e) \Downarrow (e_2', e_2'') \qquad e_2'' \Downarrow e_2''}{\pi_2\ \gamma_2(e) \Downarrow e_2''}$$

  Thus we have

  $$(e_1', \pi_1\ \gamma_2(e)) \in \phi(\gamma_1(X))$$

  $$(e_1'', \pi_2\ \gamma_2(e)) \in \phi((\gamma_1, e_1'/x)(Y))$$

  Which brings us the conclusion by the definition of Σ.

- **Case** IUNITETA: $\overline{\Gamma \vdash e \equiv e' : 1}$
  By induction $(\gamma_1(e), \gamma_2(e)) \in \phi(1)$, so
  $$\gamma_1(e) \Downarrow () \wedge \gamma_2(e') \Downarrow ()$$
  and $((), ()) \in \phi(1)$.
- **Case** IGBETA: $\overline{\Gamma \vdash \mathsf{G}\,(\mathsf{G}^{-1}\,e) \equiv e : \mathsf{G}\,A}$
  By induction, $(\gamma_1(e), \gamma_2(e)) \in \phi(\gamma_1(A))$. Thus there is (t_1, t_2) such that
  $$\gamma_1(e) \Downarrow \mathsf{G}\,t_1$$
  $$\gamma_2(e) \Downarrow \mathsf{G}\,t_2$$
  $$((\epsilon, t_1), (\epsilon, t_2)) \in \psi(\gamma_1(A))$$
  Hence there is $((\sigma_1, u_1), (\sigma_2, u_2)) \in \psi(\gamma_1(A))$ such that
  $$\langle \epsilon; t_1 \rangle \Downarrow \langle \sigma_1; u_1 \rangle \wedge \langle \epsilon; t_2 \rangle \Downarrow \langle \sigma_2; u_2 \rangle$$
  $$\frac{\gamma_1(e) \Downarrow \mathsf{G}\,e_1' \qquad \langle \epsilon; t_1 \rangle \Downarrow \langle \sigma_1; u_1 \rangle}{\left\langle \epsilon; \mathsf{G}^{-1}\,\gamma_1(e) \right\rangle \Downarrow \langle \sigma_1; u_1 \rangle}$$
  Thus, we have
  $$((\epsilon, \mathsf{G}^{-1}\,\gamma_1(e)), (\epsilon, t_2)) \in \psi(\gamma_1(A))$$
  So from the definition of G,
  $$(\mathsf{G}\,(\mathsf{G}^{-1}\,\gamma_1(e)), \mathsf{G}\,t_2) \in \phi(\gamma_1(A))$$
  and we can conclude by recalling $(\gamma_2(e), \mathsf{G}\,t_2) \in \phi(\gamma_1(A))$.
- **Case** IALLETA: $\dfrac{\Gamma, x : X \vdash e \equiv e' : Y}{\Gamma \vdash e \equiv e' : \forall x : X.\ Y}$
  Since $x \notin \mathrm{FV}(e, e')$, the result follows directly from the induction hypothesis which tells us $\forall (e'', e''') \in \phi(\gamma_1(X)), (\gamma_1(e), \gamma_2(e')) \in \phi(\gamma_1([e''/x]\forall x : X.\ Y))$
- **Case** IALLBETA: $\dfrac{\Gamma \vdash e \equiv e' : \forall x : X.\ Y \qquad \Gamma \vdash t : X}{\Gamma \vdash e \equiv e' : [t/x]Y}$
  By induction $(\gamma_1(e), \gamma_2(e')) \in \phi(\gamma_1(\forall x : X.\ Y))$ and $(\gamma_1(t), \gamma_2(t)) \in \phi(\gamma_1(X))$.
  Thus we have values $(v_1, v_2) \in \phi(\gamma_1(\forall x : X.\ Y))$ such that $\gamma_1(e) \Downarrow v_1$, $\gamma_2(e') \Downarrow v_2$ and $(v_1, v_2) \in \phi([\gamma_1(t)/x]\gamma_1(Y))$.
  Hence $(\gamma_1(e), \gamma_2(e')) \in \phi([\gamma_1(t)/x]\gamma_1(Y))$.
- **Case** IEXBETA: $\dfrac{\Gamma \vdash e \equiv e' : [t/x]Y \qquad \Gamma \vdash t : X}{\Gamma \vdash e \equiv e' : \exists x : X.\ Y}$
  By induction
  $$(\gamma_1(e), \gamma_2(e')) \in \phi(\gamma_1([t/x]Y))$$
  $$(\gamma_1(t), \gamma_2(t)) \in \phi(\gamma_1(X))$$
  which gives us our result by taking $\gamma_1(t)$ as our witness.

- **Case** IExEta: $\dfrac{\Gamma, x : X, y : Y \vdash e \equiv e' : Z \qquad x \notin FV(e, e', Z)}{\Gamma, y : \exists x : X.\ Y \vdash e \equiv e' : Z}$
  Let $\gamma \in [\![\Gamma]\!]$ and $(t_1, t_2) \in \phi(\gamma_1(\exists x : X.\ Y))$.
  By this second hypothesis, there exists $(t', t') \in \phi(\gamma_1(X))$ such that $(t_1, t_2) \in \phi(\gamma_1([t'/x]Y))$. Thus $((\gamma_1, t'/x, t_1/y), (\gamma_2, t'/x, t_2/y)) \in [\![\Gamma]\!]$. By induction, since $x \notin FV(e, e', Z)$,

  $$([t_1/y]\gamma_1(e), [t_2/y]\gamma_2(e)) \in \phi([t_1/y]\gamma_1(Z))$$

  which is what we wanted.

- **Case** IFixBeta: $\overline{\Gamma \vdash (\mathsf{fix}\ f\ x = e)\ e' \equiv [(\mathsf{fix}\ f\ x = e)/f, e'/x]e : Z}$
  Let $\gamma \in [\![\Gamma]\!]$.
  By induction, $\gamma((\mathsf{fix}\ f\ x = e)\ e') \in \phi(\gamma_1(Z))$.
  By induction, $\gamma([(\mathsf{fix}\ f\ x = e)/f, e'/x]e) \in \phi(\gamma_1(Z))$.
  So $(\mathsf{fix}\ f\ x = \gamma(e))\ \gamma(e') \in \phi(\gamma_1(Z))$.
  So, $[(\gamma(\mathsf{fix}\ f\ x = e))/f, \gamma(e')/x]\gamma(e) \in \phi(\gamma_1(Z))$.
  Hence $[(\gamma_i(\mathsf{fix}\ f\ x = e))/f, \gamma_i(e')/x]\gamma_i(e) \Downarrow v_i$ such that $(v_1, v_2) \in \phi(\gamma_1(Z))$.
  By evaluation rules, $(\mathsf{fix}\ f\ x = \gamma_i(e))\ \gamma_i(e') \Downarrow v_i$.
  Hence $((\mathsf{fix}\ f\ x = \gamma_1(e))\ \gamma_1(e'), [(\gamma_2(\mathsf{fix}\ f\ x = e))/f, \gamma_2(e')/x]\gamma_2(e)) \in \phi(\gamma_1(Z))$.

- **Case** IReflect: $\dfrac{\Gamma \vdash p : e =_X e'}{\Gamma \vdash e \equiv e' : X}$
  The statement of the induction hypothesis and the conclusion are the same thing.

- **Case** K: $\dfrac{\Gamma \vdash p : e =_X e \qquad \Gamma \vdash q : e =_X e}{\Gamma \vdash p \equiv q : e =_X e}$
  By induction

  $$(\gamma_1(p), \gamma_1(p)) \in \phi(\gamma_1(e =_X e))$$
  $$(\gamma_1(q), \gamma_2(q)) \in \phi(\gamma_1(e =_X e))$$

  Thus, we have

  $$\gamma_1(p) \Downarrow \mathsf{refl} \wedge \gamma_2(q) \Downarrow \mathsf{refl}$$
  $$(\gamma_1(e), \gamma_1(e)) \in \phi(\gamma_1(X))$$

  Thus by compatibility of evalutation with PERs, we have $(\gamma_1(p), \gamma_2(q)) \in \phi(\gamma_1(e =_X e))$.

- **Case** IReflex: $\dfrac{\Gamma \vdash e : X}{\Gamma \vdash e \equiv e : X}$
  The statement of the induction hypothesis and the conclusion are the same thing.

- **Case** ITrans: $\dfrac{\Gamma \vdash e \equiv e' : X \qquad \Gamma \vdash e' \equiv e'' : X}{\Gamma \vdash e \equiv e'' : X}$
  Let $\gamma \in [\![\Gamma]\!]$. We also know $[\![\Gamma]\!]$ to be reflexive, thus by induction:
  - $(\gamma_1(e), \gamma_1(e')) \in \phi(\gamma_1(X))$
  - $(\gamma_1(e'), \gamma_2(e'')) \in \phi(\gamma_1(X))$

  Since PERs are transitive, $(\gamma_1(e), \gamma_2(e'')) \in \phi(\gamma_1(X))$.

- **Case** ILolliCong: $\dfrac{\Gamma \vdash A \equiv A' : L_i \qquad \Gamma \vdash B \equiv B' : L_i}{\Gamma \vdash A \multimap B \equiv A' \multimap B' : L_i}$
  By induction, we have

  $$(\gamma_1(A), \gamma_2(A')) \in \phi(L_i)$$
  $$(\gamma_1(B), \gamma_2(B')) \in \phi(L_i)$$

But we know that the L-component of $T_i(\phi(L_i))$ is $\phi(L_i)$. Thus, by the definition of T_i we have our result

$$(\gamma_1(A \multimap B), \gamma_2(A' \multimap B')) \in \phi(L_i)$$

- **Case** ITENSORCONG: $\dfrac{\Gamma \vdash A \equiv A' : L_i \qquad \Gamma \vdash B \equiv B' : L_i}{\Gamma \vdash A \otimes B \equiv A' \otimes B' : L_i}$
  Similar to ILOLLICONG.
- **Case** IWITHCONG: $\dfrac{\Gamma \vdash A \equiv A' : L_i \qquad \Gamma \vdash B \equiv B' : L_i}{\Gamma \vdash A \,\&\, B \equiv A' \,\&\, B' : L_i}$
  Similar to ILOLLICONG.
- **Case** ITCONG: $\dfrac{\Gamma \vdash A \equiv A' : L_i}{\Gamma \vdash \mathsf{T}\,A \equiv \mathsf{T}\,A' : L_i}$
  Similar to ILOLLICONG.
- **Case** IPICONG: $\dfrac{\Gamma \vdash X \equiv X' : U_i \qquad \Gamma, x : X \vdash Y \equiv Y' : U_i}{\Gamma \vdash \Pi x : X.\ Y \equiv \Pi x : X'.\ Y' : U_i}$
  Let $(e_1, e_2) \in \phi(\gamma_1(X))$.
  By definition, $(\gamma, (e, e')/x) \in [\![\Gamma, x : X]\!]$ By induction, we have

  $$(\gamma_1(X), \gamma_2(X')) \in \phi(U_i)$$

  $$((\gamma_1, e_1/x)(B), (\gamma_2, e_2/x)(B')) \in \phi(U_i)$$

  We know that the U-component of $\mathcal{T}_i$ is $\phi(U)$.
  Thus, by universal quantification of (e_1, e_2) and the stability under T_i,

  $$(\gamma_1(\Pi x : X.\ Y), \gamma_2(\Pi x : X'.\ Y')) \in \phi(U_i)$$

- **Case** ILPICONG: $\dfrac{\Gamma \vdash X \equiv X' : U_i \qquad \Gamma, x : X; \Delta \vdash A \equiv A' : L_i}{\Gamma \vdash \Pi x : X.\ A \equiv \Pi x : X'.\ A' : L_i}$
  Similar to IPICONG.
- **Case** ISIGMACONG: $\dfrac{\Gamma \vdash X \equiv X' : U_i \qquad \Gamma, x : X \vdash Y \equiv Y' : U_i}{\Gamma \vdash \Sigma x : X.\ Y \equiv \Sigma x : X'.\ Y' : U_i}$
  Similar to IPICONG.
- **Case** IALLCONG: $\dfrac{\Gamma \vdash X \equiv X' : U_i \qquad \Gamma, x : X \vdash Y \equiv Y' : U_i}{\Gamma \vdash \forall x : X.\ Y \equiv \forall x : X'.\ Y' : U_i}$
  Similar to IPICONG.
- **Case** IEXCONG: $\dfrac{\Gamma \vdash X \equiv X' : U_i \qquad \Gamma, x : X \vdash Y \equiv Y' : U_i}{\Gamma \vdash \exists x : X.\ Y \equiv \exists x : X'.\ Y' : U_i}$
  Similar to IPICONG.
- **Case** LALLCONG: $\dfrac{\Gamma \vdash X \equiv X' : U_i \qquad \Gamma, x : X \vdash Y \equiv Y' : L_i}{\Gamma \vdash \forall x : X.\ Y \equiv \forall x : X'.\ Y' : L_i}$
  Similar to IPICONG.
- **Case** LEXCONG: $\dfrac{\Gamma \vdash X \equiv X' : U_i \qquad \Gamma, x : X \vdash Y \equiv Y' : L_i}{\Gamma \vdash \exists x : X.\ Y \equiv \exists x : X'.\ Y' : L_i}$
  Similar to IPICONG.
- **Case** IFCONG: $\dfrac{\Gamma \vdash X \equiv X' : U_i \qquad \Gamma, x : X \vdash A \equiv A' : L_i}{\Gamma \vdash \mathsf{F} x : X.\ A \equiv \mathsf{F} x : X'.\ A' : U_i}$
  Similar to IPICONG.

- **Case** IPtrCong: $\dfrac{\Gamma \vdash X \equiv X' : \mathsf{U}_i \qquad \Gamma \vdash e_1 \equiv e_1' : \mathsf{Loc}}{\Gamma \vdash e_1 \mapsto X \equiv e_1' \mapsto X' : \mathsf{U}_i}$
  By induction
  $$(\gamma_1(X), \gamma_2(X')) \in \phi(\mathsf{U}_i)$$
  $$(\gamma_1(e_1), \gamma_2(e_1')) \in \phi(\mathsf{Loc})$$
  We know that the U-component of $\mathcal{T}_i$ is $\phi(U)$.
  Thus we can conclude thanks to the stability under T_i of $\mathcal{T}_i$.
- **Case** IEqCong: $\dfrac{\Gamma \vdash X \equiv X' : \mathsf{U}_i \qquad \Gamma \vdash e_1 \equiv e_2 : X \qquad \Gamma \vdash e_1' \equiv e_2' : X'}{\Gamma \vdash e_1 =_X e_2 \equiv e_1' =_{X'} e_2' : \mathsf{U}_i}$
  Similar to IPtrCong.
- **Case** IFunCong: $\dfrac{\Gamma, x : X \vdash e \equiv e' : Y}{\Gamma \vdash \lambda x : X.\ e \equiv \lambda x : X.\ e' : \Pi x : X.\ Y}$
  Notice that for all $e'' \in \phi(\gamma_1(X))$, $(\gamma, e''/x) \in [\![\Gamma, x : X]\!]$.
  Thus by induction
  $$\forall e'' \in \phi(\gamma_1(X)), ([e_1''/x]\gamma_1(e), [e_2''/x]\gamma_2(e)) \in \phi([e_1''/x]\gamma_1(Y))$$
  which gives us the result we want by definition of the Π operator in the semantics.
- **Case** IAppCong: $\dfrac{\Gamma \vdash e_1 \equiv e_1' : \Pi x : X.\ Y \qquad \Gamma \vdash e_2 \equiv e_2' : X}{\Gamma \vdash e_1\ e_2 \equiv e_1'\ e_2' : Y[e_2/x]}$
  By induction
  $$(\gamma_1(e_1), \gamma_2(e_1')) \in \phi(\gamma_1(\Pi x : X.\ Y))$$
  $$(\gamma_1(e_2), \gamma_2(e_2')) \in \phi(\gamma_1(X))$$
  Thus there exists (u_1, u_2) such that
  $$\gamma_1(e_1) \Downarrow \lambda x.\ u_1 \wedge \gamma_2(e_1') \Downarrow \lambda x.\ u_2$$
  $$([\gamma_1(e_2)/x]u_1, [\gamma_2(e_2')/x]u_2) \in \phi([\gamma_1(e_2)/x]\gamma_1(Y))$$
  There exists (v_1, v_2) such that
  $$[\gamma_1(e_2)/x]u_1 \Downarrow v_1 \wedge [\gamma_2(e_2')/x]u_2 \Downarrow v_2$$
  $$(v_1, v_2) \in \phi(\gamma_1([e_2/x]Y))$$
  So we can build
  $$\dfrac{\gamma_1(e_1) \Downarrow \lambda x : X.\ u_1 \qquad [e_2/x]u_1 \Downarrow v_1}{e_1\ e_2 \Downarrow v_1} \qquad\qquad \dfrac{\gamma_2(e_1') \Downarrow \lambda x : X.\ u_2 \qquad [e_2'/x]u_2 \Downarrow v_2}{e_1'\ e_2' \Downarrow v_2}$$
  And conclude.
- **Case** IPairCong: $\dfrac{\Gamma \vdash e_1 \equiv e_1' : X \qquad \Gamma \vdash e_2 \equiv e_2' : Y[e_1/x]}{\Gamma \vdash (e_1, e_2) \equiv (e_1', e_2') : \Sigma x : X.\ Y}$
  Similar to IFunCong.
- **Case** IFstCong: $\dfrac{\Gamma \vdash e \equiv e' : \Sigma x : X.\ Y}{\Gamma \vdash \pi_1\, e \equiv \pi_1\, e' : X}$
  Similar to IAppCong.

- **Case** ISNDCONG: $\dfrac{\Gamma \vdash e \equiv e' : \Sigma x : X.\ Y}{\Gamma \vdash \pi_2\, e \equiv \pi_2\, e' : Y[\pi_1\, e/x]}$
  Similar to IAPPCONG.
- **Case** : $\dfrac{\Gamma, x : X \vdash e \equiv e' : Y \qquad x \notin FV(e, e')}{\Gamma \vdash e \equiv e' : \forall x : X.\ Y}$
  Let $\gamma \in [\![\Gamma]\!]$. Then, for every $(t, t') \in \phi(\gamma_1(X))$, $((\gamma_1, t/x), (\gamma_2, t'/x)) \in [\![\Gamma, x : X]\!]$, thus we get the expected result thanks to the induction hypothesis.
- **Case** : $\dfrac{\Gamma \vdash e \equiv e' : [e''/x]Y}{\Gamma \vdash e \equiv e' : \exists x : X.\ Y}$
  We get the expected result directly from the induction hypothesis.

5. If $\Gamma \vdash A$ linear then $\gamma(A) \in L(\gamma_1(X))$.

   We case analyze the derivation of $\Gamma \vdash A$ linear:

   - **Case** LTP: $\dfrac{\Gamma \vdash A : \mathsf{L}_i}{\Gamma \vdash A \text{ linear}}$
     By induction, $\gamma(A) \in \phi(\mathsf{L}_i)$ at some $\mathcal{T}_i$, so $\gamma(A) \in L$ in $T_{i+1}(\mathcal{T}_{i+1}) = \mathcal{T}_i$.

6. If $\Gamma \vdash A \equiv B$ linear then $(\gamma_1(A), \gamma_2(B)) \in L(\gamma_1(X))$.

   We case analyze the derivation of $\Gamma \vdash A \equiv B$ linear:

   - **Case** LTPEQ: $\dfrac{\Gamma \vdash A \equiv B : \mathsf{L}_i}{\Gamma \vdash A \equiv B \text{ linear}}$
     By induction, $\gamma(A, B) \in \phi(\mathsf{L}_i)$ at some $\mathcal{T}_i$, so $\gamma(A, B) \in L$ in $T_{i+1}(\mathcal{T}_{i+1}) = \mathcal{T}_i$.

7. If $\Gamma; \Delta \vdash e : A$ then $\gamma(\delta(e), \sigma) \in \psi(\gamma_1(X))$.

   We case analyze the derivation of $\Gamma; \Delta \vdash e : A$:

   - **Case** LHYP: $\dfrac{}{\Gamma; a : A \vdash a : A}$
     Let $\gamma \in [\![\Gamma]\!]$ and $((\sigma_1, \delta_1), (\sigma_2, \delta_2)) \in [\![\gamma_1(\Delta)]\!]$.
     Then we have by definition $((\sigma_1, \delta_1(a)), (\sigma_2, \delta_2(a))) \in \psi(\gamma_1(A))$, which is what we require.
   - **Case** LEQ: $\dfrac{\Gamma; \Delta \vdash e : B \qquad \Gamma \vdash A \equiv B \text{ linear}}{\Gamma; \Delta \vdash e : A}$
     By induction, we have:
     - $(\gamma_1(A), \gamma_2(B)) \in L$
     - $(\delta_1(e)) \in \psi(\gamma_1(Y))$
     - $(\gamma_1(A), \gamma_2(A)) \in I$ since A is a linear type

     Thus we have $\psi(\gamma_1(A)) = \psi(\gamma_2(A)) = \psi(\gamma_1(B))$. Then $((\sigma_1, \delta_1(\gamma_1(e))), (\sigma_1, \delta_1(\gamma_1(e)))) \in \psi(\gamma_1(A))$.
   - **Case** LONEI: $\dfrac{}{\Gamma; \cdot \vdash () : \mathsf{I}}$
     Straightforward.
   - **Case** LONEE: $\dfrac{\Gamma; \Delta \vdash e : \mathsf{I} \qquad \Gamma; \Delta' \vdash e' : C}{\Gamma; \Delta, \Delta' \vdash \mathsf{let}\ () = e\ \mathsf{in}\ e' : C}$
     Begin by separating $((\sigma_1, \delta_1), (\sigma_2, \delta_2)) \in [\![\Delta, \Delta']\!]$ into $((\sigma_1, \delta_1), (\sigma_2, \delta_2)) \in [\![\Delta]\!]$ and $((\sigma_1', \delta_1'), (\sigma_2', \delta_2')) \in [\![\Delta']\!]$ (we will do that implicitely from now on).
     By our first induction hypothesis $((\sigma_1, \gamma_1(\delta_1(e))), (\sigma_2, \gamma_2(\delta_2(e)))) \in \psi(\mathsf{I})$, so

$$\langle \sigma_1; \delta_1(\gamma_1(e)) \rangle \Downarrow \langle \epsilon; () \rangle$$

$$\langle \sigma_2; \delta_2(\gamma_2(e)) \rangle \Downarrow \langle \epsilon; () \rangle$$

By our second induction hypothesis, $((\sigma_1, \gamma_1(\delta_1'(e))), (\sigma_2, \gamma_2(\delta_2'(e)))) \in \psi(\gamma_1(C))$, so

$$\langle \sigma_1'; \delta_1'(\gamma_1(e')) \rangle \Downarrow \langle \sigma_1''; \nu_1 \rangle$$

$$\langle \sigma_2'; \delta_2'(\gamma_2(e')) \rangle \Downarrow \langle \sigma_2''; \nu_2 \rangle$$

$$((\sigma_1'', \nu_1), (\sigma_2'', \nu_2)) \in \psi(\gamma_1(C))$$

Thus, we have $(\sigma_i \cdot \sigma_i', (\delta_i, \delta_i')(\mathsf{let}\ () = e\ \mathsf{in}\ e')) = (\sigma_i \cdot \sigma_i', \mathsf{let}\ () = \delta_i(\gamma_i(e))\ \mathsf{in}\ \delta_i'(\gamma_i(e')))$ which evaluates to (σ_i'', ν_i) for $i = 1, 2$.
Therefore the conclusion follows by closure under evaluation of CPERs.

- **Case** LTENSORI: $\dfrac{\Gamma; \Delta \vdash e : A \qquad \Gamma; \Delta' \vdash e' : B}{\Gamma; \Delta, \Delta' \vdash (e, e') : A \otimes B}$
  By induction
  – $((\sigma_1, \delta_1(\gamma_1(e))), (\sigma_2, \delta_2(\gamma_2(e)))) \in \psi(\gamma_1(A))$
  – $((\sigma_1', \delta_1'(\gamma_1(e'))), (\sigma_2', \delta_2'(\gamma_2(e'))) \in \psi(\gamma_1(B))$
  Thus the conclusion follows immediately from the definition of $\hat{\otimes}$.
- **Case** LTENSORE: $\dfrac{\Gamma; \Delta \vdash e : A \otimes B \qquad \Gamma; \Delta', a : A, b : B \vdash e' : C}{\Gamma; \Delta, \Delta' \vdash \mathsf{let}\ (a, b) = e\ \mathsf{in}\ e' : C}$
  Our first induction hypothesis yields $((\sigma_1, \delta_1(e)), (\sigma_2, \delta_2(e))) \in \psi(\gamma_1(A \otimes B))$. Thus

$$\langle \sigma_1; \delta_1(\gamma_1(e)) \rangle \Downarrow \langle \sigma_1'' \cdot \sigma_1'''; (e_1'', e_1''') \rangle$$

$$\langle \sigma_2; \delta_2(\gamma_2(e)) \rangle \Downarrow \langle \sigma_2'' \cdot \sigma_2'''; (e_2'', e_2''') \rangle$$

$$((\sigma_1'', e_1''), (\sigma_2'', e_2'')) \in \psi(\gamma_1(A))$$

$$((\sigma_1''', e_1'''), (\sigma_2''', e_2''')) \in \psi(\gamma_1(B))$$

  From ou second induction hypothesis, we get

$$((\sigma_1' \cdot \sigma_1'' \cdot \sigma_1''', (\delta_1', e_1''/a, e_1'''/b)(e')), (\sigma_2' \cdot \sigma_2'' \cdot \sigma_2''', (\delta_2', e_2''/a, e_2'''/b)(e'))) \in \psi(\gamma_1(C))$$

  by checking that

$$((\sigma_1' \cdot \sigma_1'' \cdot \sigma_1''', (\delta_1', e_1''/a, e_1'''/b)), (\sigma_2' \cdot \sigma_2'' \cdot \sigma_2''', (\delta_2', e_2''/a, e_2'''/b))) \in [\![\Delta', a : A, b : B]\!]$$

  with the obvious decomposition.
  We can then evaluate these and deduce that $(\sigma_i \cdot \sigma_i', \gamma_i(\mathsf{let}\ (a, b) = \delta_i(e)\ \mathsf{in}\ \delta_i'(e')))$ yields the same evaluation for $i = 1, 2$ to conclude.
- **Case** LFUNI: $\dfrac{\Gamma; \Delta, a : A \vdash e : B}{\Gamma; \Delta \vdash \lambda a.\ e : A \multimap B}$
  Let $((\sigma_1', t_1), (\sigma_2', t_2)) \in \psi(\gamma_1(A))$ with $\sigma_1 \# \sigma_1'$ and $\sigma_2 \# \sigma_2'$.
  We then have

$$((\sigma_1 \cdot \sigma_1', (\delta_1, t_1/a)), (\sigma_2 \cdot \sigma_2', (\delta_2, t_2/a))) \in [\![\gamma_1(\Delta, a : A)]\!]$$

  Then, by induction

$$((\sigma_1 \cdot \sigma_1', \gamma_1((\delta_1([t_1/a]e_1))), (\sigma_2 \cdot \sigma_2', \gamma_2((\delta_2([t_2/a]e_2)))) \in \psi(\gamma_1(B))$$

  Which is what we wanted.

- **Case** LFUNE: $\dfrac{\Gamma; \Delta \vdash e : A \multimap B \qquad \Gamma; \Delta' \vdash e' : A}{\Gamma; \Delta, \Delta' \vdash e\ e' : B}$

  By induction

$$((\sigma_1, \gamma_1(e)), (\sigma_2, \gamma_2(e))) \in \psi(\gamma_1(A \multimap B))$$
$$((\sigma_1', \gamma_1(e')), (\sigma_2', \gamma_2(e'))) \in \psi(\gamma_1(A))$$

  We thus have some $((\sigma_1'', e_1''), (\sigma_2'', e_2''))$ such that

$$\langle \sigma_1; \gamma_1(e) \rangle \Downarrow \langle \sigma_1''; \lambda x.\ e_1'' \rangle \wedge \langle \sigma_2; \gamma_2(e) \rangle \Downarrow \langle \sigma_2''; \lambda x.\ e_2'' \rangle$$

$$((\sigma_1'', \lambda x.\ e_1''), (\sigma_2'', \lambda x.\ e_2'')) \in \psi(\gamma_1(A \multimap B)) \text{ and thus}$$
$$((\sigma_1'' \cdot \sigma_1', [\gamma_1(e')/x]e_1''), (\sigma_2'' \cdot \sigma_2', [\gamma_2(e')/x]e_2'')) \in \psi(\gamma_1(B))$$

  From which we have $((\sigma_1''', e_1'''), (\sigma_2''', e_2''')) \in \psi(\gamma_1(B))$ such that

$$\langle \sigma_1'' \cdot \sigma_1'; [\gamma_1(e')/x]e_1'' \rangle \Downarrow \langle \sigma_1'''; e_1''' \rangle \wedge \langle \sigma_2'' \cdot \sigma_2'; [\gamma_2(e')/x]e_2'' \rangle \Downarrow \langle \sigma_2'''; e_2''' \rangle$$

  We can then build the following derivations

$$\dfrac{\langle \sigma_1 \cdot \sigma_1'; \gamma_1(e) \rangle \Downarrow \langle \sigma_1'' \cdot \sigma_1'; \lambda x.\ e_1'' \rangle \qquad \langle \sigma_1'' \cdot \sigma_1'; [\gamma_1(e')/x]e_1'' \rangle \Downarrow \langle \sigma_1'''; e_1''' \rangle}{\langle \sigma_1 \cdot \sigma_1'; \gamma_1(e\ e') \rangle \Downarrow \langle \sigma_1'''; e_1''' \rangle}$$

$$\dfrac{\langle \sigma_2 \cdot \sigma_2'; \gamma_2(e) \rangle \Downarrow \langle \sigma_2'' \cdot \sigma_2'; \lambda x.\ e_2'' \rangle \qquad \langle \sigma_2'' \cdot \sigma_2'; [\gamma_2(e')/x]e_2'' \rangle \Downarrow \langle \sigma_2'''; e_2''' \rangle}{\langle \sigma_2 \cdot \sigma_2'; \gamma_2(e\ e') \rangle \Downarrow \langle \sigma_2'''; e_2''' \rangle}$$

  And conclude.

- **Case** LPII: $\dfrac{\Gamma, x : X; \Delta \vdash e : A}{\Gamma; \Delta \vdash \hat{\lambda} x.\ e : \Pi x : X.\ A}$

  Let $(t_1, t_2) \in \phi(\gamma_1(X))$.
  We have $((\gamma_1, t_1/x), (\gamma_2, t_2/x)) \in \phi(\gamma_1(X))$.
  Notice that x is not a free variable of Δ.
  Thus, by induction $((\sigma_1, [t_1/x]\gamma_1(\delta_1(e))), (\sigma_2, [t_2/x]\gamma_2(\delta_2(e)))) \in \psi([t_1/x]\gamma_1(A))$. We then know that there exists $((\sigma_1', v_1), (\sigma_2', v_2)) \in \psi([t_1/x]\gamma_1(A))$ such that

$$\langle \sigma_1; [t_1/x]\gamma_1(\delta_1(e)) \rangle \Downarrow \langle \sigma_1'; v_1 \rangle \wedge \langle \sigma_2; [t_2/x]\gamma_2(\delta_2(e)) \rangle \Downarrow \langle \sigma_2'; v_2 \rangle$$

  So we can derive

$$\left\langle \sigma_1; \hat{\lambda} x.\ \gamma_1(\delta_1(e))\ t_1 \right\rangle \Downarrow \langle \sigma_1'; v_1 \rangle \wedge \left\langle \sigma_2; \hat{\lambda} x.\ \gamma_2(\delta_2(e))\ t_2 \right\rangle \Downarrow \langle \sigma_2'; v_2 \rangle$$

  And conclude by closure of CPERs under evaluation.

- **Case** LPIE: $\dfrac{\Gamma; \Delta \vdash e : \Pi x : X.\ A \qquad \Gamma \vdash e' : X}{\Gamma; \Delta \vdash e\ e' : [e'/x]A}$

  By induction, we have

$$\forall (t_1, t_2) \in \phi(\gamma_1(X)), (\gamma_1(\delta_1(e))\ t_1, \gamma_2(\delta_2(e))\ t_2) \in \phi([t_1/x]\gamma_1(Y))$$

$$(\gamma_1(e'), \gamma_2(e')) \in \phi(\gamma_1(X))$$

  Thus, by applying the first hypothesis to the second, we have what we need.

$$(\gamma_1(\delta_1(e)\ e'), \gamma_2(\delta_2(e)\ e')) \in \phi([\gamma_1(e')/x]\gamma_1(Y))$$

- **Case** : $\dfrac{\Gamma, x : X; \Delta \vdash e : Y \qquad x \notin FV(e)}{\Gamma; \Delta \vdash e : \forall x : X.\, Y}$
  By induction and $\Gamma, x : X$ ok, we have $\forall (e_1', e_2') \in \phi(\gamma_1(X)), ((\sigma_1, \delta_1((\gamma_1, e_1'/x)(e))), (\sigma_2, \delta_2((\gamma_2, e_2'/x)(e)))) \in \psi([e_1'/x]\gamma_1(Y))$, so since x is free in e, $((\sigma_1, \delta_1(\gamma_1(e))), (\sigma_2, \delta_2(\gamma_2(e)))) \in \psi([e_1'/x]\gamma_1(Y))$ which directly implies that

  $$((\sigma_1, \delta_1(\gamma_1(e))), (\sigma_2, \delta_2(\gamma_2(e)))) \in \psi(\forall x : \gamma_1(X).\, \gamma_1(Y))$$

- **Case** : $\dfrac{\Gamma; \Delta \vdash e : \forall x : X.\, Y \qquad \Gamma \vdash e' : X}{\Gamma; \Delta \vdash e : [e'/x]Y}$
  By induction, $((\sigma_1, \gamma_1(\delta_1(e))), (\sigma_2, \delta_2(\gamma_2(e)))) \in \psi(\gamma_1(\forall x : X.\, Y))$, which means there exits $(e_1'', e_2'') \in \psi(\gamma_1(\forall x : X.\, Y))$ such that for every $((\sigma_1', e_1'''), (\sigma_2', e_2''')) \in \phi(\gamma_1(X))$, we have

  $$\langle \sigma_1; \gamma_1(e) \rangle \Downarrow \langle \sigma_1'; e_1'' \rangle \wedge \langle \sigma_2; \gamma_2(e) \rangle \Downarrow \langle \sigma_2'; e_2'' \rangle$$
  $$((\sigma_1', e_1''), (\sigma_2', e_2'')) \in \psi([e_1'''/x]\gamma_1(Y))$$

  Thus by compatibility with reduction, it means

  $$((\sigma_1, \gamma_1(e)), (\sigma_2, \gamma_2(e))) \in \psi([e_1'''/x]\gamma_1(Y))$$

  Now we can take $(e_1''', e_2''') := (\gamma_1(e'), \gamma_2(e'))$ and conclude.

- **Case** : $\dfrac{\Gamma, x : X; \Delta, y : Y \vdash e : Z \qquad x \notin FV(e)}{\Gamma; \Delta, y : \exists x : X.\, Y \vdash e : Z}$
  Let $(\gamma_1, \gamma_2) \in [\![\Gamma]\!]$ and $((\sigma_1', e_1'), (\sigma_2', e_2')) \in \psi(\gamma_1(\exists x : X.\, Y))$.
  That last fact tells us there exists some $(e_1'', e_2'') \in \phi(\gamma_1(X))$ such that $((\sigma_1', e_1'), (\sigma_2', e_2') \in \phi(\gamma_1([e_1''/x]Y))$ modulo a bit of reasoning with reductions.
  Thus, by noticing that $((\gamma_1, e_1''/x), (\gamma_2, e_2''/x)) \in [\![\Gamma, x : X, y : Y]\!]$ and $((\sigma_1 \cdot \sigma_1', (\delta_1, e_1'/y)), (\sigma_1 \cdot \sigma_1', (\delta_1, e_1'/y))) \in [\![\gamma_1(\Delta)]\!]$, by induction (and $x \notin FV(e)$) we have

  $$((\sigma_1 \cdot \sigma_1', \delta_1(\gamma_1([e_1'/y]e))), (\sigma_2 \cdot \sigma_2', \delta_2(\gamma_2([e_2'/y]e)))) \in \phi(\gamma_1(Z))$$

- **Case** : $\dfrac{\Gamma, x : X \vdash Y \text{ linear} \qquad \Gamma \vdash e' : X \qquad \Gamma; \Delta \vdash e : [e'/x]Y}{\Gamma; \Delta \vdash e : \exists x : X.\, Y}$
  By induction, we have:
  - $(\gamma_1(e'), \gamma_2(e')) \in \phi(\gamma_1(X))$
  - $((\sigma_1, (\gamma_1, \gamma_1(e')/x)(\delta_1(e))), (\sigma_2, (\gamma_2, \gamma_2(e')/x)(\delta_2(e)))) \in \psi((\gamma_1, \gamma_1(e')/x)(Y))$

  Which gives us the result once we notice $x \notin FV(e)$ thanks to typing.

- **Case** LWithI: $\dfrac{\Gamma; \Delta \vdash e_1 : A_1 \qquad \Gamma; \Delta \vdash e_2 : A_2}{\Gamma; \Delta \vdash (e_1, e_2) : A_1 \,\&\, A_2}$
  Similarly to the LTensorI case, the result follows directly from the induction hypothesis and the definition of the semantic $\&$.

- **Case** LWithEFst: $\dfrac{\Gamma; \Delta \vdash e : A \,\&\, B}{\Gamma; \Delta \vdash \pi_1\, e : A}$
  By induction $((\sigma_1, \delta_1(\gamma_1(e))), (\sigma_2, \delta_2(\gamma_2(e)))) \in \psi(\gamma_1(A \,\&\, B))$, so there is some $((\sigma_1', (v_1, w_1,))\ (\sigma_2', (v_2, w_2,))) \in \psi(A \,\&\, B)$ such that

  $$\langle \sigma_1; \delta_1(\gamma_1(e)) \rangle \Downarrow \langle \sigma_1'; (v_1, w_1) \rangle \wedge \langle \sigma_2; \delta_2(\gamma_2(e)) \rangle \Downarrow \langle \sigma_2'; (v_2, w_2) \rangle$$

  Thus, we have $((\sigma_1', v_1), (\sigma_2', v_2)) \in \psi(\gamma_1(A))$ and

  $$\langle \sigma_1; \pi_1\, \gamma_1(\delta_1(e)) \rangle \Downarrow \langle \sigma_1'; v_1 \rangle \wedge \langle \sigma_2; \pi_1\, \gamma_2(\delta_2(e)) \rangle \Downarrow \langle \sigma_2'; v_2 \rangle$$

  Thus, we have the expected result.

- **Case** LWITHESNDI: $\dfrac{\Gamma; \Delta \vdash e : A \,\&\, B}{\Gamma; \Delta \vdash \pi_2\, e : B}$
  By induction $((\sigma_1, \delta_1(\gamma_1(e))), (\sigma_2, \delta_2(\gamma_2(e)))) \in \psi(\gamma_1(A \,\&\, B))$, so there is some $((\sigma_1', (v_1, w_1,)) \, (\sigma_2', (v_2, w_2,))) \in \psi(A \,\&\, B)$ such that

$$\langle \sigma_1; \delta_1(\gamma_1(e)) \rangle \Downarrow \langle \sigma_1'; (v_1, w_1) \rangle \wedge \langle \sigma_2; \delta_2(\gamma_2(e)) \rangle \Downarrow \langle \sigma_2'; (v_2, w_2) \rangle$$

  Thus, we have $((\sigma_1', w_1), (\sigma_2', w_2)) \in \psi(\gamma_1(B))$ and

$$\langle \sigma_1; \pi_2\, \gamma_1(\delta_1(e)) \rangle \Downarrow \langle \sigma_1'; w_1 \rangle \wedge \langle \sigma_2; \pi_2\, \gamma_2(\delta_2(e)) \rangle \Downarrow \langle \sigma_2'; w_2 \rangle$$

  Thus, we have the expected result.

- **Case** LFI: $\dfrac{\Gamma \vdash e : X \qquad \Gamma; \Delta \vdash t : [e/x]A}{\Gamma; \Delta \vdash \mathsf{F}\,(e, t) : \mathsf{F}x : X.\ A}$
  Induction gives us:
  - $(\gamma_1(e), \gamma_2(e)) \in \phi(\gamma_1(X))$
  - $((\sigma_1, \delta_1(\gamma_1(t))), (\sigma_2, \delta_2(\gamma_2(t)))) \in \psi(\gamma_1([e/x]A))$

  which directly gives us the conclusion.

- **Case** LFE: $\dfrac{\Gamma; \Delta \vdash e : \mathsf{F}x : X.\ A \qquad \Gamma, x : X; \Delta', a : A \vdash e' : C}{\Gamma; \Delta, \Delta' \vdash \mathsf{let}\ \mathsf{F}\,(x, a) = e\ \mathsf{in}\ e' : C}$
  By induction, $((\sigma_1, \gamma_1(\delta_1(e))), (\sigma_2, \gamma_2(\delta_2(e)))) \in \psi(\gamma_1(\mathsf{F}x : X.\ A))$, thus

$$\langle \sigma_1; (\delta_1(\gamma_1(e))) \rangle \Downarrow \langle \sigma_1''; \mathsf{F}\,(e_1'', e_1''') \rangle$$

$$\langle \sigma_2; (\delta_2(\gamma_2(e))) \rangle \Downarrow \langle \sigma_2''; \mathsf{F}\,(e_2'', e_2''') \rangle$$

  In particular, $(e_1'', e_2'') \in \phi(\gamma_1(X))$ and $(e_1''', e_2''') \in \psi(\gamma_1(A))$. Notice that by $\Gamma \vdash \Delta'$ ok, x is not a free variable in Δ'. Then we have

$$((\gamma_1, e_1''/x), (\gamma_2, e_2''/x)) \in [\![\Gamma, x : X]\!]$$

$$(\sigma_1' \cdot \sigma_1'', (\delta_1', e_1'''/a)), (\sigma_2' \cdot \sigma_2'', (\delta_2', e_2'''/a))) \in [\![(\gamma_1, e_1''/x)(\Delta, a : A)]\!]$$

  By our second induction hypothesis, we can check that

$$(\sigma_1' \cdot \sigma_1'', (\delta_1', e_1'''/a)(e'), \sigma_2' \cdot \sigma_2'', (\delta_2', e_2'''/a)(e')) \in \psi((\gamma_1, e_1''/x)(C))$$

  We then have

$$\langle \sigma_1' \cdot \sigma_1''; \gamma_1((\delta_1', e_1''/x, e_1'''/a)(e')) \rangle \Downarrow \langle \sigma_1'''; v_1 \rangle$$
$$\langle \sigma_2' \cdot \sigma_2''; \gamma_2((\delta_2', e_2''/x, e_2'''/a)(e')) \rangle \Downarrow \langle \sigma_2'''; v_2 \rangle$$
$$((\sigma_1''', v_1), (\sigma_2''', v_2)) \in \psi(\gamma_1(C))$$

  From which we can construct derivations

$$\dfrac{\langle \sigma_1; \gamma_1(\delta_1(e)) \rangle \Downarrow \langle \sigma_1''; \mathsf{F}\,(e_1'', e_1''') \rangle \qquad \langle \sigma_1'' \cdot \sigma_1'; [e_1''/x, e_1'''/a]\gamma_1(\delta_1(e_1')) \rangle \Downarrow \langle \sigma_1'''; u \rangle}{\langle \sigma_1 \cdot \sigma_1'; \mathsf{let}\ \mathsf{F}\,(x, a) = \gamma_1(\delta_1(e))\ \mathsf{in}\ \gamma_1(\delta_1(e')) \rangle \Downarrow \langle \sigma_1'''; v_1 \rangle}$$

$$\dfrac{\langle \sigma_2; \gamma_2(\delta_2(e)) \rangle \Downarrow \langle \sigma_2''; \mathsf{F}\,(e_2'', e_2''') \rangle \qquad \langle \sigma_2'' \cdot \sigma_2'; [e_2''/x, e_2'''/a]\gamma_2(\delta_2(e_2')) \rangle \Downarrow \langle \sigma_2'''; u \rangle}{\langle \sigma_2 \cdot \sigma_2'; \mathsf{let}\ \mathsf{F}\,(x, a) = \gamma_2(\delta_2(e))\ \mathsf{in}\ \gamma_2(\delta_2(e')) \rangle \Downarrow \langle \sigma_2'''; v_2 \rangle}$$

  And we can conclude thanks to the closure of CPERs under evaluation.

- **Case** LGE: $\dfrac{\Gamma \vdash e : \mathsf{G}\, A}{\Gamma; \cdot \vdash \mathsf{G}^{-1}\, e : A}$
Our inductive hypothesis tells us that
$$(\gamma_1(e), \gamma_2(e)) \in \phi(\gamma_1(\mathsf{G}\, A))$$
Therefore $\gamma_1(e) \Downarrow \mathsf{G} e_1'$ and $\gamma_2(e) \Downarrow \mathsf{G} e_2'$ and by closure of CPERs under evaluation and definition of G, $(e_1', e_2') \in \psi(\gamma_1(A))$.
Hence
$$\langle \epsilon; e_1' \rangle \Downarrow \langle \sigma_1; v_1 \rangle$$
$$\langle \epsilon; e_2' \rangle \Downarrow \langle \sigma_2; v_2 \rangle$$
and
$$((\sigma_1, v_1), (\sigma_2, v_2)) \in \psi(\gamma_1(A))$$
From there, we can build
$$\dfrac{\gamma_2(e) \Downarrow \mathsf{G} e_2' \qquad \langle \epsilon; e_2' \rangle \Downarrow \langle \sigma'; v_2 \rangle}{\left\langle \sigma; \mathsf{G}^{-1}\, \gamma_2(e) \right\rangle \Downarrow \langle \sigma_2; v_2 \rangle} \qquad \dfrac{\gamma_1(e) \Downarrow \mathsf{G} e_1' \qquad \langle \epsilon; e_1' \rangle \Downarrow \langle \sigma'; v_1 \rangle}{\left\langle \sigma; \mathsf{G}^{-1}\, \gamma_1(e) \right\rangle \Downarrow \langle \sigma_1; v_1 \rangle}$$
and deduce the expected conclusion by closure of CPERs under evaluation.

- **Case** LTI: $\dfrac{\Gamma; \Delta \vdash e : A}{\Gamma; \Delta \vdash \mathsf{val}\, e : \mathsf{T}\, A}$
By induction
$$((\sigma_1, \gamma_1(\delta_1(e))), (\sigma_2, \gamma_2(\delta_2(e)))) \in \psi(\gamma_1(A))$$
Let $\sigma_{f1} \# \sigma_1$ and $\sigma_{f2} \# \sigma_2$. We have
$$\langle \sigma_1 \cdot \sigma_{f1}; \mathsf{val}\, \gamma_1(\delta_1(e)) \rangle \rightsquigarrow \langle \sigma_1 \cdot \sigma_{f1}; \mathsf{val}\, \gamma_1(\delta_1(e)) \rangle$$
$$\langle \sigma_2 \cdot \sigma_{f2}; \mathsf{val}\, \gamma_2(\delta_2(e)) \rangle \rightsquigarrow \langle \sigma_2 \cdot \sigma_{f2}; \mathsf{val}\, \gamma_2(\delta_2(e)) \rangle$$
Thus we are trivially in $\phi(\mathsf{T}\, A)$.

- **Case** LTLet: $\dfrac{\Gamma; \Delta \vdash e : \mathsf{T}\, A \qquad \Gamma; \Delta', a : A \vdash e' : \mathsf{T}\, C}{\Gamma; \Delta, \Delta' \vdash \mathsf{let\ val}\, a = e\ \mathsf{in}\ e' : \mathsf{T}\, C}$
Let $\sigma_{f1} \# \sigma_1 \cdot \sigma_1'$ and $\sigma_{f2} \# \sigma_2 \cdot \sigma_2'$.
By induction,
$$(\sigma_1, \delta_1(\gamma_1(e))) \in \psi(\gamma_1(A))$$
$$(\sigma_2, \delta_2(\gamma_2(e))) \in \psi(\gamma_1(A))$$
thus we have $((\sigma_1'', e_1''), (\sigma_2'', e_2''))$ such that
$$\langle \sigma_1 \cdot \sigma_1' \cdot \sigma_{f1}; \delta_1(\gamma_1(e))) \rangle \rightsquigarrow \langle \sigma_1'' \cdot \sigma_1' \cdot \sigma_{f1}; \mathsf{val}\, e_1'' \rangle$$
$$\langle \sigma_2 \cdot \sigma_2' \cdot \sigma_{f2}; \delta_2(\gamma_2(e))) \rangle \rightsquigarrow \langle \sigma_2'' \cdot \sigma_2' \cdot \sigma_{f2}; \mathsf{val}\, e_2'' \rangle$$
$$((\sigma_1'', e_1''), (\sigma_2'', e_2'')) \in \psi(\gamma_1(A))$$
Thus
$$(\sigma_1'' \cdot \sigma_1', (\delta_1', e_1''/a)) \in [\![\gamma_1(\Delta', a : A)]\!]$$
$$(\sigma_2'' \cdot \sigma_2', (\delta_2', e_2''/a)) \in [\![\gamma_1(\Delta', a : A)]\!]$$
By our second induction hypothesis,
$$((\sigma_1'' \cdot \sigma_1', \gamma_1((\delta_1, e_1''/a)(e'))), (\sigma_2'' \cdot \sigma_2', \gamma_2((\delta_2, e_2''/a)(e')))) \in \psi(\mathsf{T}\, (\gamma_1(C)))$$

Hence we have $((\sigma_1''', e_1'''), (\sigma_2''', e_2'''))$ such that

$$\langle \sigma_1'' \cdot \sigma_1' \cdot \sigma_{f1}; \gamma_1(\delta_1'(e)))\rangle \rightsquigarrow \langle \sigma_1''' \cdot \sigma_{f1}; \mathsf{val}\, e_1''' \rangle$$
$$\langle \sigma_2'' \cdot \sigma_2' \cdot \sigma_{f2}; \gamma_2(\delta_2'(e)))\rangle \rightsquigarrow \langle \sigma_2''' \cdot \sigma_{f2}; \mathsf{val}\, e_2''' \rangle$$
$$((\sigma_1''', e_1'''), (\sigma_2''', e_2''')) \in \psi(\gamma_1(A))$$

Therefore, we can deduce the following reductions

$$\langle \sigma_1 \cdot \sigma_1' \cdot \sigma_{f1}; \gamma_1(\mathsf{let\ val}\ a = \delta_1(e)\ \mathsf{in}\ \delta_1'(e'))\rangle \rightsquigarrow \langle \sigma_1''' \cdot \sigma_{f1}; \mathsf{val}\, e_1''' \rangle$$
$$\langle \sigma_2 \cdot \sigma_2' \cdot \sigma_{f2}; \gamma_2(\mathsf{let\ val}\ a = \delta_2(e)\ \mathsf{in}\ \delta_2'(e'))\rangle \rightsquigarrow \langle \sigma_2''' \cdot \sigma_{f2}; \mathsf{val}\, e_2''' \rangle$$

and conclude.

- **Case** LNEW: $\dfrac{\Gamma \vdash e : X}{\Gamma; \cdot \vdash \mathsf{new}_X\, e : \mathsf{T}\,(\mathsf{F}x : \mathsf{Loc}.\ [x \mapsto X])}$

  Let σ_{f1} and σ_{f2} be arbitrary stores and a location $l \notin \mathrm{dom}(\sigma_{f1}) \cup \mathrm{dom}(\sigma_{f2})$.
  By our induction hypothesis

  $$(\gamma_1(e), \gamma_2(e)) \in \phi(\gamma_1(X))$$

  We thus have a pair of values $(v_1, v_2) \in \phi(\gamma_1(X))$ such that

  $$\gamma_1(e) \Downarrow v_1 \wedge \gamma_2(e) \Downarrow v_2$$

  Then the reduction relation gives us

  $$\langle \sigma_{f1}; \gamma_1(\delta_1(\mathsf{new}_X\, e))\rangle \rightsquigarrow \langle \sigma_{f1}, l : v_1; \mathsf{val}\, \mathsf{F}\,(l, *)\rangle$$
  $$\langle \sigma_{f2}; \gamma_2(\delta_2(\mathsf{new}_X\, e))\rangle \rightsquigarrow \langle \sigma_{f2}, l : v_2; \mathsf{val}\, \mathsf{F}\,(l, *)\rangle$$

  We can check that

  $$(([l : v_1], \mathsf{F}\,(l, *)), ([l : v_2], \mathsf{F}\,(l, *))) \in \psi(\gamma_1(\mathsf{F}x : \mathsf{Loc}.\ [[x \mapsto X]]))$$

  to conclude.

- **Case** LFREE: $\dfrac{\Gamma \vdash e : \mathsf{Loc} \qquad \Gamma; \Delta \vdash t : [e \mapsto X]}{\Gamma; \Delta \vdash \mathsf{free}\,(e, t) : \mathsf{T}\,\mathsf{I}}$

  Let σ_f and σ_g be heaps such that $\sigma_f \# \sigma_1$ and $\sigma_g \# \sigma_2$.
  By induction

  $$(\gamma_1(e), \gamma_2(e)) \in \phi(\mathsf{Loc})$$
  $$((\sigma_1, \gamma_1(\delta_1(t))), (\sigma_2, \gamma_2(\delta_2(t)))) \in \psi([\gamma_1(e) \mapsto \gamma_1(X)])$$

  Then we have l, (v_1, v_2) such that

  $$\langle \sigma_1; \gamma_1(\delta_1(t))\rangle \Downarrow \langle [l : v_1]; *\rangle$$
  $$\langle \sigma_2; \gamma_2(\delta_2(t))\rangle \Downarrow \langle [l : v_2]; *\rangle$$
  $$(\gamma_1(e_1), l) \in \mathsf{Loc}$$
  $$\gamma_1(e) \Downarrow l \wedge \gamma_2(e) \Downarrow l$$

  From there we can build the following derivations

  $$\dfrac{\gamma_1(e) \Downarrow l \qquad \langle \sigma_1 \cdot \sigma_f; \delta_1(t)\rangle \Downarrow \langle [l : v_1] \cdot \sigma_f; *\rangle}{\langle \sigma_1 \cdot \sigma_f; \gamma_1(\mathsf{free}\,(e, \delta_1(t)))\rangle \rightsquigarrow \langle \sigma_f; ()\rangle} \qquad \dfrac{\gamma_2(\delta_2(e)) \Downarrow l \qquad \langle \sigma_2; \delta_2(t)\rangle \Downarrow \langle [l : v_2]; *\rangle}{\langle \sigma_2 \cdot \sigma_g; \gamma_2(\mathsf{free}\,(e, \delta_2(t)))\rangle \rightsquigarrow \langle \sigma_g; ()\rangle}$$

  And check that since $((\epsilon, ()), (\epsilon, ())) \in \psi(\mathsf{I})$, we have the required result by definition of T.

- **Case** LGET: $$\frac{\Gamma \vdash e : \mathsf{Loc} \qquad \Gamma;\Delta \vdash t : [e \mapsto X] \qquad \Gamma, x : X; \Delta', a : [e \mapsto X] \vdash t' : C}{\Gamma;\Delta,\Delta' \vdash \mathsf{let}\ (x,p) = \mathsf{get}(a,t)\ \mathsf{in}\ t' : C}$$
  By induction
  $$(\gamma_1(e), \gamma_2(e)) \in \phi(\mathsf{Loc})$$
  $$((\sigma_1, \gamma_1(\delta_1(t))), (\sigma_2, \gamma_2(\delta_2(t)))) \in \psi([\gamma_1(e) \mapsto \gamma_1(e')])$$
  Then we have l, $(v_1, v_2) \in \phi(\gamma_1(X))$ such that
  $$\langle \sigma_1; \gamma_1(\delta_1(t)) \rangle \Downarrow \langle [l : v_1]; * \rangle$$
  $$\langle \sigma_2; \gamma_2(\delta_2(t)) \rangle \Downarrow \langle [l : v_2]; * \rangle$$
  $$(\gamma_1(e_1), l) \in \mathsf{Loc}$$
  $$\gamma_1(e) \Downarrow l \wedge \gamma_2(e) \Downarrow l$$
  Thus
  $$((\gamma_1, v_1/x), (\gamma_2, v_2/x)) \in [\![\Gamma, x : X, p : x =_X e']\!]$$
  Let us denote that pair of substitution (γ_1', γ_2').
  We then have
  $$(((\sigma_1', l : v_1), (\delta_1', */a)), ((\sigma_2', l : v_2), (\delta_2', */a))) \in [\![\gamma_1'(\Delta', a : [e \mapsto X])]\!]$$
  Then by our last induction hypothesis
  $$((((\sigma_1', l : v_1), \gamma_1'((\delta_1', */a)(t'))), ((\sigma_2', l : v_2), \gamma_2'((\delta_2', */a)(t')))) \in [\![\gamma_1'(C)]\!]$$
  Thus
  $$\langle \sigma_1', l : v_1; (\delta_1', */a)(t') \rangle \Downarrow \langle \sigma_1''; t_1'' \rangle$$
  $$\langle \sigma_2', l : v_2; (\delta_2', */a)(t') \rangle \Downarrow \langle \sigma_2''; t_2'' \rangle$$
  $$((\sigma_1'', t_1''), (\sigma_2'', t_2'')) \in \phi(\gamma_1'(C))$$
  Notice that since Γ ok and $\Gamma \vdash C$ type, $\gamma_1'(C) = \gamma_1(C)$ and $\gamma_i'(t) = \gamma_i(t)$ for $i = 1, 2$.
  Now we can build derivations
  $$\frac{\gamma_1(e) \Downarrow l \qquad \langle \sigma_1; \gamma_1(\delta_1(t)) \rangle \Downarrow \langle \sigma', l : v_1; * \rangle \qquad \langle \sigma, l : v_1; [v_1/x, */c]\gamma_1(\delta_1'(e'')) \rangle \Downarrow \langle \sigma'; t_1'' \rangle}{\langle \sigma; \gamma_1(\mathsf{let}\ (x,p) = \mathsf{get}(c, \delta_1(e))\ \mathsf{in}\ \delta_1(e')\delta_1'(e'')) \rangle \Downarrow \langle \sigma; t_1'' \rangle}$$
  $$\frac{\gamma_2(e) \Downarrow l \qquad \langle \sigma_2; \gamma_2(\delta_2(t)) \rangle \Downarrow \langle \sigma', l : v_2; * \rangle \qquad \langle \sigma, l : v_2; [v_2/x, */c]\gamma_2(\delta_2'(e'')) \rangle \Downarrow \langle \sigma'; t_2'' \rangle}{\langle \sigma; \gamma_2(\mathsf{let}\ (x,p) = \mathsf{get}(c, \delta_2(e))\ \mathsf{in}\ \delta_2(e')\delta_2'(e'')) \rangle \Downarrow \langle \sigma; t_2'' \rangle}$$
  And conclude.
- **Case** LSET: $$\frac{\Gamma \vdash e : \mathsf{Loc} \qquad \Gamma;\Delta \vdash t : [e \mapsto X] \qquad \Gamma \vdash e'' : Y}{\Gamma;\Delta \vdash e :=_t e'' : \mathsf{T}\,([e \mapsto Y])}$$
  Let σ_f and σ_g be heaps such that $\sigma_1 \# \sigma_f$ and $\sigma_2 \# \sigma_g$.
  By induction, there exists a location l and values (v_1, v_2) such that
  $$\gamma_1(e) \Downarrow l \wedge \gamma_2(e) \Downarrow l$$
  $$\langle \sigma_1; \gamma_1(\delta_1(t)) \rangle \Downarrow \langle [l : v_1]; * \rangle$$
  $$\langle \sigma_2; \gamma_2(\delta_2(t)) \rangle \Downarrow \langle [l : v_2]; * \rangle$$
  $$\gamma_1(e'') \Downarrow v_1' \wedge \gamma_2(e'') \Downarrow v_2'$$

$$(v_1', v_2') \in \phi(\gamma_1(Y))$$

We build the derivations

$$\frac{\gamma_1(e) \Downarrow l \qquad \gamma_1(e'') \Downarrow v_1' \qquad \langle \sigma_1 \cdot \sigma_f; \gamma_1(\delta_1(t)) \rangle \Downarrow \langle \sigma_f, l : v_1; * \rangle}{\langle \sigma_1 \cdot \sigma_f; \gamma_1(\delta_1(e :=_{e''} t)) \rangle \rightsquigarrow \langle \sigma_f, l : v_1'; * \rangle}$$

$$\frac{\gamma_2(e) \Downarrow l \qquad \gamma_2(e'') \Downarrow v_2' \qquad \langle \sigma_2 \cdot \sigma_g; \gamma_2(\delta_2(t)) \rangle \Downarrow \langle l : v_2; * \rangle}{\langle \sigma_2 \cdot \sigma_g; \gamma_2(\delta_2(e :=_{e''} t)) \rangle \rightsquigarrow \langle \sigma_g, l : v_2'; * \rangle}$$

- **Case** LIRR: $\dfrac{\Gamma; \Delta \vdash e \div A}{\Gamma; \Delta \vdash * : [A]}$

  This is a direct consequence of the induction hypothesis.

- **Case** LIRRUNIT: $\dfrac{\Gamma; \Delta \vdash e : [\mathsf{I}] \qquad \Gamma; \Delta' \vdash e' : C}{\Gamma; \Delta, \Delta' \vdash \mathsf{let}\ [] = e\ \mathsf{in}\ e' : C}$

  By induction, we have

  $$((\sigma_1, \gamma_1(\delta_1(e))), (\sigma_2, \gamma_2(\delta_2(e)))) \in \psi(\gamma_1([\mathsf{I}]))$$
  $$((\sigma_1', \gamma_1(\delta_1'(e'))), (\sigma_2', \gamma_2(\delta_2'(e')))) \in \psi(\gamma_1(C))$$

  Thus there exists $((\sigma_1, v_1), (\sigma_2, v_2)) \in \psi(\gamma_1(C))$ such that

  $$\langle \sigma_1'; \gamma_1(\delta_1'(e')) \rangle \Downarrow \langle \sigma_1''; v_1 \rangle$$
  $$\langle \sigma_2'; \gamma_2(\delta_2'(e')) \rangle \Downarrow \langle \sigma_2''; v_2 \rangle$$

  Thus, we have

  $$\langle \sigma_1' \cdot \sigma_1; \gamma_1(\mathsf{let}\ [] = \delta_1(e)\ \mathsf{in}\ \delta_1'(e')) \rangle \Downarrow \langle \sigma_1'' \cdot \sigma_1; v_1 \rangle$$
  $$\langle \sigma_2' \cdot \sigma_2; \gamma_2(\mathsf{let}\ [] = \delta_2(e)\ \mathsf{in}\ \delta_2'(e')) \rangle \Downarrow \langle \sigma_2'' \cdot \sigma_2; v_2 \rangle$$

  And we can thus conclude by compatibility of CPERs with evaluation.

- **Case** LIRRPAIR: $\dfrac{\Gamma; \Delta \vdash e : [A \otimes B] \qquad \Gamma; \Delta', a : [A], b : [B] \vdash e' : C}{\Gamma; \Delta, \Delta' \vdash \mathsf{let}\ [a, b] = e\ \mathsf{in}\ e' : C}$

  By induction, we have

  $$((\sigma_1, \gamma_1(\delta_1(e))), (\sigma_2, \gamma_2(\delta_2(e)))) \in \psi(\gamma_1([A \otimes B]))$$

  Thus we can split the σ_i into σ_i'' and σ_i''' such that

  $$((\sigma_1, *), (\sigma_1', *)) \in \psi(\gamma_1(A))$$
  $$((\sigma_2, *), (\sigma_2', *)) \in \psi(\gamma_1(B))$$

  Hence, we have

  $$((\sigma_1', \gamma_1(\delta_1'([*/a, */b]e'))), (\sigma_2', \gamma_2(\delta_2'([*/a, */b]e')))) \in \psi(\gamma_1(C))$$

  Thus there exists $((\sigma_1'', v_1), (\sigma_2'', v_2)) \in \psi(\gamma_1(C))$ such that

  $$\langle \sigma_1'; \gamma_1(\delta_1'(e')) \rangle \Downarrow \langle \sigma_1''; v_1 \rangle$$
  $$\langle \sigma_2'; \gamma_2(\delta_2'(e')) \rangle \Downarrow \langle \sigma_2''; v_2 \rangle$$

  Thus, we have

  $$\langle \sigma_1' \cdot \sigma_1; \gamma_1(\mathsf{let}\ [a, b] = \delta_1(e)\ \mathsf{in}\ \delta_1'(e')) \rangle \Downarrow \langle \sigma_1'' \cdot \sigma_1; v_1 \rangle$$
  $$\langle \sigma_2' \cdot \sigma_2; \gamma_2(\mathsf{let}\ [a, b] = \delta_2(e)\ \mathsf{in}\ \delta_2'(e')) \rangle \Downarrow \langle \sigma_2'' \cdot \sigma_2; v_2 \rangle$$

  And we can thus conclude by compatibility of CPERs with evaluation.

8. If $\Gamma; \Delta \vdash e_1 \equiv e_2 : A$ then $((\sigma_1, \gamma_1(\delta_1(e_1))), (\sigma_2, \gamma_2(\delta_2(e_2)))) \in \psi(\gamma_1(A))$.

   We case analyze the derivation of $\Gamma; \Delta \vdash e_1 \equiv e_2 : A$:

   - **Case** LREFLEX: $\dfrac{\Gamma; \Delta \vdash t : A}{\Gamma; \Delta \vdash t \equiv t : A}$

     The induction hypothesis directly solves this case.

   - **Case** LTRANS: $\dfrac{\Gamma; \Delta \vdash t \equiv t' : A \qquad \Gamma; \Delta \vdash t' \equiv t'' : A}{\Gamma; \Delta \vdash t \equiv t'' : A}$

     Intanciating the first induction hypothesis with (γ_1, γ_1), $((\sigma_1, \delta_1), (\sigma_1, \delta_1))$ and the second with (γ_1, γ_2), $((\sigma_1, \delta_1), (\sigma_2, \delta_2))$ solve this case by transitivity in CPERs.

   - **Case** IGETA: $\overline{\Gamma; \cdot \vdash \mathsf{G}^{-1}\,(\mathsf{G}\,t) \equiv t : A}$

     By induction $((\epsilon, \gamma_1(t)), (\epsilon, \gamma_2(t))) \in \psi(\gamma_1(A))$, we have

     $$\langle \epsilon; \gamma_1(\delta_1(t)) \rangle \Downarrow \langle \sigma_1'; u_1 \rangle$$

     $$\langle \epsilon; \gamma_2(\delta_2(t)) \rangle \Downarrow \langle \sigma_2'; u_2 \rangle$$

     $$((\sigma_1', u_1), (\sigma_2', u_2)) \in \psi(\gamma_1(A))$$

     We can then build the following derivation

     $$\dfrac{\mathsf{G}\,\delta_1(\gamma_1(t)) \Downarrow \mathsf{G}\,\delta_1(\gamma_1(t)) \qquad \langle \epsilon; \delta_1(\gamma_1(t)) \rangle \Downarrow \langle \sigma_1'; u_1 \rangle}{\left\langle \epsilon; \mathsf{G}^{-1}\,(\mathsf{G}\,\delta_1(\gamma_1(t))) \right\rangle \Downarrow \langle \sigma_1'; u_1 \rangle}$$

     Thus, by closure under evaluation, $((\epsilon, \mathsf{G}^{-1}\,(\mathsf{G}\,\delta_1(\gamma_1(t)))), (\epsilon, \delta_2(\gamma_2(t)))) \in \psi(\gamma_1(A))$

   - **Case** LFUNBETA: $\overline{\Gamma; \Delta \vdash (\lambda x.\ e)\ e' \equiv [e'/x]e : C}$

     By induction

     $$((\sigma_1, \gamma_1([e'/x]e)), (\sigma_2, \gamma_2([e'/x]e))) \in \psi(\gamma_1(C))$$

     So we have $((\sigma_1', v_1), (\sigma_2', v_2))$ such that

     $$\langle \sigma_1; \gamma_1([e'/x]e)) \rangle \Downarrow \langle \sigma_1'; v_1 \rangle \wedge \langle \sigma_2; \gamma_2([e'/x]e)) \rangle \Downarrow \langle \sigma_2'; v_2 \rangle$$

     We can build

     $$\dfrac{\langle \sigma_1; \lambda x.\ \gamma_1(e) \rangle \Downarrow \langle \sigma_1; \lambda x.\ \gamma_1(e) \rangle \qquad \langle \sigma_1; \gamma_1([e'/x]e) \rangle \Downarrow \langle \sigma_1'; v_1 \rangle}{\langle \sigma_1; \gamma_1(\lambda x.\ e\ e') \rangle \Downarrow \langle \sigma_1'; v_1 \rangle}$$

     And conclude.

   - **Case** LFUNETA: $\overline{\Gamma; \Delta \vdash e \equiv \lambda x.\ e\ x : A \multimap B}$

     Let $((\sigma_1', e_1'), (\sigma_2', e_2')) \in \psi(\gamma_1(A))$.

     By induction, we have

     $$((\sigma_1, \delta_1(\gamma_1(e))), (\sigma_2, \delta_2(\gamma_2(e)))) \in \psi(\gamma_1(A \multimap B))$$

     Hence there exists $((\sigma_1'', e_1''), (\sigma_2'', e_2''))$ such that

     $$\langle \sigma_1; \gamma_1(\delta_1(e)) \rangle \Downarrow \langle \sigma_1''; \lambda x.\ e_1'' \rangle \wedge \langle \sigma_2; \gamma_2(\delta_2(e)) \rangle \Downarrow \langle \sigma_2''; \lambda x.\ e_2'' \rangle$$

     $$((\sigma_1'', \lambda x.\ e_1''), (\sigma_2'', \lambda x.\ e_2'')) \in \psi(\gamma_1(A \multimap B))$$

Thus, we have $((\sigma_1'' \cdot \sigma_1', \lambda x.\ e_1''\ e_1'), (\sigma_2'' \cdot \sigma_2', \lambda x.\ e_2''\ e_2')) \in \psi(\gamma_1(B))$. Hence

$$\langle \sigma_1'' \cdot \sigma_1'; [e_1'/x]e_1'' \rangle \Downarrow \langle \sigma_1'''; v_1 \rangle \wedge \langle \sigma_2'' \cdot \sigma_2'; [e_2'/x]e_2'' \rangle \Downarrow \langle \sigma_2'''; v_2 \rangle$$

$$((\sigma_1''', v_1), (\sigma_2''', v_2)) \in \psi(\gamma_1(B))$$

We can then build

$$\frac{\langle \sigma_1 \cdot \sigma_1'; \delta_1(\gamma_1(e)) \rangle \Downarrow \langle \sigma_1'' \cdot \sigma_1'; \lambda x.\ e_1'' \rangle \qquad \langle \sigma_1'' \cdot \sigma_1'; [e_1'/x]e_1'' \rangle \Downarrow \langle \sigma_1'''; v_1 \rangle}{\langle \sigma_1 \cdot \sigma_1'; \delta_1(\gamma_1(e))\ e_1' \rangle \Downarrow \langle \sigma_1'''; v_1 \rangle}$$

$$\frac{\langle \sigma_2 \cdot \sigma_2'; \lambda x.\ \delta_2(\gamma_2(e))\ x \rangle \Downarrow \langle \sigma_2 \cdot \sigma_2'; \lambda x.\ \delta_2(\gamma_2(e))\ x \rangle \qquad \dfrac{\langle \sigma_2 \cdot \sigma_2'; \delta_2(\gamma_2(e)) \rangle \Downarrow \langle \sigma_2'' \cdot \sigma_2'; \lambda x.\ e_2'' \rangle \qquad \langle \sigma_2'' \cdot \sigma_2'; [e_2'/x]e_2'' \rangle \Downarrow \langle \sigma_2'''; v_2 \rangle}{\langle \sigma_2 \cdot \sigma_2'; \delta_2(\gamma_2(e))\ e_2' \rangle \Downarrow \langle \sigma_2'''; v_2 \rangle}}{\langle \sigma_2 \cdot \sigma_2'; \lambda x.\ \delta_2(\gamma_2(e))\ e_2' \rangle \Downarrow \langle \sigma_2'''; v_2 \rangle}$$

And conclude.

- **Case** LONEBETA: $\overline{\Gamma; \Delta \vdash \mathsf{let}\ ()=()\ \mathsf{in}\ e \equiv e : C}$
  By induction $((\sigma_1, \gamma_1(\delta_1(e))), (\sigma_2, \gamma_2(\delta_2(e)))) \in \psi(\gamma_1(C))$.
  Thus we have $((\sigma_1', e_1'), (\sigma_2', e_2')) \in \psi(\gamma_1(C))$ such that

$$\langle \sigma_1; \gamma_1(\delta_1(e)) \rangle \Downarrow \langle \sigma_1'; e_1' \rangle \wedge \langle \sigma_2; \gamma_2(\delta_2(e)) \rangle \Downarrow \langle \sigma_2'; e_2' \rangle$$

  We can build

$$\frac{\langle \sigma_1; () \rangle \Downarrow \langle \sigma_1; () \rangle \qquad \langle \sigma_1; \gamma_1(\delta_1(e)) \rangle \Downarrow \langle \sigma_1'; e_1' \rangle}{\langle \sigma; \mathsf{let}\ ()=()\ \mathsf{in}\ \gamma_1(\delta_1(e)) \rangle \Downarrow \langle \sigma_1'; e_1' \rangle}$$

  And conclude.

- **Case** LONEETA: $\dfrac{\Gamma; \Delta \vdash t : \mathsf{I} \qquad \Gamma; \Delta', x : \mathsf{I} \vdash t' : C}{\Gamma; \Delta, \Delta' \vdash \mathsf{let}\ () = t\ \mathsf{in}\ [()/x]t' \equiv [t/x]t' : C}$
  By induction

$$((\sigma_1, \gamma_1(\delta_1(t))), (\sigma_2, \gamma_2(\delta_2(t)))) \in \psi(\gamma_1(\mathsf{I}))$$

$$((\sigma_1', \gamma_1(\delta_1'([t/x]t'))), (\sigma_2', \gamma_2(\delta_2'([t/x]t')))) \in \psi(\gamma_1(C))$$

  Hence, there exists $((\sigma_1'', ()), (\sigma_2'', ())) \in \psi(\mathsf{I})$ such that

$$\langle \sigma_1; \gamma_1(\delta_1(t)) \rangle \Downarrow \langle \sigma_1''; (e_1, t_1) \rangle$$

$$\langle \sigma_2; \gamma_2(\delta_2(t)) \rangle \Downarrow \langle \sigma_2''; (e_2, t_2) \rangle$$

  Hence $((\sigma_1'', ()), (\sigma_2, \gamma_2(\delta_2(t)))) \in \psi(\mathsf{I})$.
  Thus $((\sigma_1' \cdot \sigma_1'', (\delta_1', ()/x)), (\sigma_2' \cdot \sigma_2, (\delta_2', \gamma_2(\delta_2(t))/x))) \in [\![\gamma_1(\Delta, x : \mathsf{I})]\!]$.
  Therefore, by our second induction hypothesis

$$((\sigma_1' \cdot \sigma_1'', [()/x]\gamma_1(\delta_1'(t'))), (\sigma_2' \cdot \sigma_2, \gamma_2(\delta_2'([\delta_2(t)/y]t')))) \in \psi(\gamma_1(C))$$

  Hence, there exists $((\sigma_1''', u_1), (\sigma_2''', u_2))$ such that

$$\langle \sigma_1' \cdot \sigma_1''; [()/x]\gamma_1(\delta_1'(t')) \rangle \Downarrow \langle \sigma_1'''; u_1 \rangle$$

$$\langle \sigma_2' \cdot \sigma_2; \gamma_2(\delta_2'([\delta_2(t)/x]t')) \rangle \Downarrow \langle \sigma_2'''; u_2 \rangle$$

  Thus we can build the following derivation

$$\frac{\langle \sigma_1 \cdot \sigma_1'; \gamma_1(\delta_1(t)) \rangle \Downarrow \langle \sigma_1'' \cdot \sigma_1'; () \rangle \qquad \langle \sigma_1'' \cdot \sigma_1'; [()/x]\gamma_1(\delta_1'(t')) \rangle \Downarrow \langle \sigma_1'''; u_1 \rangle}{\langle \sigma_1 \cdot \sigma_1'; \mathsf{let}\ (a, b) = \gamma_1(\delta_1(t))\ \mathsf{in}\ \gamma_1(\delta_1'([()/x]t')) \rangle \Downarrow \langle \sigma_1'''; u_1 \rangle}$$

  And conclude.

- **Case** LTensorBeta: $\dfrac{\Gamma;\Delta \vdash [t_1/a, t_2/b]t' : C}{\Gamma;\Delta \vdash \mathsf{let}\ (a, b) = (t_1, t_2)\ \mathsf{in}\ t' \equiv [t_1/a, t_2/b]t' : C}$
By induction

$$((\sigma_1, \gamma_1(\delta_1([t_1/a, t_2/b]t'))), (\sigma_2, \gamma_2(\delta_2([t_1/a, t_2/b]t')))) \in \psi(\gamma_1(C))$$

Thus, there exists $((\sigma'_1, e_1), (\sigma'_2, e_2)) \in \psi(\gamma_1(C))$ such that

$$\langle \sigma_1; \delta_1(\gamma_1([t_1/a, t_2/b]t')) \rangle \Downarrow \langle \sigma'_1; e_1 \rangle \wedge \langle \sigma_2; \delta_2(\gamma_2([t_1/a, t_2/b]t')) \rangle \Downarrow \langle \sigma'_2; e_2 \rangle$$

Thus we can build the following evaluation tree and conclude.

$$\dfrac{\langle \sigma_1; \gamma_1(\delta_1((t_1, t_2))) \rangle \Downarrow \langle \sigma_1; \gamma_1(\delta_1((t_1, t_2))) \rangle \qquad \langle \sigma_1; [t_1/a, t_2/b]\gamma_1(\delta_1(t')) \rangle \Downarrow \langle \sigma'_1; e_1 \rangle}{\langle \sigma_1; \gamma_1(\delta_1(\mathsf{let}\ (a, b) = (t_1, t_2)\ \mathsf{in}\ t')) \rangle \Downarrow \langle \sigma'_1; e_1 \rangle}$$

- **Case** LTensorEta: $\dfrac{\Gamma;\Delta \vdash t : A \otimes B \qquad \Gamma;\Delta', x : A \otimes B \vdash t' : C}{\Gamma;\Delta, \Delta' \vdash \mathsf{let}\ (a, b) = t\ \mathsf{in}\ [(a, b)\,/x]t' \equiv [t/x]t' : C}$
By induction

$$((\sigma_1, \gamma_1(\delta_1(t))), (\sigma_2, \gamma_2(\delta_2(t)))) \in \psi(\gamma_1(A \otimes B))$$
$$((\sigma'_1, \gamma_1(\delta'_1([t/x]t'))), (\sigma'_2, \gamma_2(\delta'_2([t/x]t')))) \in \psi(\gamma_1(C))$$

Hence, there exists $((\sigma''_1, (e_1, t_1)), (\sigma''_2, (e_2, t_2))) \in \psi(\gamma_1(A \otimes B))$ such that

$$\langle \sigma_1; \gamma_1(\delta_1(t)) \rangle \Downarrow \langle \sigma''_1; (e_1, t_1) \rangle$$
$$\langle \sigma_2; \gamma_2(\delta_2(t)) \rangle \Downarrow \langle \sigma''_2; (e_2, t_2) \rangle$$

Hence $((\sigma''_1, (e_1, t_1)), (\sigma_2, \gamma_2(\delta_2(t)))) \in \psi(\gamma_1(A \otimes B))$.
Thus $((\sigma'_1 \cdot \sigma''_1, (\delta'_1, (e_1, t_1)\,/x)), (\sigma'_2 \cdot \sigma_2, (\delta'_2, \gamma_2(\delta_2(t))/x))) \in [\![\gamma_1(\Delta, x : A \otimes B)]\!]$.
Therefore, by our second induction hypothesis

$$((\sigma'_1 \cdot \sigma''_1, [(e_1, t_1)\,/x]\gamma_1(\delta'_1(t'))), (\sigma'_2 \cdot \sigma_2, \gamma_2(\delta'_2([\delta_2(t)/y]t')))) \in \psi(\gamma_1(C))$$

Hence, there exists $((\sigma'''_1, u_1), (\sigma'''_2, u_2))$ such that

$$\langle \sigma'_1 \cdot \sigma''_1; [(e_1, t_1)\,/x]\gamma_1(\delta'_1(t')) \rangle \Downarrow \langle \sigma'''_1; u_1 \rangle$$
$$\langle \sigma'_2 \cdot \sigma_2; \gamma_2(\delta'_2([\delta_2(t)/x]t')) \rangle \Downarrow \langle \sigma'''_2; u_2 \rangle$$

Thus we can build the following derivation

$$\dfrac{\langle \sigma_1 \cdot \sigma'_1; \gamma_1(\delta_1(t)) \rangle \Downarrow \langle \sigma''_1 \cdot \sigma'_1; (e_1, t_1) \rangle \qquad \langle \sigma''_1 \cdot \sigma'_1; [\gamma_1(\delta_1((e_1, t_1)))/x]\gamma_1(\delta'_1(t')) \rangle \Downarrow \langle \sigma'''_1; u_1 \rangle}{\langle \sigma_1 \cdot \sigma'_1; \mathsf{let}\ (a, b) = \gamma_1(\delta_1(t))\ \mathsf{in}\ \gamma_1(\delta'_1([(a, b)\,/x]t')) \rangle \Downarrow \langle \sigma'''_1; u_1 \rangle}$$

- **Case** LFBeta: $\dfrac{}{\Gamma;\Delta \vdash \mathsf{let}\ \mathsf{F}\,(x, a) = \mathsf{F}\,(e, t)\ \mathsf{in}\ t' \equiv [e/x, t/a]t' : C}$ By induction

$$((\sigma_1, \gamma_1(\delta_1([e/x, t/a]t'))), (\sigma_2, \gamma_2(\delta_2([e/x, t/a]t')))) \in \psi(\gamma_1(C))$$

Thus, there exists $((\sigma'_1, u_1), (\sigma'_2, u_2)) \in \psi(\gamma_1(C))$ such that

$$\langle \sigma_1; \delta_1(\gamma_1([e/x, t/a]t')) \rangle \Downarrow \langle \sigma'_1; u_1 \rangle \wedge \langle \sigma_2; \delta_2(\gamma_2([e/x, t/a]t')) \rangle \Downarrow \langle \sigma'_2; u_2 \rangle$$

Thus we can build the following evaluation tree and conclude.

$$\dfrac{\langle \sigma_1; \gamma_1(\delta_1(\mathsf{F}\,(e, t))) \rangle \Downarrow \langle \sigma_1; \gamma_1(\delta_1(\mathsf{F}\,(e, t))) \rangle \qquad \langle \sigma_1; [\gamma_1(e)/x, \gamma_1(\delta_1(t))/a]\gamma_1(\delta_1(t')) \rangle \Downarrow \langle \sigma'_1; u_1 \rangle}{\langle \sigma_1; \gamma_1(\delta_1(\mathsf{let}\ \mathsf{F}\,(e, t) = \mathsf{F}\,(e, t)\ \mathsf{in}\ t')) \rangle \Downarrow \langle \sigma'_1; u_1 \rangle}$$

- **Case** LFEta: $\dfrac{\Gamma; \Delta \vdash t : \mathsf{F}x : X.\ A \qquad \Gamma; \Delta', y : \mathsf{F}x : X.\ A \vdash t' : C}{\Gamma; \Delta \vdash \mathsf{let}\ \mathsf{F}\,(x, a) = t\ \mathsf{in}\ [\mathsf{F}\,(x, a)\,/y]t' \equiv [t/y]t' : C}$
  By induction
  $$((\sigma_1, \gamma_1(\delta_1(t))), (\sigma_2, \gamma_2(\delta_2(t)))) \in \psi(\gamma_1(\mathsf{F}x : X.\ A))$$
  $$((\sigma_1', \gamma_1(\delta_1'([\delta_1(t)/y]t'))), (\sigma_2', \gamma_2(\delta_2'([\delta_1(t)/y]t')))) \in \psi(\gamma_1(C))$$
  Hence, there exists $((\sigma_1'', \mathsf{F}\,(e_1, t_1)), (\sigma_2'', \mathsf{F}\,(e_2, t_2))) \in \psi(\gamma_1(\mathsf{F}x : X.\ A))$ such that
  $$\langle \sigma_1; \gamma_1(\delta_1(t)) \rangle \Downarrow \langle \sigma_1''; \mathsf{F}\,(e_1, t_1) \rangle$$
  $$\langle \sigma_2; \gamma_2(\delta_2(t)) \rangle \Downarrow \langle \sigma_2''; \mathsf{F}\,(e_2, t_2) \rangle$$
  Hence $((\sigma_1'', \mathsf{F}\,(e_1, t_1)), (\sigma_2, \gamma_2(\delta_2(t)))) \in \psi(\gamma_1(\mathsf{F}x : X.\ A))$.
  Thus $((\sigma_1' \cdot \sigma_1'', (\delta_1', \mathsf{F}\,(e_1, t_1)\,/y)), (\sigma_2' \cdot \sigma_2, (\delta_2', \gamma_2(\delta_2(t))/y))) \in [\![\gamma_1(\Delta, y : \mathsf{F}x : X.\ A)]\!]$.
  Therefore, by our second induction hypothesis
  $$((\sigma_1' \cdot \sigma_1'', [\mathsf{F}\,(e_1, t_1)\,/y]\gamma_1(\delta_1'(t'))), (\sigma_2' \cdot \sigma_2, \gamma_2(\delta_2'([\delta_2(t)/y]t')))) \in \psi(\gamma_1(C))$$
  Hence, there exists $((\sigma_1''', u_1), (\sigma_2''', u_2))$ such that
  $$\langle \sigma_1' \cdot \sigma_1''; [\mathsf{F}\,(e_1, t_1)\,/y]\gamma_1(\delta_1'(t')) \rangle \Downarrow \langle \sigma_1'''; u_1 \rangle$$
  $$\langle \sigma_2' \cdot \sigma_2; \gamma_2(\delta_2'([\delta_2(t)/y]t')) \rangle \Downarrow \langle \sigma_2'''; u_2 \rangle$$
  Thus we can build the following derivation
  $$\dfrac{\langle \sigma_1 \cdot \sigma_1'; \gamma_1(\delta_1(t)) \rangle \Downarrow \langle \sigma_1'' \cdot \sigma_1'; \mathsf{F}\,(e_1, t_1) \rangle \qquad \langle \sigma_1'' \cdot \sigma_1'; [\mathsf{F}\,(e_1, t_1)\,/y]\gamma_1(\delta_1'(t')) \rangle \Downarrow \langle \sigma_1'''; u_1 \rangle}{\langle \sigma_1 \cdot \sigma_1'; \mathsf{let}\ \mathsf{F}\,(x, a) = \gamma_1(\delta_1(t))\ \mathsf{in}\ \gamma_1(\delta_1'([\mathsf{F}\,(x, a)\,/y]t')) \rangle \Downarrow \langle \sigma_1'''; u_1 \rangle}$$
  And conclude.
- **Case** LPiBeta: $\overline{\Gamma; \Delta \vdash (\hat\lambda x.\ e)\ e' \equiv [e'/x]e : C}$
  By induction:
  - $((\sigma_1, \delta_1(\gamma_1([e'/x]e))), (\sigma_2, \delta_2(\gamma_2([e'/x]e)))) \in \psi(\gamma_1(C))$
  - $((\sigma_1, \delta_1(\gamma_1((\hat\lambda x.\ e)\ e'))), (\sigma_2, \delta_2(\gamma_2((\lambda x.\ e)\ e')))) \in \psi(\gamma_1(Z))$

  This means that we have $((\sigma_1', e_1''), (\sigma_2', e_2''))$ such that
  $$\langle \sigma_1; \gamma_1([e'/x]\delta_1(e)) \rangle \Downarrow \langle \sigma_1'; e_1'' \rangle$$
  $$\langle \sigma_2; \gamma_2([e'/x]\delta_2(e)) \rangle \Downarrow \langle \sigma_2'; e_2'' \rangle$$
  We then build
  $$\dfrac{\left\langle \sigma_1; \hat\lambda x.\ \delta_1(\gamma_1(e)) \right\rangle \Downarrow \left\langle \sigma_1; \hat\lambda x.\ \delta_1(\gamma_1(e)) \right\rangle \qquad \langle \sigma_1; [\gamma_1(e')/x]\delta_1(\gamma_1(e)) \rangle \Downarrow \langle \sigma_1'; e_1'' \rangle}{\left\langle \sigma_1; \gamma_1(\hat\lambda x.\ \delta_1(e)\ e') \right\rangle \Downarrow \langle \sigma_1'; e_1'' \rangle}$$
  Since our CPER $\psi(\gamma_1(Z))$ is closed under evaluation, we have the expected result.
- **Case** LPiEta: $\overline{\Gamma; \Delta \vdash e \equiv \hat\lambda x.\ e\ x : \Pi x : X.\ A}$
  By induction
  $$((\sigma_1, \delta_1(\gamma_1(e))), (\sigma_2, \delta_2(\gamma_2(e)))) \in \psi(\Pi x : \gamma_1(X).\ \gamma_1(A))$$
  $$((\sigma_1, \gamma_1(\delta_1(\hat\lambda x.\ e))), (\sigma_2, \delta_2(\gamma_2(\hat\lambda x.\ e)))) \in \psi(\Pi x : \gamma_1(X).\ \gamma_1(A))$$

Let $(e_1', e_2') \in \phi(\gamma_1(X))$.
We know that there is $((\sigma_1', \nu_1), (\sigma_2', \nu_2)) \in \phi((\gamma_1, e_1')/x)(A))$ such that

$$\langle \sigma_1; \delta_1(\gamma_1(e))\ e_1' \rangle \Downarrow \langle \sigma_1'; \nu_1 \rangle$$

$$\langle \sigma_2; \delta_2(\gamma_2(e))\ e_2' \rangle \Downarrow \langle \sigma_2'; \nu_2 \rangle$$

We can use this to build a derivation

$$\frac{\langle \sigma_1; \hat{\lambda} x.\ \delta_2(\gamma_2(e))\ x \rangle \Downarrow \langle \sigma_1; \lambda x : A.\ \delta_2(\gamma_2(e))\ x \rangle \qquad \langle \sigma_1; \delta_2(\gamma_2(e))\ e_2' \rangle \Downarrow \langle \sigma_1'; \nu_2 \rangle}{\langle \sigma_1; \lambda x.\ \delta_2(\gamma_2(e))\ x\ e_2' \rangle \Downarrow \langle \sigma_1'; \nu_2 \rangle}$$

Then we have $((\sigma_1, \delta_1(\gamma_1(e))\ e_1')(\sigma_2, \delta_2(\gamma_2(\lambda x.\ e))\ e_2')) \in \phi((\gamma_1, e_1'/x)Y)$ which is what we need.

- **Case** LALLETA: $\dfrac{\Gamma, x : X; \Delta \vdash e \equiv e' : Y}{\Gamma; \Delta \vdash e \equiv e' : \forall x : X.\ Y}$

  Since $x \notin FV(e, e')$, the result follows directly from the induction hypothesis which tells us $\forall (e'', e''') \in \phi(\gamma_1(X)), ((\sigma_1, \delta_1(\gamma_1(e))), (\sigma_2, \delta_2(\gamma_2(e')))) \in \psi(\gamma_1([e''/x]\forall x : X.\ Y))$

- **Case** IALLBETA: $\dfrac{\Gamma; \Delta \vdash e \equiv e' : \forall x : X.\ Y \qquad \Gamma \vdash t : X}{\Gamma; \Delta \vdash e \equiv e' : [t/x]Y}$

  By induction $((\sigma_1, \delta_1(\gamma_1(e))), (\sigma_2, \delta_2(\gamma_2(e')))) \in \psi(\gamma_1(\forall x : X.\ Y))$ and $(\gamma_1(t), \gamma_2(t)) \in \phi(\gamma_1(X))$.
  Thus we have $((\sigma_1, \nu_1), (\sigma_2, \nu_2)) \in \psi(\gamma_1(\forall x : X.\ Y))$ such that $\langle \sigma_1; \gamma_1(e) \rangle \Downarrow \langle \sigma_1'; \nu_1 \rangle$, $\langle \sigma_2; \gamma_2(e') \rangle \Downarrow \langle \sigma_2'; \nu_2 \rangle$ and $((\sigma_1', \nu_1), (\sigma_2', \nu_2)) \in \psi([\gamma_1(t)/x]\gamma_1(Y))$.
  Hence $((\sigma_1, \delta_1(\gamma_1(e))), (\sigma_2, \delta_2(\gamma_2(e')))) \in \phi([\gamma_1(t)/x]\gamma_1(Y))$.

- **Case** IEXBETA: $\dfrac{\Gamma; \Delta \vdash e \equiv e' : [t/x]Y \qquad \Gamma \vdash t : X}{\Gamma; \Delta \vdash e \equiv e' : \exists x : X.\ Y}$

  By induction

  $$((\sigma_2, \gamma_1(\delta_1(e))), (\sigma_2, \gamma_2(\delta_2(e')))) \in \phi(\gamma_1([t/x]Y))$$

  $$(\gamma_1(t), \gamma_2(t)) \in \phi(\gamma_1(X))$$

  which gives us our result by taking $\gamma_1(t)$ as our witness.

- **Case** IEXETA: $\dfrac{\Gamma, x : X; \Delta, y : Y \vdash e \equiv e' : Z \qquad x \notin FV(e, e', Z)}{\Gamma \vdash \Delta, y : \exists x : X.\ Y \equiv e : e'Z}$

  Let $((\sigma_1, \delta_1), (\sigma_2, \delta_2)) \in \psi(\gamma_1(Z))$ and$((\sigma_1', t_1), (\sigma_2', t_2)) \in \psi(\gamma_1(\exists x : X.\ Y))$.
  By this second hypothesis, there exists $(t', t') \in \phi(\gamma_1(X))$ such that $(t_1, t_2) \in \phi(\gamma_1([t'/x]Y))$. Thus $((\gamma_1, t'/x), (\gamma_2, t'/x)) \in [\![\Gamma]\!]$ and $(((\sigma_1 \cdot \sigma_1'), (\delta_1, t_1/y)), ((\sigma_2 \cdot \sigma_2'), (\delta_2, t_2/y))) \in [\![\Delta, y : Y]\!]$ By induction, since $x \notin FV(e, e', Z)$,

  $$([t_1/y]\gamma_1(\delta_1(e)), [t_2/y]\gamma_2(\delta_2(e))) \in \psi([t_1/y]\gamma_1(Z))$$

  which is what we wanted.

- **Case** LWITHBETAFST: $\overline{\Gamma; \Delta \vdash \pi_1\,(e, e') \equiv e : A}$

  By induction, $((\sigma_1, \delta_1(\gamma_1(e))), (\sigma_1, \delta_1(\gamma_1(e)))) \in \psi(\gamma_1(A))$, so there exists $((\sigma_1', \nu_1), (\sigma_2', \nu_2))$ such that

  $$\langle \sigma_1; \delta_1(\gamma_1(e)) \rangle \Downarrow \langle \sigma_1'; \nu_1 \rangle \wedge \langle \sigma_2; \delta_2(\gamma_2(e)) \rangle \Downarrow \langle \sigma_2'; \nu_2 \rangle$$

  $$((\sigma_1, \nu_1), (\sigma_2, \nu_2)) \in \psi(\gamma_1(A))$$

  Thus we can build the following and conclude.

  $$\frac{\langle \sigma_1; \delta_1(\gamma_1((e, e'))) \rangle \Downarrow \langle \sigma_1; \delta_1(\gamma_1((e, e'))) \rangle \qquad \langle \sigma_1; \gamma_1(e) \rangle \Downarrow \langle \sigma_1'; \nu_1 \rangle}{\langle \sigma_1; \delta_1(\gamma_1(\pi_1\,(e, e'))) \rangle \Downarrow \langle \sigma_1'; \nu_1 \rangle}$$

- **Case** LWITHBETASND: $\overline{\Gamma; \Delta \vdash \pi_2\,(e, e') \equiv e' : B}$
  By induction, $((\sigma_1, \delta_1(\gamma_1(e'))), (\sigma_1, \delta_1(\gamma_1(e')))) \in \psi(\gamma_1(B))$, so there exists $((\sigma_1', v_1), (\sigma_2', v_2))$ such that
  $$\langle \sigma_1; \delta_1(\gamma_1(e')) \rangle \Downarrow \langle \sigma_1'; v_1 \rangle \wedge \langle \sigma_2; \delta_2(\gamma_2(e')) \rangle \Downarrow \langle \sigma_2'; v_2 \rangle$$
  $$((\sigma_1, v_1), (\sigma_2, v_2)) \in \psi(\gamma_1(B))$$
  Thus we can build the following and conclude.
  $$\frac{\langle \sigma_1; \delta_1(\gamma_1((e, e'))) \rangle \Downarrow \langle \sigma_1; \delta_1(\gamma_1((e, e'))) \rangle \qquad \langle \sigma_1; \gamma_1(e) \rangle \Downarrow \langle \sigma_1'; v_1 \rangle}{\langle \sigma_1; \delta_1(\gamma_1(\pi_2\,(e, e'))) \rangle \Downarrow \langle \sigma_1'; v_1 \rangle}$$

- **Case** LWITHETA: $\overline{\Gamma; \Delta \vdash e \equiv (\pi_1\, e, \pi_2\, e) : A \,\&\, B}$
  By induction $((\sigma_1, \gamma_1(e)), (\sigma_2, \gamma_2(e))) \in \psi(\gamma_1(A \,\&\, B))$, so there exists e_1', e_2', e_1'', e_2'' such that
  $$\langle \sigma_1; \gamma_1(e) \rangle \Downarrow \langle \sigma_1'; (e_1', e_1'') \rangle$$
  $$\langle \sigma_2; \gamma_2(e) \rangle \Downarrow \langle \sigma_2'; (e_2', e_2'') \rangle$$
  $$((\sigma_1', e_1'), (\sigma_2', e_2')) \in \psi(\gamma_1(A))$$
  $$((\sigma_1', e_1''), (\sigma_2', e_2'')) \in \psi(\gamma_1(B))$$
  It suffices to show that $((\sigma_1', (e_1', e_1'')), (\sigma_2, \gamma_2((\pi_1\, e, \pi_2\, e)))) \in \psi(\gamma_1(A \,\&\, B))$
  We can build the following derivations
  $$\frac{\langle \sigma_2; \gamma_2(e) \rangle \Downarrow \langle \sigma_2'; (e_2', e_2'') \rangle \qquad \langle \sigma_2'; e_2' \rangle \Downarrow \langle \sigma_2'; e_2' \rangle}{\langle \sigma_2; \pi_1\, \gamma_2(e) \rangle \Downarrow \langle \sigma_2'; e_2' \rangle}$$
  $$\frac{\langle \sigma_2; \gamma_2(e) \rangle \Downarrow \langle \sigma_2'; (e_2', e_2'') \rangle \qquad \langle \sigma_2'; e_2'' \rangle \Downarrow \langle \sigma_2'; e_2'' \rangle}{\langle \sigma_2; \pi_2\, \gamma_2(e) \rangle \Downarrow \langle \sigma_2'; e_2'' \rangle}$$
  Thus we have
  $$((\sigma_1', e_1'), (\sigma_2, \pi_1\, \gamma_2(e))) \in \psi(\gamma_1(A))$$
  $$((\sigma_1', e_1''), (\sigma_2, \pi_2\, \gamma_2(e))) \in \psi(\gamma_1(B))$$
  Which brings us the conclusion by the definition of the semantics of $\&$.

- **Case** LTBETA: $\overline{\Gamma; \Delta \vdash \mathsf{let\ val}\, x = \mathsf{val}\, t \;\mathsf{in}\; t' \equiv [t/x]t' : \mathsf{T}\, C}$
  Let σ_f and σ_g be heaps such that $\sigma_f \# \sigma_1$ and $\sigma_g \# \sigma_2$.
  By induction $((\sigma_1, \gamma_1(\delta_1([t/x]t'))), (\sigma_2, \gamma_2(\delta_2([t/x]t')))) \in \psi(\mathsf{T}\, \gamma_1(C))$.
  Thus, there exists $((\sigma_1', v_1), (\sigma_2', v_2)) \in \psi(\gamma_1(C))$ such that
  $$\langle \sigma_1 \cdot \sigma_f; \gamma_1(\delta_1([t/x]t')) \rangle \rightsquigarrow \langle \sigma_1' \cdot \sigma_f; \mathsf{val}\, v_1 \rangle$$
  $$\langle \sigma_2 \cdot \sigma_g; \gamma_2(\delta_2([t/x]t')) \rangle \rightsquigarrow \langle \sigma_2' \cdot \sigma_g; \mathsf{val}\, v_2 \rangle$$
  Hence we can conclude with the following derivation.
  $$\frac{\langle \sigma_1 \cdot \sigma_f; \mathsf{val}\, t \rangle \rightsquigarrow \langle \sigma_1 \cdot \sigma_f; \mathsf{val}\, t \rangle \qquad \langle \sigma_1 \cdot \sigma_f; [t/x]t' \rangle \rightsquigarrow \langle \sigma_1' \cdot \sigma_f; \mathsf{val}\, v_1 \rangle}{\langle \sigma_1 \cdot \sigma_f; \mathsf{let\ val}\, x = \mathsf{val}\, t \;\mathsf{in}\; t' \rangle \rightsquigarrow \langle \sigma_1' \cdot \sigma_f; \mathsf{val}\, v_1 \rangle}$$

- **Case** LTETA: $\overline{\Gamma; \Delta \vdash \mathsf{let\ val}\, x = t \ \mathsf{in\ val}\, x \equiv t : \mathsf{T}\, C}$
  Let σ_f and σ_g be heaps such that $\sigma_f \# \sigma_1$ and $\sigma_g \# \sigma_2$.
  By induction $((\sigma_1, \gamma_1(\delta_1(t))), (\sigma_2, \gamma_2(\delta_2(t)))) \in \psi(\mathsf{T}\,\gamma_1(C))$.
  Thus, there exists $((\sigma_1', v_1), (\sigma_2', v_2)) \in \psi(\gamma_1(C))$ such that

$$\langle \sigma_1 \cdot \sigma_f; \gamma_1(\delta_1(t)) \rangle \rightsquigarrow \langle \sigma_1' \cdot \sigma_f; \mathsf{val}\, v_1 \rangle$$

$$\langle \sigma_2 \cdot \sigma_g; \gamma_2(\delta_2(t)) \rangle \rightsquigarrow \langle \sigma_2' \cdot \sigma_g; \mathsf{val}\, v_2 \rangle$$

  Hence we can conclude with the following derivation.

$$\frac{\langle \sigma_1 \cdot \sigma_f; \gamma_1(\delta_1(t)) \rangle \rightsquigarrow \langle \sigma_1' \cdot \sigma_f; \mathsf{val}\, v_1 \rangle \qquad \langle \sigma_1 \cdot \sigma_f; \mathsf{val}\, v_1 \rangle \rightsquigarrow \langle \sigma_1 \cdot \sigma_f; \mathsf{val}\, v_1 \rangle}{\langle \sigma_1 \cdot \sigma_f; \gamma_1(\delta_1(\mathsf{let\ val}\, x = t \ \mathsf{in\ val}\, x)) \rangle \rightsquigarrow \langle \sigma_1' \cdot \sigma_f; \mathsf{val}\, v_1 \rangle}$$

- **Case** LTASSOC: $\dfrac{\Gamma; \Delta \vdash t_1 : \mathsf{T}\, A \qquad \Gamma; \Delta', x : A \vdash t_2 : \mathsf{T}\, B \qquad \Gamma; \Delta'', y : B \vdash t_3 : \mathsf{T}\, C}{\Gamma; \Delta, \Delta', \Delta'' \vdash \mathsf{let\ val}\, y = (\mathsf{let\ val}\, x = t_1 \ \mathsf{in}\ t_2) \ \mathsf{in}\ t_3 \equiv \mathsf{let\ val}\, x = t_1 \ \mathsf{in\ let\ val}\, y = t_2 \ \mathsf{in}\ t_3 : \mathsf{T}\, C}$
  Let σ_f and σ_g be heaps such that $\sigma_f \# \sigma_1 \cdot \sigma_1' \cdot \sigma_1''$ and $\sigma_g \# \sigma_2 \cdot \sigma_2' \cdot \sigma_2''$.
  By induction $((\sigma_1, \gamma_1(\delta_1(t_1))), (\sigma_2, \gamma_2(\delta_2(t_1)))) \in \psi(\mathsf{T}\,\gamma_1(A))$, thus there is $((\sigma_1''', u_1), (\sigma_2''', u_2)) \in \psi(\gamma_1(A))$ such that

$$\langle \sigma_1 \cdot \sigma_1' \cdot \sigma_1'' \cdot \sigma_f; \delta_1(\gamma_1(t_1)) \rangle \rightsquigarrow \langle \sigma_1''' \cdot \sigma_1' \cdot \sigma_1'' \cdot \sigma_f; \mathsf{val}\, u_1 \rangle$$

$$\langle \sigma_2 \cdot \sigma_2' \cdot \sigma_2'' \cdot \sigma_f; \delta_2(\gamma_2(t_1)) \rangle \rightsquigarrow \langle \sigma_2''' \cdot \sigma_2' \cdot \sigma_2'' \cdot \sigma_g; \mathsf{val}\, u_2 \rangle$$

  Notice that

$$((\sigma_1' \cdot \sigma_1''', (\delta_1', u_1/x)), (\sigma_2' \cdot \sigma_2''', (\delta_2', u_2/x))) \in [\![\Delta', x : A]\!]$$

  Thus by induction $((\sigma_1' \cdot \sigma_1''', \gamma_1(\delta_1'(t_2))), (\sigma_2' \cdot \sigma_2''', \gamma_2(\delta_2'(t_2)))) \in \psi(\mathsf{T}\,\gamma_1(B))$, thus there is $((\sigma_1'''', v_1), (\sigma_2'''', v_2)) \in \psi(\gamma_1(B))$ such that

$$\langle \sigma_1''' \cdot \sigma_1' \cdot \sigma_1'' \cdot \sigma_f; [u_1/x]\delta_1'(\gamma_1(t_2)) \rangle \rightsquigarrow \langle \sigma_1'''' \cdot \sigma_1'' \cdot \sigma_f; \mathsf{val}\, v_1 \rangle$$

$$\langle \sigma_2''' \cdot \sigma_2' \cdot \sigma_2'' \cdot \sigma_f; [u_2/x]\delta_2'(\gamma_2(t_2)) \rangle \rightsquigarrow \langle \sigma_2'''' \cdot \sigma_2'' \cdot \sigma_g; \mathsf{val}\, v_2 \rangle$$

  Notice that

$$((\sigma_1'' \cdot \sigma_1'''', (\delta_1'', v_1/x)), (\sigma_2'' \cdot \sigma_2'''', (\delta_2'', v_2/x))) \in [\![\Delta'', y : B]\!]$$

  Thus by induction $((\sigma_1'' \cdot \sigma_1'''', \gamma_1(\delta_1''(t_3))), (\sigma_2'' \cdot \sigma_2'''', \gamma_2(\delta_2''(t_3)))) \in \psi(\mathsf{T}\,\gamma_1(C))$, thus there is $((\sigma_1''''', w_1), (\sigma_2''''', w_2)) \in \psi(\gamma_1(B))$ such that

$$\langle \sigma_1'''' \cdot \sigma_1'' \cdot \sigma_f; [v_1/y]\delta_1''(\gamma_1(t_3)) \rangle \rightsquigarrow \langle \sigma_1''''' \cdot \sigma_f; \mathsf{val}\, w_1 \rangle$$

$$\langle \sigma_2'''' \cdot \sigma_2'' \cdot \sigma_f; [v_2/y]\delta_2''(\gamma_2(t_3)) \rangle \rightsquigarrow \langle \sigma_2''''' \cdot \sigma_g; \mathsf{val}\, w_2 \rangle$$

  We can then build the following derivations to conclude.

$$\frac{\dfrac{\langle \sigma_1 \cdot \sigma_1' \cdot \sigma_1'' \cdot \sigma_f; \gamma_1(\delta_1(t_1)) \rangle \rightsquigarrow \langle \sigma_1''' \cdot \sigma_1' \cdot \sigma_1'' \cdot \sigma_f; \mathsf{val}\, u_1 \rangle \qquad \langle \sigma_1''' \cdot \sigma_1' \cdot \sigma_1'' \cdot \sigma_f; [u_1/x]\gamma_1(\delta_1'(t_2)) \rangle \rightsquigarrow \langle \sigma_1'''' \cdot \sigma_1'' \cdot \sigma}{\langle \sigma_1 \cdot \sigma_1' \cdot \sigma_1'' \cdot \sigma_f; \gamma_1(\mathsf{let\ val}\, x = \delta_1(t_1) \ \mathsf{in}\ \delta_1'(t_2)) \rangle \rightsquigarrow \langle \sigma_1'''' \cdot \sigma_1'' \cdot \sigma_f; \mathsf{val}\, v_1 \rangle}}{\langle \sigma_1 \cdot \sigma_1' \cdot \sigma_1'' \cdot \sigma_f; \gamma_1(\mathsf{let\ val}\, y = (\mathsf{let\ val}\, x = \delta_1(t_1) \ \mathsf{in}\ \delta_1'(t_2)) \ \mathsf{in}\ \delta_1''(t_3)}$$

$$\frac{\langle \sigma_2 \cdot \sigma_2'\sigma_2'' \cdot \sigma_g; \gamma_2(\delta_2(t_1)) \rangle \rightsquigarrow \langle \sigma_2''' \cdot \sigma_2' \cdot \sigma_2'' \cdot \sigma_g; \mathsf{val}\, u_2 \rangle \qquad \dfrac{\langle \sigma_2''' \cdot \sigma_2' \cdot \sigma_2'' \cdot \sigma_g; [u_2/x]\gamma_2(\delta_2'(t_2)) \rangle \rightsquigarrow \langle \sigma_2'''' \cdot \sigma_2'' \cdot \sigma_g}{\langle \sigma_2''' \cdot \sigma_2' \cdot \sigma_2'' \cdot \sigma_g; [u_2/x]\gamma_2(\mathsf{let\ v}}}{\langle \sigma_2 \cdot \sigma_2' \cdot \sigma_2'' \cdot \sigma_g; \gamma_2(\mathsf{let\ val}\, x = \delta_2(t_1) \ \mathsf{in\ let\ val}\, y = \delta_2'(t_2) \ \mathsf{in}\ \delta_2''(t_3))}$$

- **Case** LTENSORCONG: $\dfrac{\Gamma;\Delta \vdash t_1 \equiv t_1' : A \qquad \Gamma;\Delta' \vdash t_2 \equiv t_2' : B}{\Gamma;\Delta,\Delta' \vdash (t_1,t_2) \equiv (t_1',t_2') : A \otimes B}$
  By induction
  $$((\sigma_1, \gamma_1(\delta_1(t_1))), (\sigma_2, \gamma_2(\delta_2(t_1')))) \in \psi(\gamma_1(A))$$
  $$((\sigma_1', \gamma_1(\delta_1'(t_2))), (\sigma_2', \gamma_2(\delta_2'(t_2')))) \in \psi(\gamma_1(B))$$
  And with $\sigma_1 \# \sigma_1'$ and $\sigma_2 \# \sigma_2'$, we have all we need.
- **Case** LTENSORECONG: $\dfrac{\Gamma;\Delta \vdash t_1 \equiv t_1' : A \otimes B \qquad \Gamma;\Delta', a : A, b : B \vdash t_2 \equiv t_2' : C}{\Gamma;\Delta,\Delta' \vdash \mathsf{let}\ (a,b) = t_1\ \mathsf{in}\ t_2 \equiv \mathsf{let}\ (a,b) = t_1'\ \mathsf{in}\ t_2' : C}$
  By our first induction hypothesis,
  $$((\sigma_1, \gamma_1(\delta_1(t_1))), (\sigma_2, \gamma_2(\delta_2(t_1')))) \in \psi(\gamma_1(\delta_1(A \otimes B)))$$
  Therefore, there exists $((\sigma_1'' \cdot \sigma_1''', (u_1, u_1')), (\sigma_2'' \cdot \sigma_2''', (u_2, u_2')) \in \psi(\gamma_1(A \otimes B))$ such that
  $$\langle \sigma_1; \gamma_1(\delta_1(t_1)) \rangle \Downarrow \langle \sigma_1'' \cdot \sigma_1'''; (u_1, u_1') \rangle$$
  $$\langle \sigma_2; \gamma_2(\delta_2(t_1')) \rangle \Downarrow \langle \sigma_2'' \cdot \sigma_2'''; (u_2, u_2') \rangle$$
  $$((\sigma_1'', u_1), (\sigma_2'', u_2)) \in \psi(\gamma_1(A))$$
  $$((\sigma_1''', u_1'), (\sigma_2''', u_2')) \in \psi(\gamma_1(B))$$
  In particular, it means that
  $$((\sigma_1'' \cdot \sigma_1''' \cdot \sigma_1', (\delta_1', u_1/a, u_1'/b)), (\sigma_2'' \cdot \sigma_2''' \cdot \sigma_2', (\delta_2', u_2/a, u_2'/b))) \in [\![\gamma_1(\Delta', a : A, b : B)]\!]$$
  Thus, by our second induction hypothesis
  $$((\sigma_1'' \cdot \sigma_1''' \cdot \sigma_1', [u_1/a, u_1'/b]\gamma_1(\delta_1'(t_2))), (\sigma_2'' \cdot \sigma_2''' \cdot \sigma_2', [u_2/a, u_2'/b]\gamma_2(\delta_2'(t_2')))) \in \psi(\gamma_1(C))$$
  We then have $((\sigma_1'''', v_1), (\sigma_2'''', v_2)) \in \psi(\gamma_1(C))$ such that
  $$\langle \sigma_1' \cdot \sigma_1'' \cdot \sigma_1'''; \gamma_1(\delta_1'([u_1/a, u_1'/b]t_2)) \rangle \Downarrow \langle \sigma_1'''; v_1 \rangle$$
  $$\langle \sigma_2' \cdot \sigma_2'' \cdot \sigma_2'''; \gamma_2(\delta_2'([u_2/a, u_2'/b]t_2')) \rangle \Downarrow \langle \sigma_2'''; v_2 \rangle$$
  Hence we can build the following derivations and conclude.

$$\dfrac{\langle \sigma_1 \cdot \sigma_1'; \gamma_1(\delta_1(t_1)) \rangle \Downarrow \langle \sigma_1'' \cdot \sigma_1''' \cdot \sigma_1'; (u_1, u_1') \rangle \qquad \langle \sigma_1'' \cdot \sigma_1''' \cdot \sigma_1'; [u_1/a, u_1'/b]\gamma_1(\delta_1'(t_2)) \rangle \Downarrow \langle \sigma_1''''; v_1 \rangle}{\langle \sigma_1 \cdot \sigma_1'; \gamma_1(\delta_1(\delta_1'(\mathsf{let}\ (a,b) = t_1\ \mathsf{in}\ t_2))) \rangle \Downarrow \langle \sigma_1''''; v_1 \rangle}$$

$$\dfrac{\langle \sigma_2 \cdot \sigma_2'; \gamma_2(\delta_2(t_1')) \rangle \Downarrow \langle \sigma_2'' \cdot \sigma_2''' \cdot \sigma_2'; (u_2, u_2') \rangle \qquad \langle \sigma_2'' \cdot \sigma_2''' \cdot \sigma_2'; [u_2/a, u_2'/b]\gamma_2(\delta_2'(t_2')) \rangle \Downarrow \langle \sigma_2''''; v_2 \rangle}{\langle \sigma_2 \cdot \sigma_2'; \gamma_2(\delta_2(\delta_2'(\mathsf{let}\ (a,b) = t_1'\ \mathsf{in}\ t_2'))) \rangle \Downarrow \langle \sigma_2''''; v_2 \rangle}$$

- **Case** LFUNCONG: $\dfrac{\Gamma;\Delta, x : A \vdash t \equiv t' : B}{\Gamma;\Delta \vdash \lambda x : A.\ t \equiv \lambda x : A.\ t' : A \multimap B}$
  Let $((\sigma_1', u), (\sigma_2', u')) \in \psi(\gamma_1(A))$ such that $\sigma_1' \# \sigma_1$ and $\sigma_2' \# \sigma_2$.
  We can notice
  $$((\sigma_1 \cdot \sigma_1', (\delta_1, u/x)), (\sigma_2 \cdot \sigma_2', (\delta_2, u'/x))) \in [\![\gamma_1(\Delta, x : A)]\!]$$
  Thus, by induction
  $$((\sigma_1 \cdot \sigma_1', [u/x]\gamma_1(t)), (\sigma_2 \cdot \sigma_2', [u'/x]\gamma_2(t'))) \in \psi(\gamma_1(B))$$
  Which is what we need to conclude that
  $$((\sigma_1, \lambda x.\ t), (\sigma_2, \lambda x.\ t')) \in \psi(\gamma_1(A \multimap B))$$

- **Case** LAppCong: $\dfrac{\Gamma;\Delta \vdash t_1 \equiv t_1' : A \multimap B \qquad \Gamma;\Delta' \vdash t_2 \equiv t_2' : A}{\Gamma;\Delta,\Delta' \vdash t_1\ t_2 \equiv t_1'\ t_2' : B}$
  Our first induction hypothesis states

  $$\forall((\sigma,t),(\sigma',t')) \in \psi(\gamma_1(A)), \sigma\#\sigma_1 \Rightarrow \sigma'\#\sigma_2 \Rightarrow ((\sigma_1{\cdot}\sigma, \gamma_1(\delta_1(t_1))\ t)(\sigma_2{\cdot}\sigma, \gamma_2(\delta_2(t_2))\ t')) \in \psi(\gamma_1(B))$$

  Thus we can apply it to our second induction hypothesis

  $$((\sigma_1', \delta_1'(\gamma_1(t_2))), (\sigma_2', \delta_2'(\gamma_2(t_2')))) \in \psi(\gamma_1(A))$$

  to get the expected conclusion.
- **Case** LFICong: $\dfrac{\Gamma \vdash e \equiv e' : X \qquad \Gamma;\Delta \vdash t \equiv t' : A[e/x]}{\Gamma;\Delta \vdash \mathsf{F}\,(e,t) \equiv \mathsf{F}\,(e',t') : \mathsf{F}x : X.\ A}$
  By induction, $(\gamma_1(e), \gamma_2(e')) \in \phi(\gamma_1(X))$ and $((\sigma_1, \delta_1(\gamma_1(t))), (\sigma_2, \delta_2(\gamma_2(t')))) \in \psi(\gamma_1(A[e/x]))$.
  These are exactly the hypothesis we need to conclude.
- **Case** LFECong: $\dfrac{\Gamma;\Delta \vdash t_1 \equiv t_1' : \mathsf{F}x : X.\ A \qquad \Gamma, x : X;\Delta', a : A \vdash t_2 \equiv t_2' : B}{\Gamma;\Delta,\Delta' \vdash \mathsf{let}\ \mathsf{F}\,(x,a) = t_1\ \mathsf{in}\ t_2 \equiv \mathsf{let}\ \mathsf{F}\,(x,a) = t_1'\ \mathsf{in}\ t_2' : B}$
  By our first induction hypothesis,

  $$((\sigma_1, \gamma_1(\delta_1(t_1))), (\sigma_2, \gamma_2(\delta_2(t_1')))) \in \psi(\gamma_1(\delta_1(\mathsf{F}x : X.\ A)))$$

  Therefore, there exists $((\sigma_1'', \mathsf{F}\,(e_1,u_1)), (\sigma_2'', \mathsf{F}\,(e_2,u_2))) \in \psi(\gamma_1(\mathsf{F}x : X.\ A))$ such that

  $$\langle \sigma_1; \gamma_1(\delta_1(t_1)) \rangle \Downarrow \langle \sigma_1''; \mathsf{F}\,(e_1,u_1) \rangle$$
  $$\langle \sigma_2; \gamma_2(\delta_2(t_1')) \rangle \Downarrow \langle \sigma_2''; \mathsf{F}\,(e_2,u_2) \rangle$$
  $$(e_1, e_2) \in \phi(\gamma_1(X))$$
  $$((\sigma_1'', [e_1/x]u_1)(\sigma_2'', [e_2/x]u_2)) \in \psi(\gamma_1([e_1/x]A))$$

  In particular, it means that
  $$((\gamma_1, e_1/x), (\gamma_2, e_2/x)) \in [\![\Gamma, x : X]\!]$$
  and (recall that x is not a free variable of Δ' by $\Gamma \vdash \Delta'$ ok)

  $$((\sigma_1'' \cdot \sigma_1', (\delta_1', u_1/a)), (\sigma_2'' \cdot \sigma_2', (\delta_2', u_2/a))) \in [\![(\gamma_1, e_1/x)(\Delta', a : A)]\!]$$

  Thus, by our second induction hypothesis

  $$((\sigma_1' \cdot \sigma_1'', \gamma_1(\delta_1'([e_1/x, u_1/a]t_2))), (\sigma_2' \cdot \sigma_2'', \gamma_2(\delta_2'([e_2/x, u_2/a]t_2')))) \in \psi(\gamma_1([e_1/x]B))$$

  Now since $\Gamma \vdash B$ linear, $[e_1/x]B = B$.
  We then have $((\sigma_1''', v_1), (\sigma_2''', v_2)) \in \psi(\gamma_1(B))$ such that

  $$\langle \sigma_1' \cdot \sigma_1''; \gamma_1(\delta_1'([e_1/x, u_1/a]t_2)) \rangle \Downarrow \langle \sigma_1'''; v_1 \rangle$$
  $$\langle \sigma_2' \cdot \sigma_2''; \gamma_2(\delta_2'([e_2/x, u_2/a]t_2')) \rangle \Downarrow \langle \sigma_2'''; v_2 \rangle$$

  We can then build the following derivations and conclude.

  $$\dfrac{\langle \sigma_1 \cdot \sigma_1'; \gamma_1(\delta_1(t_1)) \rangle \Downarrow \langle \sigma_1''; \mathsf{F}\,(e_1,u_1) \rangle \qquad \langle \sigma_1'' \cdot \sigma_1'; [e_1/x, u_1/a]\gamma_1(\delta_1'(t_2)) \rangle \Downarrow \langle \sigma_1'''; v_1 \rangle}{\langle \sigma_1 \cdot \sigma_1'; \gamma_1(\delta_1(\delta_1'(\mathsf{let}\ \mathsf{F}\,(x,a) = t_1\ \mathsf{in}\ t_2))) \rangle \Downarrow \langle \sigma_1'''; v_1 \rangle}$$

  $$\dfrac{\langle \sigma_2 \cdot \sigma_2'; \gamma_2(\delta_2(t_1')) \rangle \Downarrow \langle \sigma_2''; \mathsf{F}\,(e_2,u_2) \rangle \qquad \langle \sigma_2'' \cdot \sigma_2'; [e_2/x, u_2/a]\gamma_2(\delta_2'(t_2')) \rangle \Downarrow \langle \sigma_2'''; v_2 \rangle}{\langle \sigma_2 \cdot \sigma_2'; \gamma_2(\delta_2(\delta_2'(\mathsf{let}\ \mathsf{F}\,(x,a) = t_1'\ \mathsf{in}\ t_2'))) \rangle \Downarrow \langle \sigma_2'''; v_2 \rangle}$$

- **Case** LVALCONG: $\dfrac{\Gamma; \Delta \vdash e \equiv e' : A}{\Gamma; \Delta \vdash \mathsf{val}\, e \equiv \mathsf{val}\, e' : \mathsf{T}\, A}$
  Let σ_f and σ_g be heaps such that $\sigma_f \# \sigma_1$ and $\sigma_g \# \sigma_2$.
  By induction $((\sigma_1, \gamma_1(\delta_1(e))), (\sigma_2, \gamma_2(\delta_2(e')))) \in \psi(\mathsf{T}\, \gamma_1(A))$, so thanks to the linear evalutation frame property, there exists $((\sigma_1', v_1), (\sigma_2', v_2)) \in \psi(\gamma_1(A))$ such that

$$\langle \sigma_1 \cdot \sigma_f; \gamma_1(\delta_1(e)) \rangle \Downarrow \langle \sigma_1' \cdot \sigma_f; \mathsf{val}\, v_1 \rangle$$

$$\langle \sigma_2 \cdot \sigma_g; \gamma_2(\delta_2(e')) \rangle \Downarrow \langle \sigma_2' \cdot \sigma_g; \mathsf{val}\, v_2 \rangle$$

  Thus we have the following derivations

$$\dfrac{\langle \sigma_1 \cdot \sigma_f; \gamma_1(\delta_1(e)) \rangle \rightsquigarrow \langle \sigma_1' \cdot \sigma_f; \mathsf{val}\, v_1 \rangle}{\langle \sigma_1 \cdot \sigma_f; \gamma_1(\delta_1(e)) \rangle \rightsquigarrow \langle \sigma_1' \cdot \sigma_f; \mathsf{val}\, v_1 \rangle} \qquad \dfrac{\langle \sigma_2 \cdot \sigma_g; \gamma_2(\delta_2(e')) \rangle \rightsquigarrow \langle \sigma_2' \cdot \sigma_g; \mathsf{val}\, v_2 \rangle}{\langle \sigma_2 \cdot \sigma_g; \gamma_2(\delta_2(e')) \rangle \rightsquigarrow \langle \sigma_2' \cdot \sigma_g; \mathsf{val}\, v_2 \rangle}$$

  Thus we can conclude thanks to the definition of $\psi(\gamma_1(\mathsf{T}\, A))$.

- **Case** LLETCONG: $\dfrac{\Gamma; \Delta \vdash e_1 \equiv e_1' : \mathsf{T}\, A \qquad \Gamma; \Delta', a : A \vdash e_2 \equiv e_2' : \mathsf{T}\, C}{\Gamma; \Delta, \Delta' \vdash \mathsf{let\ val}\, a = e_1\ \mathsf{in}\ e_2 \equiv \mathsf{let\ val}\, a = e_1'\ \mathsf{in}\ e_2' : \mathsf{T}\, C}$
  Let σ_f and σ_g be heaps such that $\sigma_f \# \sigma_1 \cdot \sigma_1'$ and $\sigma_g \# \sigma_2 \cdot \sigma_2'$.
  By induction $((\sigma_1, \gamma_1(\delta_1(e_1))), (\sigma_2, \gamma_2(\delta_2(e_1')))) \in \psi(\mathsf{T}\, \gamma_1(A))$, so there exists $((\sigma_1'', v_1), (\sigma_2'', v_2)) \in \psi(\gamma_1(A))$ such that

$$\langle \sigma_1 \cdot \sigma_1' \cdot \sigma_f; \gamma_1(\delta_1(e_1)) \rangle \rightsquigarrow \langle \sigma_1'' \cdot \sigma_1' \cdot \sigma_f; \mathsf{val}\, v_1 \rangle$$

$$\langle \sigma_2 \cdot \sigma_2' \cdot \sigma_g; \gamma_2(\delta_2(e_1')) \rangle \rightsquigarrow \langle \sigma_2'' \cdot \sigma_2' \cdot \sigma_g; \mathsf{val}\, v_2 \rangle$$

  Thus, we have

$$((\sigma_1' \cdot \sigma_1'', (\delta_1', v_1/a)), (\sigma_2' \cdot \sigma_2'', (\delta_2', v_2/a))) \in [\![\gamma_1(\Delta', a : A)]\!]$$

  So by induction

$$((\sigma_1' \cdot \sigma_1'', \gamma_1(\delta_1'([v_1/a]e_2))), (\sigma_2' \cdot \sigma_2'', \gamma_2(\delta_2'([v_2/a]e_2')))) \in \psi(\mathsf{T}\, (\gamma_1(C)))$$

  Therefore, there is $((\sigma_1''', w_1), (\sigma_2''', w_2)) \in \psi(\gamma_1(A))$ such that

$$\langle \sigma_1' \cdot \sigma_1'' \cdot \sigma_f; \gamma_1(\delta_1([v_1/a]e_2)) \rangle \rightsquigarrow \langle \sigma_1''' \cdot \sigma_f; \mathsf{val}\, w_1 \rangle$$

$$\langle \sigma_2' \cdot \sigma_2'' \cdot \sigma_g; \gamma_2(\delta_2([v_2/a]e_2')) \rangle \rightsquigarrow \langle \sigma_2''' \cdot \sigma_g; \mathsf{val}\, w_2 \rangle$$

  Thus we have the following derivations

$$\dfrac{\langle \sigma_1 \cdot \sigma_1' \cdot \sigma_f; \gamma_1(\delta_1(e_1)) \rangle \rightsquigarrow \langle \sigma_1'' \cdot \sigma_1' \cdot \sigma_f; \mathsf{val}\, v_1 \rangle \quad \langle \sigma_1'' \cdot \sigma_1' \cdot \sigma_f; [v_1/x]\gamma_1(\delta_1(e_2)) \rangle \rightsquigarrow \langle \sigma_1''' \cdot \sigma_f; \mathsf{val}\, w_1 \rangle}{\langle \sigma_1 \cdot \sigma_1' \cdot \sigma_f; \gamma_1(\delta_1(\mathsf{let\ val}\, x = e_1\ \mathsf{in}\ e_2)) \rangle \rightsquigarrow \langle \sigma_1''' \cdot \sigma_f; \mathsf{val}\, w_1 \rangle}$$

$$\dfrac{\langle \sigma_2 \cdot \sigma_2' \cdot \sigma_g; \gamma_2(\delta_2(e_1')) \rangle \rightsquigarrow \langle \sigma_2'' \cdot \sigma_2' \cdot \sigma_g; \mathsf{val}\, v_2 \rangle \quad \langle \sigma_2'' \cdot \sigma_2' \cdot \sigma_g; [v_2/x]\gamma_2(\delta_2(e_2')) \rangle \rightsquigarrow \langle \sigma_2''' \cdot \sigma_g; \mathsf{val}\, w_2 \rangle}{\langle \sigma_2 \cdot \sigma_2' \cdot \sigma_g; \gamma_2(\delta_2(\mathsf{let\ val}\, x = e_1'\ \mathsf{in}\ e_2')) \rangle \rightsquigarrow \langle \sigma_2''' \cdot \sigma_g; \mathsf{val}\, w_2 \rangle}$$

  Thus we can conclude thanks to the definition of $\psi(\gamma_1(\mathsf{T}\, C))$.

- **Case** LNEWCONG: $\dfrac{\Gamma \vdash e \equiv e' : X}{\Gamma; \cdot \vdash \mathsf{new}_X\, e \equiv \mathsf{new}_X\, e' : \mathsf{T}\, (\mathsf{F} x : \mathsf{Loc}.\ [x \mapsto X])}$
  By induction, $(\gamma_1(e), \gamma_1(e')) \in \phi(\gamma_1(X)$.
  Thus we have $(v_1, v_2) \in \phi(\gamma_1(X))$ such that

$$\gamma_1(e) \Downarrow v_1 \wedge \gamma_2(e) \Downarrow v_2$$

Let σ_f and σ_g be heaps.
There exists some location $l \notin \text{dom}(\sigma_f) \cup \text{dom}(\sigma_g)$. Hence we have the following

$$\frac{\gamma_1(e) \Downarrow v_1 \qquad l \notin \text{dom}(\sigma_f)}{\langle \sigma_f; \mathsf{new}_X\, \gamma_1(e)\rangle \rightsquigarrow \langle \sigma_f, l : v_1; \mathsf{val}\, \mathsf{F}\,(l, *)\rangle}$$

$$\frac{\gamma_2(e) \Downarrow v_2 \qquad l \notin \text{dom}(\sigma_g)}{\langle \sigma_g; \mathsf{new}_X\, \gamma_2(e)\rangle \rightsquigarrow \langle \sigma_g, l : v_2; \mathsf{val}\, \mathsf{F}\,(l, *)\rangle}$$

It is easy to check that $(([l : v_1], \mathsf{F}\,(l, *)), ([l : v_2], \mathsf{F}\,(l, *))) \in \psi(\gamma_1(\mathsf{F}x : \mathsf{Loc}.\ [x \mapsto X]))$ with $(v_1, v_2) \in \phi(\gamma_1(X))$ and conclude.

- **Case** LFREECONG: $\dfrac{\Gamma \vdash e \equiv e' : \mathsf{Loc} \qquad \Gamma; \Delta \vdash t \equiv t' : [e \mapsto e_0]}{\Gamma; \Delta \vdash \mathsf{free}\,(e, t) \equiv \mathsf{free}\,(e', t') : \mathsf{I}}$
  Let σ_f and σ_g be heap such that $\sigma_1 \# \sigma_f$ and $\sigma_2 \# \sigma_g$.
  By our first induction hypothesis $(\gamma_1(e), \gamma_2(e')) \in \psi(\mathsf{Loc})$, so there is some $l \in \mathsf{Loc}$ such that

  $$\gamma_1(e) \Downarrow l \wedge \gamma_2(e') \Downarrow l$$

  By our second induction hypothesis $((\sigma_1, \gamma_1(\delta_1(t))), (\sigma_2, \gamma_2(\delta_2(t')))) \in \psi(\gamma_1([e \mapsto X]))$, we have values $(v_1, v_2) \in \phi(\gamma_1(X))$ such that

  $$\langle \sigma_1; \gamma_1(\delta_1(t))\rangle \Downarrow \langle [l : v_1]; *\rangle \langle \sigma_2; \gamma_2(\delta_2(t'))\rangle \Downarrow \langle [l : v_2]; *\rangle$$

  Then we can build the following derivations

  $$\frac{\gamma_1(e) \Downarrow l \qquad \langle \sigma_1 \cdot \sigma_f; \gamma_1(\delta_1(t'))\rangle \Downarrow \langle \sigma_f \cdot l : v_1; *\rangle}{\langle \sigma_1 \cdot \sigma_f; \gamma_1(\mathsf{free}\,(e, \delta_1(t)))\rangle \rightsquigarrow \langle \sigma_f; ()\rangle}$$

  $$\frac{\gamma_2(e) \Downarrow l \qquad \langle \sigma_g \cdot \sigma_2; \gamma_2(\delta_2(t'))\rangle \Downarrow \langle \sigma_g, l : v_2; *\rangle}{\langle \sigma_1 \cdot \sigma_g; \gamma_2(\mathsf{free}\,(e, \delta_2(t)))\rangle \rightsquigarrow \langle \sigma_g; ()\rangle}$$

  Notice that $((\epsilon, ()), (\epsilon, ())) \in \psi(\mathsf{I})$ and conclude.

- **Case** LGETCONG: $\dfrac{\Gamma \vdash e \equiv e' : \mathsf{Loc} \qquad \Gamma; \Delta \vdash t_1 \equiv t_1' : [e \mapsto X] \qquad \Gamma, x : X; \Delta', a : [e \mapsto X] \vdash t_2 \equiv t_2' : C}{\Gamma; \Delta, \Delta' \vdash \mathsf{let}\ (x, a) = \mathsf{get}(e, t_1)\ \mathsf{in}\ t_2 \equiv \mathsf{let}\ (x, a) = \mathsf{get}(e', t_1')\ \mathsf{in}\ t_2' : C}$
  By our first induction hypothesis $(\gamma_1(e), \gamma_2(e')) \in \psi(\mathsf{Loc})$, so there is some $l \in \mathsf{Loc}$ such that

  $$\gamma_1(e) \Downarrow l \wedge \gamma_2(e') \Downarrow l$$

  By our second induction hypothesis $((\sigma_1, \gamma_1(\delta_1(t_1))), (\sigma_2, \gamma_2(\delta_2(t_1')))) \in \psi(\gamma_1([e \mapsto X]))$, we have values $(v_1, v_2) \in \phi(\gamma_1(X))$ such that

  $$\langle \sigma_1; \gamma_1(\delta_1(t_1))\rangle \Downarrow \langle [l : v_1]; *\rangle \langle \sigma_2; \gamma_2(\delta_2(t_1'))\rangle \Downarrow \langle [l : v_2]; *\rangle$$

  Now we can notice that

  $$((\gamma_1, v_1/x), (\gamma_2, v_2/x)) \in [\![\Gamma, x : X]\!]$$

  Let us denote that new substitution γ'. We also have

  $$((\sigma_1' \cdot [l : v_1], (\delta_1', */a)), (\sigma_2' \cdot [l : v_2], (\delta_2', */a))) \in [\![\gamma_1'(\Delta', a : [e \mapsto X])]\!]$$

  Let us denote the substitution δ''.
  By our third induction hypothesis $(\gamma_1'(\delta_1''(t_2)), \gamma_2'(\delta_2''(t_2'))) \in \psi(\gamma_1(C))$, we have $((\sigma_1'', u_1), (\sigma_2'', u_2)) \in \psi(\gamma_1'(C))$ such that

  $$\langle \sigma_1', l : v_1; \gamma_1'(\delta_1''(t_2))\rangle \Downarrow \langle \sigma_1''; u_1\rangle$$

$$\langle \sigma_2', l : v_2; \gamma_2'(\delta_2''(t_2')) \rangle \Downarrow \langle \sigma_2''; u_2 \rangle$$

Then we can build the following derivations

$$\frac{\gamma_1(e) \Downarrow l \qquad \langle \sigma_1 \cdot \sigma_1'; \gamma_1(\delta_1(t_1)) \rangle \Downarrow \langle \sigma_1', l : v; * \rangle \qquad \langle \sigma_1', l : v; [v/x, */a]\gamma_1(\delta_1(t_2)) \rangle \Downarrow \langle \sigma_1''; u_1 \rangle}{\langle \sigma_1' \cdot \sigma_1; \gamma_1(\mathsf{let}\ (x, a) = \mathsf{get}(e, \delta_1(t_1))\ \mathsf{in}\ \delta_1(t_2)) \rangle \Downarrow \langle \sigma_1''; u_1 \rangle}$$

$$\frac{\gamma_2(e') \Downarrow l \qquad \langle \sigma_2 \cdot \sigma_2'; \gamma_2(\delta_2(t_1')) \rangle \Downarrow \langle \sigma_2', l : v; * \rangle \qquad \langle \sigma_2', l : v; [v/x, */a]\gamma_2(\delta_2(t_2')) \rangle \Downarrow \langle \sigma_2''; u_2 \rangle}{\langle \sigma_2' \cdot \sigma_2; \gamma_2(\mathsf{let}\ (x, a) = \mathsf{get}(e', \delta_2(t_1'))\ \mathsf{in}\ \delta_2(t_2')) \rangle \Downarrow \langle \sigma_2''; u_2 \rangle}$$

And conclude.

- **Case** LASSIGNCONG: $\dfrac{\Gamma \vdash e_1 \equiv e_1' : \mathsf{Loc} \qquad \Gamma; \Delta \vdash t_1 \equiv t_1' : [e \mapsto X] \qquad \Gamma \vdash e_2 \equiv e_2' : Y}{\Gamma; \Delta \vdash e_1 :=_t e_2 \equiv e_1' :=_{t'} e_2' : \mathsf{T}([e \mapsto Y])}$

  Let σ_f and σ_g be heaps such that $\sigma_f \# \sigma_1$ and $\sigma_g \# \sigma_2$

  By our first induction hypothesis $(\gamma_1(e_1), \gamma_2(e_1')) \in \psi(\mathsf{Loc})$, so there is some $l \in \mathsf{Loc}$ such that

  $$\gamma_1(e_1) \Downarrow l \wedge \gamma_2(e_1') \Downarrow l$$

  By our second induction hypothesis $((\sigma_1, \gamma_1(\delta_1(t_1))), (\sigma_2, \gamma_2(\delta_2(t_1')))) \in \psi(\gamma_1([e \mapsto X]))$, we have values $(v_1, v_2) \in \phi(\gamma_1(X))$ such that

  $$\langle \sigma_1; \gamma_1(\delta_1(t_1)) \rangle \Downarrow \langle [l : v_1]; * \rangle \langle \sigma_2; \gamma_2(\delta_2(t_1')) \rangle \Downarrow \langle [l : v_2]; * \rangle$$

  And our third hypothesis $(\gamma_1(e_2), \gamma_2(e_2')) \in \phi(\gamma_1(Y))$, we have $(u_1, u_2) \in \phi(\gamma_1(Y))$ such that $\gamma_1(e_2) \Downarrow u_1 \wedge \gamma_2(e_2') \Downarrow u_2$

  Then we can build the following derivations

  $$\frac{\gamma_1(e_1) \Downarrow l \qquad \gamma_1(e_2) \Downarrow u_1 \qquad \langle \sigma_1 \cdot \sigma_f; \gamma_1(\delta_1(t_1)) \rangle \Downarrow \langle \sigma_f, l : v_1; * \rangle}{\left\langle \sigma_1 \cdot \sigma_f; \gamma_1(e_1 :=_{\delta_1(t_1)} e_2) \right\rangle \rightsquigarrow \langle \sigma_f, l : u_1; * \rangle}$$

  $$\frac{\gamma_2(e_1') \Downarrow l \qquad \gamma_2(e_2') \Downarrow u_1 \qquad \langle \sigma_2 \cdot \sigma_g; \gamma_2(\delta_2(t_1')) \rangle \Downarrow \langle \sigma_g, l : v_2; * \rangle}{\left\langle \sigma_2 \cdot \sigma_g; \gamma_2(e_1' :=_{\delta_2(t_1')} e_2') \right\rangle \rightsquigarrow \langle \sigma_g, l : u_2; * \rangle}$$

  And conclude.

- **Case** : $\dfrac{\Gamma, x : X; \Delta \vdash e \equiv e' : Y \qquad x \notin FV(e, e')}{\Gamma; \Delta \vdash e \equiv e' : \forall x : X.\ Y}$

  Let $\gamma \in [\![\Gamma]\!]$. Then, for every $(t, t') \in \phi(\gamma_1(X))$, $((\gamma_1, t/x), (\gamma_2, t'/x)) \in [\![\Gamma, x : X]\!]$, thus we get the expected result thanks to the induction hypothesis.

- **Case** : $\dfrac{\Gamma; \Delta \vdash e \equiv e' : [e''/x]Y}{\Gamma; \Delta \vdash e \equiv e' : \exists x : X.\ Y}$

  We get the expected result directly from the induction hypothesis.

- **Case** LIRREQ: $\dfrac{}{\Gamma; \Delta \vdash e \equiv e' : [A]}$

  By induction, $((\sigma_1, \delta_1(\gamma_1(e))), (\sigma_1, \delta_1(\gamma_1(e)))) \in \psi(\gamma_1([A]))$ $((\sigma_2, \delta_2(\gamma_2(e'))), (\sigma_2, \delta_2(\gamma_2(e')))) \in \psi(\gamma_1([A]))$ So we have $((\sigma_1', *), (\sigma_2', *)) \in \psi(\gamma_1([A]))$ such that

  $$\langle \sigma_1; \gamma_1(\delta_1(e)) \rangle \Downarrow \langle \sigma_1'; * \rangle \wedge \langle \sigma_2; \gamma_2(\delta_2(e')) \rangle \Downarrow \langle \sigma_2'; * \rangle$$

  Thus there exists a such that $((\sigma_1', a)(\sigma_1', a)) \in \psi(\gamma_1(A))$.

9. If $\Gamma; \Delta \vdash e \div A$ then there exists (t, t') such that for all $\gamma \in [\![\Gamma]\!]$, for all $((\sigma_1, \delta_1), (\sigma_2, \delta_2)) \in [\![\gamma_1(\Delta)]\!]$, $((\sigma_1, \delta_1(\gamma_1(t))), (\sigma_2, \delta_2(\gamma_2(t')))) \in \psi(\gamma_1(A))$.

   We case analyze the derivation of $\Gamma; \Delta \vdash e \div A$:

- **Case** : $\dfrac{\Gamma; \Delta \vdash e : A}{\Gamma; \Delta \vdash e \div A}$
  The induction hypothesis tells us that (e, e) gives us the result.
- **Case** : $\dfrac{\Gamma; \Delta \vdash e \div A \qquad \Gamma; \Delta', x : A \vdash e' \div C}{\Gamma; \Delta, \Delta' \vdash \text{let } [x] = e \text{ in } e' \div C}$
  By our first induction hypothesis, there exists (t_1, t_2) such that for every γ, δ, σ, $((\sigma_1, \delta_1(\gamma_1((t_1)))), (\sigma_2, \delta_2(\gamma_2(t_2)$ $\psi(\gamma_1(A))$.
  Thus we have So, by our second induction hypothesis, there exists (t'_1, t'_2) such that

$$((\sigma'_1 \cdot \sigma_1, \gamma_1(\delta'_1([\delta_1(t_1)/x]t'_1))), (\sigma'_2 \cdot \sigma_2, \delta'_2(\gamma_2([\delta_2(t_2)/x]t'_2)))) \in \psi(\gamma_1(C))$$

  for every $((\sigma'_1, \delta'_1), (\sigma'_2, \delta'_2)) \in [\![\Delta']\!]$

$$((\sigma_1 \cdot \sigma'_1, \delta'_1, a_1/x), (\sigma_2 \cdot \sigma'_2, \delta'_2, a_2/x)) \in [\![\Delta', x : A]\!]$$

  Thus, $([t_1/x]t'_1, [t_2/x]t'_2)$ fullfills our desiterata.

□

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