The (relative) monad–theory correspondence

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(TallCat 11.11.21)
Contents

1. Context & motivation
2. Theories & relative monads
3. Extending relative monads
4. Algebras & models
5. Relative monadicity & optheories
6. The structure—semantics adjunction
7. Examples
Algebraic theories

Denote by $\text{IF}$ the free category with strict finite coproducts on a single object.

An algebraic theory is an identity-on-objects functor

$$k : \text{IF} \rightarrow B$$

strictly preserving finite coproducts.
Algebraic theories

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Sifted-cocontinuous monads

A colimit in $\text{Set}$ is sifted if it commutes with finite products.

A monad on $\text{Set}$ is sifted-cocontinuous if it preserves sifted colimits.

These are the same as finitary monads on $\text{Set}$. 
Algebraic theories

Denote by $\mathcal{IF}$ the free category with strict finite coproducts on a single object.

An algebraic theory is an identity-on-objects functor $k : \mathcal{IF} \to B$ strictly preserving finite coproducts.

Sifted-cocontinuous monads

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A monad on Set is sifted-cocontinuous if it preserves sifted colimits.

These are the same as finitary monads on Set.
Why is there a correspondence between theories and monads?
Why is there a correspondence between theories and monads?

(And why do we care?)
Caring

There are two main reasons we might care about this problem.

- One is **theoretical**: the correspondence is deep and surprising, and understanding it would shed light on a fundamental categorical phenomenon.

- The other is **practical**: the correspondence gives us two perspectives on categorical algebra, each of which is useful in different contexts.
Theories - - - - - - - - - Monads
Theories — Relative monads —- Monads
What is a relative monad?

Let $j : A \to E$ be a functor. A $j$-relative monad consists of

- a function $t : |A| \to |E|;$
- for each $a \in A$, a morphism $\eta_a : ja \to ta;$
- for each $a, b \in A$, and morphism $f : ja \to tb$, a morphism $f^* : ta \to tb$, satisfying unitality and associativity laws.
What is a relative monad?

Let $j : A \rightarrow E$ be a functor. A $j$-relative monad $^1$ consists of

- a function $t : \lvert A \rvert \rightarrow \lvert E \rvert$;
- for each $a \in A$, a morphism $\eta_a : ja \rightarrow ta$;
- for each $a, b \in A$, and morphism $f : ja \rightarrow tb$, a morphism $f^t : ta \rightarrow tb$, satisfying unitality and associativity laws.

When $j$ is the identity, this is the same as a monad.
Every relative monad $T$ is induced canonically by two relative adjunctions.¹

Kleisli resolution (initial)

Eilenberg–Moore resolution (terminal)
What is a relative adjunction?

Let $j: A \to E$ be a functor. A $j$-relative adjunction $\triangleright$ consists of functors $L: A \to B$ and $r: B \to E$

along with a natural isomorphism $B(lx, y) \cong E(jx, ry)$
Thus, the following are in bijection:

- \( j \)-relative monads;
- \( j \)-relative Kleisli resolutions;
- \( j \)-relative Eilenberg–Moore resolutions.
Thus, the following are in bijection:

- $\text{j-relative monads}$;
- $\text{j-relative Kleisli resolutions}$;
- $\text{j-relative Eilenberg-Moore resolutions}$. 
A characterisation of Kleisli inclusions

A functor $K : A \rightarrow B$ is the Kleisli inclusion for a j-relative monad if and only if

- $K$ has a right j-relative adjoint;
- $K$ is identity-on-objects.
A characterisation of Kleisli inclusions

A functor $\mathbf{K} : A \to B$ is the Kleisli inclusion for a $j$-relative monad if and only if

- $\mathbf{K}$ has a right $j$-relative adjoint;
- $\mathbf{K}$ is identity-on-objects.

This looks familiar.
Let \( f: A \to B \) be a functor with small domain. Then there is a \( \mathcal{A}_A \)-relative adjunction

\[
\begin{array}{ccc}
B & \xrightarrow{N_f} & B(f-, -) \\
\downarrow f & & \downarrow \text{Id} \\
A & \xrightarrow{\mathcal{A}} & \hat{A}
\end{array}
\]

Hence, every (small) functor is a left \( \mathcal{A} \)-relative adjoint.
$\mathbf{L}$-relative adjunctions

Let $f: A \rightarrow B$ be a functor with small domain. Then there is a $\mathbf{L}_A$-relative adjunction

$$N_f = B(f-, -)$$

Hence, every (small) functor is a left $\mathbf{L}$-relative adjoint. (And vice versa.)
Furthermore, if $f$ preserves $\Psi$-colimits, this $\mathbf{A}$-relative adjunction restricts to a relative adjunction

\[
\begin{array}{ccc}
A & \xrightarrow{\Psi} & [A^\circ, \text{Set}]_\Psi \\
\downarrow & & \downarrow \Psi \\
A & \xrightarrow{\mathbf{A}} & [A^\circ, \text{Set}]_\Psi \\
\end{array}
\]
Furthermore, if $f$ preserves $\Psi$-colimits, this $\mathcal{A}$-relative adjunction restricts to a relative adjunction

$$
\begin{array}{ccc}
B & \xrightarrow{N_f} & [A^\circ, \text{Set}]_{\Psi} \\
\downarrow{\Psi} & & \downarrow{\Psi} \\
A & \xrightarrow{\mathcal{A}} & [A^\circ, \text{Set}]_{\Psi}
\end{array}
$$

In particular, every (small) functor preserving finite coproducts is left-adjoint relative to the inclusion into $\text{FinProd}(-^\circ, \text{Set}) \cong \text{Sind}(-)$. 
Furthermore, if $f$ preserves $\Psi$-colimits, this $\mathcal{A}$-relative adjunction restricts to a relative adjunction

\[
\begin{array}{ccc}
A & \xrightarrow{\bot} & [A^\circ, \mathbf{Set}]_{\Psi} \\
\downarrow F & & \downarrow N_f \\
B & \xleftarrow{-1} & [A^\circ, \mathbf{Set}]_{\Psi}
\end{array}
\]

In particular, every (small) functor preserving finite coproducts is left-adjoint relative to the inclusion into $\mathbf{FinProd}(-^\circ, \mathbf{Set}) \cong \mathbf{Sind}(-)$. (And vice versa.)
Therefore, an algebraic theory is equivalently:

- an identity-on-objects functor $\kappa : \mathcal{F} \to \mathcal{B}$ strictly preserving finite coproducts;

- an identity-on-objects functor $\kappa : \mathcal{F} \to \mathcal{B}$ which is a left $L_{\mathcal{F}}^{\text{sift}}$-relative adjoint;

- the Kleisli inclusion of a $L_{\mathcal{F}}^{\text{sift}}$-relative monad.
Therefore, an algebraic theory is equivalently:

- an identity-on-objects functor $K : \mathcal{F} \rightarrow \mathcal{B}$ strictly preserving finite coproducts;
- an identity-on-objects functor $K : \mathcal{F} \rightarrow \mathcal{B}$ which is a left $\mathcal{L}_{\mathcal{F}}^{\text{sift}}$-relative adjoint;
- the Kleisli inclusion of a $\mathcal{L}_{\mathcal{F}}^{\text{sift}}$-relative monad.

Algebraic theories are hence in bijection with $\mathcal{L}_{\mathcal{F}}^{\text{sift}}$-relative monads.
Therefore, an algebraic theory is equivalently:

- an identity-on-objects functor $\kappa : \mathcal{G} \to \mathcal{B}$ strictly preserving finite coproducts;
- an identity-on-objects functor $\kappa : \mathcal{G} \to \mathcal{B}$ which is a left $\mathcal{L}_\mathcal{G}^{\text{f.t.}}$-relative adjoint;
- the Kleisli inclusion of a $\mathcal{L}_\mathcal{G}^{\text{f.t.}}$-relative monad.

Algebraic theories are hence in bijection with $\mathcal{L}_\mathcal{G}^{\text{f.t.}}$-relative monads. ($\mathcal{L}_\mathcal{G}^{\text{f.t.}} = \mathcal{G} \hookrightarrow \text{Set}$)
Theories — Relative monads —— Monads

(Kleisli inclusions)
Monads and relative monads

Let $\Phi$ be a class of weights. There is an equivalence of categories

$$RMnd(\mathcal{K}_A^\Phi) \simeq Mnd_{\Phi}(\Phi A)$$

($A\rightarrow \Phi A$-relative monads) ($\Phi$-cocontinuous monads)
Monads and relative monads

Let $\Phi$ be a class of weights. There is an equivalence of categories

$$\text{RMnd}(\mathcal{A}^\Phi) \simeq \text{Mnd}_\Phi(\Phi \mathcal{A})$$

($\mathcal{A} \rightarrow \Phi \mathcal{A}$)-relative monads) ($\Phi$-cocontinuous monads)

Hence, $\mathcal{A}^{\text{sift}}$-relative monads are equivalent to sifted-cocontinuous monads on $\text{Sind}(\mathcal{A}^{\text{f}}) \simeq \text{Set}$, i.e. finitary monads on $\text{Set}$. 
Theories — Relative monads — Monads

(Kleisli inclusions) — (Extension via cocompletion)
How general is this story?
Theories — Relative monads — Monads
Theories — Relative monads — Monads

(Kleisli inclusions)
Theories \(\checkmark\) Relative monads \(~\) Monads

(Kleisli inclusions) \(\implies\) (Under suitable assumptions)
Theories

Let \( j: A \rightarrow E \) be a dense functor. A \( j \)-theory is a functor \( k: A \rightarrow B \) that is the identity on objects, and has a right \( j \)-relative adjoint.
Theories

Let \( j : A \to E \) be a dense functor. A \( j \)-theory is a functor \( k : A \to B \) that is the identity on objects, and has a right \( j \)-relative adjoint.

\( \text{Th}(j) \) is the full subcategory of \( A/\text{CAT} \) on the theories.
Theories

Let $j : A \to E$ be a dense functor. A $j$-theory is a functor $k : A \to B$ that is the identity on objects, and has a right $j$-relative adjoint.

$\text{Th}(j)$ is the full subcategory of $A/CAT$ on the theories.
Theories

Let $j: A \to E$ be a dense functor. A $j$-theory is a functor $k: A \to B$ that is the identity on objects, and has a right $j$-relative adjoint.

$\text{Th}(j)$ is the full subcategory of $A/\text{CAT}$ on the theories.

There is an equivalence

$$\text{Th}(j) \simeq \text{RMnd}(j)$$
...and monads?

When \( j \) is the inclusion of a category into its cocompletion under a class of weights, we obtain a correspondence between relative monads (and hence theories) and monads.
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When $j$ is the inclusion of a category into its cocompletion under a class of weights, we obtain a correspondence between relative monads (and hence theories) and monads.

But what about more general $j$?
**j-ary monads and realisable relative monads**

Let $j: A \to E$ be a functor. A monad $T$ on $E$ is **j-ary** if there exists a $j$-relative monad $T'$ with the same algebras, i.e.

$$\text{EM}(T) \cong \text{EM}(T')$$

$$\downarrow \quad \downarrow$$

$$E \quad E$$
Let $j: A \to E$ be a functor. A monad $T$ on $E$ is $j$-ary if there exists a $j$-relative monad $T'$ with the same algebras, i.e.

$$EM(T) \xrightarrow{\cong} EM(T')$$

A $j$-relative monad $T'$ is realisable if there exists a monad $T$ on $E$ with the same algebras.
There is an equivalence between the realisable $j$-relative monads and the $j$-ary monads, which commutes with taking categories of algebras.

\[ \text{RMnd}^E(j) \cong \text{Mnd}_j(E) \]

\[ \text{EM} \cong \text{CAT}/E \]
There is an equivalence between the realisable j-relative monads and the j-ary monads, which commutes with taking categories of algebras.

\[ \text{RMnd}^E(j) \cong \text{Mnd}_j(E) \]

\[ \text{EM} \cong \text{EM} \]

\[ \text{CAT}/E \]

The functor \( \text{Mnd}_j(E) \rightarrow \text{RMnd}^E(j) \) is given canonically by precomposing \( j \).
In nice settings, we can characterise the realisable relative monads and $j$-ary monads.

Prop. Let $j : A \to E$ be a functor. A $j$-relative monad $T'$ is realisable if and only if $u_{T'} : EM(T') \to E$ admits a left adjoint.
A colimit in $E$ is $j$-absolute if it is preserved by $E \xrightarrow{N_j} \hat{A}$.

That is: $\text{colim}(d); N_j \equiv \text{colim}(d; N_j)$. 
When $j = A \hookrightarrow \Phi A$ is the inclusion of a small category into its cocompletion under $\Phi$-weighted colimits, $\Phi$-weighted colimits are $j$-absolute.
**j-absolute colimits (2)**

When $j = A \hookrightarrow \Phi A$ is the inclusion of a small category into its cocompletion under $\Phi$-weighted colimits, $\Phi$-weighted colimits are $j$-absolute.

When $j$ is the identity, the $j$-absolute colimits are precisely the absolute colimits, i.e. those preserved by every functor.
**j-ary monads and realisable relative monads (4)**

In nice settings, we can characterise the realisable relative monads and j-ary monads.

**Prop.** Let $j$ be admissible and dense, and suppose that left Kan extensions along $j$ exist and are pointwise and j-absolute. Then

1. Every $j$-relative monad is realisable.
In nice settings, we can characterise the realisable relative monads and j-ary monads.

Prop. Let $j$ be admissible and dense, and suppose that left Kan extensions along $j$ exist and are pointwise and $j$-absolute. Then

1. Every $j$-relative monad is realisable.
2. A monad $T$ is $j$-ary if and only if $T \cong \text{Lan}_j(T \circ j)$ if and only if $T$ preserves $j$-absolute colimits.
A general *monad-theory correspondence*

**Theorem.** Let \( j \) be an admissible, dense functor and suppose that left Kan extensions along \( j \) exist and are pointwise and \( j \)-absolute.

Then there is an equivalence

\[
\text{Th}(j) \cong \text{Mnd}_j(E)
\]

between the categories of \( j \)-theories and the monads on \( E \) preserving \( j \)-absolute colimits.
* We can weaken these assumptions further to obtain an even more general correspondence, but lose the nice characterisations of j_ary monads.
* We can weaken these assumptions further to obtain an even more general correspondence, but lose the nice characterisations of j-ary monads.

(However, it is necessary to weaken the assumptions further to compare with some of the existing correspondences, such as that of Bourke-Garner (2019).)
What about the algebras?
**Theorem.** Let $j: A \to E$ be admissible and dense, and let $T$ be a $j$-relative monad. The following square forms a pullback.

$$
\begin{array}{ccc}
EM(T) & \xrightarrow{\underline{\text{K}}T} & \text{KI}(T) \\
\downarrow_{u_T} & & \downarrow_{\kappa_T^*} \\
E & \xrightarrow{N_j} & \hat{A}
\end{array}
$$

$$\kappa_T^* = [\kappa_T^*, \text{Set}]$$
**Theorem.** Let \( j : A \to E \) be admissible and dense, and let \( T \) be a \( j \)-relative monad. The following square forms a pullback.

\[
\begin{array}{c}
EM(T) \\
\downarrow u_T \\
E
\end{array}
\rightarrow
\begin{array}{c}
KI(T) \\
\downarrow k_T^n = [k_T^o, \text{Set}] \\
\hat{E}
\end{array}
\]

(When \( j = 1_E \), this observation is due to Linton\(^6\).)
Corollary. Let $j: A \to E$ be admissible and dense, and let $k: A \to B$ be a $j$-theory. Denote by $T$ the $j$-relative monad corresponding to $k$. The following square forms a pullback.

$$
\begin{array}{ccc}
EM(T) & \rightarrow & \hat{B} \\
\downarrow{u_T} & & \downarrow{\left[k^{op}, \text{Set}\right]} \\
E & \rightarrow & \hat{A} \\
\downarrow{N_j} & & \\
& & \\
\end{array}
$$
Corollary. Let \( j: A \to E \) be admissible and dense, and let \( k: A \to B \) be a \( j \)-theory. Denote by \( T \) the \( j \)-relative monad corresponding to \( k \). The following square forms a pullback.

\[
\begin{array}{ccc}
k\text{-Alg} := EM(T) & \to & \hat{B} \\
\downarrow u_T & & \downarrow \text{[k}^{\text{op}}, \text{Set}] \\
E & \to & \hat{A} \\
\downarrow N_j & & \\
\end{array}
\]
Algebras via cocompletion (3)

Corollary. Let $L: \mathcal{F} \to \mathcal{L}$ be an algebraic theory, i.e. a $(\mathcal{F} \to \mathbf{Set})$-theory. Denote by $T_L$ the relative monad corresponding to $L$. The following square forms a pullback.

$$
\begin{array}{ccc}
EM(T_L) & \to & \mathcal{L} \\
\downarrow u_T & & \downarrow \left[\mathcal{L}^{op}, \mathbf{Set}\right] \\
\mathbf{Set} & \to & \hat{\mathcal{F}} \\
\downarrow N_{\mathbf{Set} \downarrow \mathbf{Set}} & & \downarrow \hat{\mathcal{F}} \\
\end{array}
$$
Algebras via cocompletion (3)

Corollary. Let \( L : \mathcal{IF} \rightarrow L \) be an algebraic theory, i.e. a \((\mathcal{IF} \leftrightarrow \text{Set})\)-theory. Denote by \( T_L \) the relative monad corresponding to \( L \). The following square forms a pullback.

\[
\begin{array}{ccc}
EM(T_L) & \rightarrow & \hat{L} \\
\downarrow & & \downarrow \\
\text{FinProd}(\mathcal{IF}^{\text{op}}, \text{Set}) \cong \text{Set} & \rightarrow & \hat{\mathcal{IF}} \\
\downarrow & & \downarrow \\
\mathcal{IF}^{\text{op}} \text{Set} & \rightarrow & [\mathcal{L}^{\text{op}}, \text{Set}] \\
\end{array}
\]
Algebras via cocompletion (3)

**Corollary.** Let \( L : \mathcal{IF} \to \mathcal{L} \) be an algebraic theory, i.e. a \((\mathcal{IF} \to \text{Set})\)-theory. Denote by \( T_L \) the relative monad corresponding to \( L \). The following square forms a pullback.

\[
\begin{array}{ccc}
\text{FinProd}(\mathcal{L}^{\mathsf{op}}, \text{Set}) \cong \text{EM}(T_L) & \xrightarrow{\text{}} & \hat{\mathcal{L}} \\
\downarrow \text{u}_T & & \downarrow \\
\text{FinProd}(\mathcal{IF}^{\mathsf{op}}, \text{Set}) \cong \text{Set} & \xrightarrow{\text{N}_{\mathsf{ind}}} & \hat{\mathcal{IF}} \\
\end{array}
\]

\([\mathcal{L}^{\mathsf{op}}, \text{Set}] \xrightarrow{\text{[u]}} \text{Set} \]
Models & algebras

In other words, the models for an algebraic theory coincide with the algebras for the corresponding monad due to the pullback characterisation of the Eilenberg–Moore category for a relative monad.
Recall that the following are in bijection:

- \( j \)-relative monads;
- \( j \)-relative Kleisli resolutions;
- \( j \)-relative Eilenberg–Moore resolutions.
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- \( j \)-relative monads;
- \( j \)-relative Kleisli resolutions;
- \( j \)-relative Eilenberg-Moore resolutions.
**Relative monadicity**

**Theorem.** Let $j : A \rightarrow E$ be a dense functor. TFAE for a functor $r : X \rightarrow E$.

1. $r$ creates small $j$-absolute colimits, and has a left $j$-relative adjoint.
2. $r$ is $j$-relatively monadic.
Relative monadicity

**Theorem.** Let \( j : A \rightarrow E \) be a dense functor. TFAE for a functor \( r : X \rightarrow E \).

1. \( r \) creates small \( j \)-absolute colimits, and has a left \( j \)-relative adjoint.
2. \( r \) is \( j \)-relatively monadic.

(This is a Paré-style monadicity theorem\(^3\), rather than Beck-style\(^4\).)
\textbf{Optheories}

Let $j: A \to E$ be a dense functor. A \textit{j-optheory} is a functor $u: X \to E$ that strictly creates small \textit{j-absolute} colimits, and has a left \textit{j-relative} adjoint.
**Optheories**

Let \( j : A \rightarrow E \) be a dense functor. A \( j\)-optheory is a functor \( u : X \rightarrow E \) that strictly creates small \( j\)-absolute colimits, and has a left \( j\)-relative adjoint.

\( \text{Opth}(j) \) is the full subcategory of \( \text{CAT}/E \) on the optheories.
**Optheories**

Let $j : A \to E$ be a dense functor. A $j$-optheory is a functor $u : X \to E$ that strictly creates small $j$-absolute colimits, and has a left $j$-relative adjoint.

$\text{Opth}(j)$ is the full subcategory of $\text{CAT}/E$ on the optheories.

There is an equivalence

$$\text{Th}(j) \simeq \text{RMnd}(j) \simeq \text{Opth}(j)^{op}$$
Theories & optheories

In a suitable sense, theories and optheories are dual to one another:

- $j$-theories axiomatise the Kleisli inclusions of $j$-relative monads (i.e. initial resolutions).

- $j$-optheories axiomatise the forgetful Eilenberg-Moore functors of $j$-relative monads (i.e. terminal resolutions).
What can we say about the process of passing from theories to optheories and vice versa?
Pre(op)theories

Let $j : A \to E$ be a dense functor.

An $A$-pretheory is a functor $k : A \to B$ that is the identity on objects.

An $E$-preoptheory is a functor $u : X \to E$ that strictly creates small $j$-absolute colimits.
Pre(op)theories

Let \( j : A \to E \) be a dense functor.

An **A-pretheory** is a functor \( k : A \to B \) that is the identity on objects.

An **E-preoptheory** is a functor \( u : X \to E \) that strictly creates small \( j \)-absolute colimits.

(Op)theories are pre(op)theories with relative adjointness properties.
The structure-semantics adjunction (1)

There is an adjunction

\[ \text{Str} \quad \dashv \quad \text{AlgTh}^\circ \]

\[ (\text{CAT/Set})^* \quad \text{Sem} \]

\[ \downarrow \quad \uparrow \]

\[ \text{Str (structure) sends a functor } u : X \to \text{Set to the algebraic theory with operations } \{u^n \Rightarrow u\}_{n \in \mathbb{N}}. \]

\[ \text{Sem (semantics) sends an algebraic theory } L : \mathbf{F} \to \mathbf{L} \text{ to the forgetful functor } \text{Mod}(L) \to \text{Set}. \]
Fix a functor $j : A \to E$ between small categories. There is an adjunction\(^6\)

$$
\begin{array}{c}
A/\text{CAT} \quad \bot \\
\downarrow \quad \downarrow \\
(CAT/E)^{op}
\end{array}
$$

$u \mapsto A^{op} \xrightarrow{j^{op}} E^{op} \xrightarrow{\xi} E^{op} \xrightarrow{(u^{op})^*} \hat{X}^{op}$
The structure-semantics adjunction (3)

Fix a functor $j : A \rightarrow E$ between small categories. There are adjunctions $\overset{\text{6}}{\downarrow}$

$$\begin{array}{ccc}
X & \xrightarrow{j} & \hat{B} \\
\downarrow & & \downarrow \\
E & \xrightarrow{\epsilon} & \hat{A} \\
\end{array}$$

$$\text{Preth}(A) \dashv A/CAT \dashv (\text{CAT}/E)^{\text{op}}$$

$$u \mapsto A^{\text{op}} \xrightarrow{j^{\text{op}}} E^{\text{op}} \xrightarrow{e} E^{\text{op}} \xrightarrow{(u^{\text{op}})^*} \hat{X}^{\text{op}}$$
The structure-semantics adjunction (3)

Fix a functor $j: A \to E$ between small categories. There are adjunctions

$$
\begin{array}{c}
X \leftarrow \hat{B} \\
\downarrow^k \quad \downarrow^k' \\
E \quad \hat{A}
\end{array}
$$

$$
\begin{array}{c}
Preth(A) \perp A/CAT \perp (CAT/E)^{op}
\end{array}
$$

$$
\begin{array}{c}
u \mapsto A^{op} \xrightarrow{j^{op}} E^{op} \xrightarrow{\Delta} E^{op} \xrightarrow{(u^{op})^*} X^{op}
\end{array}
$$

takes values in preoptheories
The structure-semantics adjunction (4)

Fix a functor $j: A \to E$ between small categories. There is an adjunction

$$
\begin{array}{ccc}
\text{Sem} & & \text{Preopth}(E)^{\text{op}} \\
\downarrow & \Downarrow \bot & \downarrow \\
\text{Preth}(A) & \rightleftharpoons & \text{Str}
\end{array}
$$
The structure-semantics adjunction (4)

Fix a functor \( j : A \rightarrow E \) between small categories. There is an adjunction

\[
\text{Preth}(A) \quad \dashv \quad \text{PreOp th}(E)^{\circ p}.
\]

Furthermore, this adjunction restricts to an adjoint equivalence

\[
\text{Th}(j) \simeq \text{Op th}(j)^{\circ p}.
\]
The big picture

Fix a dense functor \( j : A \to E \).

\[
\begin{align*}
\text{Th}(j) & \cong \text{RMnd}(j) \cong \text{Opth}(j)^{op} \\
\text{Preth}(A) & \cong \text{Preopth}(E)^{op} \\
\text{Alg} & \to \text{EM} \\
\text{CAT/E} & \rightleftharpoons \text{EM} \\
\end{align*}
\]
Q

This is all good and well, but is such a general understanding useful?

Are there new examples?
Enrichment

Our approach is essentially independent of the base of enrichment. For concreteness, we develop our results in the setting of categories enriched in a bicategory $W$.

This has two advantages:

1. We capture interesting examples not captured by other frameworks.
2. We subsume other enriched settings.
Lucyshyn-Wright (2016) develops a monad-theory correspondence with respect to an ‘eleutheric system of arities’ $j : \mathcal{Y} \rightarrow \mathcal{V}$.

Lemma. A $\mathcal{Y}$-theory in the sense of Lucyshyn-Wright is precisely a $j$-theory in our sense, hence a $j$-relative monad.

Lucyshyn-Wright’s correspondence is thus an instance of ours.
Examples (2)

Bourke-Garners (2019) develop a monad-theory correspondence with respect to a dense, fully faithful functor \( j: A \to E \) with small domain and locally presentable codomain.

Lemma. An \( A \)-theory in the sense of Bourke-Garners is precisely a \( j \)-theory in our sense, hence a \( j \)-relative monad. An \( A \)-nervous monad in the sense of Bourke-Garners is precisely a \( j \)-ary monad in our sense.
Examples (2)

Bourke-Garnier's correspondence is thus an instance of ours.
Examples (3)

- Hoare’s (1987) framework for data refinement, as reframed in the setting of enrichment in the nonsymmetric monoidal category $\text{LocOrd}_1$ of small locally-ordered categories, locally-ordered functors, and lax natural transformations with a Gray tensor product by Kinoshita-Power (1995).
Examples (3)

• Hoare’s (1987) framework for data refinement, as reframed in the setting of enrichment in the nonsymmetric monoidal category $\text{LocOrd}_l$ of small locally-ordered categories, locally-ordered functors, and lax natural transformations with a Gray tensor product by Kinoshita-Power (1995).

• The internal monad-theory correspondence of Johnstone-Wraith (1977) via enrichment in the bicategory $\text{Span}(\mathcal{E})$ for a topos $\mathcal{E}$. 
Summary (1)

- The monad-theory correspondence is a combination of two phenomena: the coincidence of theories and relative monads, and the relationship between relative monads and monads with the same algebras.
Summary (1)

• The monad-theory correspondence is a combination of two phenomena: the coincidence of theories and relative monads, and the relationship between relative monads and monads with the same algebras.

• The definition of the category of models for a theory arises from the characterisation of the Eilenberg-Moore category as a pullback over presheaves on the Kleisli category.
Summary (2)

- The structure-semantics adjunction concerns the passage between pretheories and preoptheories, from which the monad-theory (-optheory) correspondence can be seen to arise.
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This perspective is general enough to encompass all existing (1-dimensional) correspondences.
References

2. Ulmer. 1968.