Higher-order Algebraic Theories and Relative Monads

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Outline

- Algebraic theories
- Second-order algebraic theories
- Higher-order algebraic theories
- Relative monads, monads, and theories
I. Algebraic Theories
First-order operators

1. \[
\frac{\Gamma \vdash a \quad \Gamma \vdash b}{\Gamma \vdash a \times b}
\]
   Multiplication

2. \[
\frac{\Gamma \vdash a}{\Gamma \vdash a^{-1}}
\]
   Inverses

3. \[
\frac{\Gamma \vdash m : M \quad \Gamma \vdash a : A}{\Gamma \vdash m \times a : A}
\]
   Actions
Algebraic theories

\[ \mathbb{L} \xrightarrow{L} \mathbb{L} \]

- identity-on-objects cartesian functor
- free cartesian category on a point
- cartesian category

(Here, ‘cartesian’ means finite products.)
Algebraic theories

The objects of $L$ are given by $X^n$ for $X$ the generating object, and $n \in \mathbb{N}$. A morphism $X^n \xrightarrow{t} X^m$ represents an $m$-tuple of terms in $n$ variables:

$$\langle x_1, \ldots, x_n + t \rangle_{sism}$$
Algebraic theories

The objects of $L$ are given by $X^n$ for $X$ the generating object, and $n \in \mathbb{N}$. A morphism $X^n \xrightarrow{t_i} X^m$ represents an $m$-tuple of terms in $n$ variables:

$$\langle x_1, \ldots, x_n + t_i \rangle_{is\ ism}$$

$n$-ary operation $\langle t_i : X^n \to X \rangle_i$
Algebraic theories

A map of algebraic theories is a commutative triangle

\[
\begin{array}{ccc}
L & \xrightarrow{F} & L' \\
\downarrow & & \downarrow \\
L & & L'
\end{array}
\]

Algebraic theories and their maps form a category Law.
Monads and theories

There is a classic equivalence between algebraic theories and (strongly) finitary monads on the category of sets.

\[
\text{Law} \quad \sim \quad \text{Mnd}_f(\text{Set}) = \text{Mnd}_{sf}(\text{Set})
\]

Finitary = preserves filtered colimits
Strongly finitary = preserves sifted colimits (sifted-cocontinuous)
Universal algebra

Equational logic

Algebraic theories

\cong

Monads on Set
II. SECOND-ORDER ALGEBRAIC THEORIES

[Fiore & Mahmoud, 2010]
Second-order operators

1. \( \Gamma, x \vdash f \quad \Gamma \vdash x_0 \)
   \[ \Gamma \vdash \frac{df}{dx}(x_0) \]
   Differential operators (cf. Plotkin 2020)

2. \( \Gamma, x \vdash P \)
   \[ \Gamma \vdash \exists x. P \]
   Logical quantifiers

3. \( \Gamma, x \vdash t \)
   \[ \Gamma \vdash \lambda x. t \]
   \( \lambda \)-abstraction
Second-order operators

4. \[ \Gamma \vdash t : A + B \quad \Gamma, a : A \vdash u : C \quad \Gamma, b : B \vdash v : C \]
   \[ \Gamma \vdash \text{case}(t, a.u, b.v) : C \]
   Coproducts, case-splittings

5. \[ \Gamma, x : X \vdash f : X \]
   \[ \Gamma \vdash \text{fix}(f) : X \]
   Fixed points
operators and equations of the theory

structural operations
Second-order theory of equality

\( \mathbb{L}_2 \) is the free cartesian category with an exponentiable object (i.e. an object such that \((-)^\times : \mathbb{L}_2 \to \mathbb{L}_2 \) exists).

Objects of \( \mathbb{L}_2 \) are given by products

\[
\underbrace{X \times X \times \cdots \times X}_{n_k}\]

with morphisms given by projection and evaluation.
Second-order theory of equality

\( \mathbb{L}_2 \) is the free cartesian category with an exponentiable object (i.e., an object such that \((-)^x : \mathbb{L}_2 \to \mathbb{L}_2 \) exists).

Objects of \( \mathbb{L}_2 \) are given by products with morphisms given by projection and evaluation.
Second-order algebraic theories

$\text{free cartesian category on an exponentiable object}$

$\text{identity-on-objects cartesian functor, preserving the exponentiable object}$

$\text{cartesian category with an exponentiable object}$
Second-order algebraic theories

\[
\begin{array}{c}
\mathbb{L}_2 \\
\downarrow \quad L \\
\quad \downarrow L
\end{array}
\]

A morphism \( X^{x^n} \times \ldots \times X^{x^n} \overset{t}{\to} \times X^{m_1} \times \ldots \times X^{m_L} \) in \( \mathbb{L} \) represents an \( L \)-tuple of terms in \( K \) metavariables and \( M_i \) variables:

\[
\langle (x^i_1, \ldots, x^i_{n^i}) x^i_1, \ldots, (x^k_1, \ldots, x^k_{n^k}) x^k, y_i, \ldots, y_{m_i+1} \rangle^i
\]

(parameterised variable) Ordinary (variable)
'Differentiate \( f(x) \) with respect to \( x \) and evaluate at \( x_0 \).'

\[ \delta(x, f(x), x_0) \]

represented by

\[ X \xrightarrow{x \times X} X \xrightarrow{\delta} X \]
Second-order algebraic theories

A map of second-order algebraic theories is a commutative triangle

\[ L \xrightarrow{F} L' \xleftarrow{G} L_2 \]

Second-order algebraic theories and their maps form a category $\text{Law}_2$. 
Second-order universal algebra

Second-order equational logic

Second-order algebraic theories

???
Ⅲ. Higher-order Algebraic Theories
operators and equations of the theory

structural operations
Higher-order theory of equality

$L_n$ is the free cartesian category on an $n$-tetrable object (i.e. an object $X$ such that $1, X, X^X, X^{X^X}, ...$ is exponentiable).

$$1 \ 2 \ 3 \ 4 \ ...$$

$\eta$
Higher-order theory of equality

$\mathbb{L}_n$ is the free cartesian category on an $n$-tetrable object (i.e. an object $X$ such that $1, X, X^x, X^{x^x}, \ldots$ is exponentiable).

$\underbrace{1 \ 2 \ 3 \ 4 \ \ldots}_n$

Intuitively, morphisms in $\mathbb{L}_n$ represent operators taking operators as operands.
Higher-order theory of equality

$L_n$ is the free cartesian category on an $n$-tetrable object (i.e. an object $X$ such that $1, X, X^X, X^{X^X}, \ldots$ is exponentiable).

We have:

$$L = L_1 \hookrightarrow L_2 \hookrightarrow \cdots \hookrightarrow L_\omega$$

↑ free cartesian category on a point

↑ free cartesian-closed category on a point
Higher-order algebraic theories

\[ \text{free cartesian category on an } n\text{-tetrable object} \]

\[ \text{cartesian category with an } n\text{-tetrable object} \]

\[ \text{identity-on-objects cartesian functor, preserving the } n\text{-tetrable object} \]
Higher-order algebraic theories

A map of $n^{th}$-order algebraic theories is a commutative triangle

\[
\begin{array}{ccc}
L & \xrightarrow{F} & L' \\
\downarrow & & \downarrow \\
L & \xrightarrow{n} & L'
\end{array}
\]

$n^{th}$-order algebraic theories and their maps form a category $\text{Law}_n$. 
Is there a monad correspondence for $n^{th}$-order algebraic theories?
The universal property of $\text{Law}_n$

Thm

$\text{Law}_n$ is locally strongly finitely presentable.

$$\text{Law}_n \simeq \text{Cart}(\mathcal{L}_{n+1}, \text{Set}) \simeq \text{Sind}(\mathcal{L}_{n+1}^\circ)$$

sifted
cocompletion

free cartesian category on an
(n+1)-tegable point
The universal property of $\text{Law}_n$ $(n=1)$

Thm

$\text{Law}_1$ is locally strongly finitely presentable.

$\text{Law} = \text{Law}_1 \cong \text{Cart}(\mathbb{L}_2, \text{Set})$

sifted cocompletion

$\cong \text{Sind}(\mathbb{L}_2^\times)$

free cartesian category on an exponentiable object
The universal property of $\text{Lawn}$

**Thm**

$L_{\text{Lawn}}$ is locally strongly finitely presentable.

$L_{\text{Lawn}} \simeq \text{Cart}(L_{n+1}, \text{Set}) 
\simeq \text{Sind}(L_{n+1}^\circ)$

Hence also:
- Locally finitely presentable
- Cocomplete
- Complete
IV. Relative Monads
Higher-order algebraic theories $\sim$ Relative monads $\sim$ Monads
Monads

subject to associativity and unitality laws
Relative monads

\[ \mathcal{E} \xrightarrow{j} \mathcal{O} \]
Relative monads

\[ T \]

\[ \eta \uparrow \]

\[ E \xrightarrow{j} D \]

\[ \text{Diagram for relative monads} \]
Relative monads

But what about multiplication?
Relative monads

A J-relative monad \((T, \eta, (-)^\ast)\) consists of

- a function \(T : 1E' \rightarrow 1E\)
- a transformation \(\eta_x : JX \rightarrow TX\)
- a transformation \((-)^\ast_x : E(JX, TY) \rightarrow E(TX, TY)\)

satisfying unitality and associativity conditions.
A $J$-relative monad $(T, \eta, (-)^*)$ consists of:

- a function $T : \mathcal{E}' \rightarrow \mathcal{E}$
- a transformation $\eta_x : JX \rightarrow TX$
- a transformation $(-)^*_x : \mathcal{E}(JX, TY) \rightarrow \mathcal{E}(TX, TY)$

satisfying unitality and associativity conditions.

Prop: a monad is precisely an $Id$-relative monad.
We will consider the functor

\[ L_{n+1}^0 \xleftarrow{\sim} \text{Sind}(L_{n+1}^0) \cong \text{Law}_n \]

(intuitively a Yoneda embedding)
We will consider the functor

\[ \mathbb{1}_{n+1} \xrightarrow{\mathbb{k}} \text{Sind}(\mathbb{1}_{n+1}) \approx \text{Law}_n \]

(intuitively a Yoneda embedding)

When \( n = 0 \):

\[ \mathbb{1}_1 \xrightarrow{\mathbb{k}} \text{Sind}(\mathbb{1}_1) \]

↑

free cocartesian category on a point
We will consider the functor

$$\mathbb{L}_{n+1}^\circ \xleftarrow{\Leftarrow} \text{Sind}(\mathbb{L}_{n+1}^\circ) \simeq \text{Law}_n$$

(intuitively a Yoneda embedding)

When $n = 0$:

$$\text{FinSet} \simeq \mathbb{L}_1^\circ \xleftarrow{\Leftarrow} \text{Sind}(\mathbb{L}_1^\circ) \simeq \text{Set}$$

↑

free cocartesian category on a point
Thm

\[ \text{Law}_{n+1} \cong \text{RMnd}_{+\text{-lin}}(\mathcal{U}_{n+1}^\circ) \]

\((n+1)^{th}\)-order algebraic theories

+ linear \hspace{1cm} (\mathcal{U}_{n+1}^\circ \hookrightarrow \text{Law}_n)\]

relative monads
Thm

\[ \mathbb{L}^{(n+1)} \xrightarrow{\xi} \text{Sind}(\mathbb{L}^{(n+1)}) \cong \text{Lawn} \]

\[ \text{Law}_{n+1} \cong \text{RMnd}_{+-\text{lin}}(\xi^{\mathbb{L}^{(n+1)}}) \]

\( (n+1) \text{th}-\text{order algebraic theories} \)
**Thm**

\[ \text{Law}_{n+1} \cong \text{RMnd}_{+-\text{lin}} (k \downarrow \text{Law}_{n+1}^\circ) \]

\((n+1)^{\text{th}}\)-order algebraic theories

\[ +\text{-linear } (k \downarrow \text{Law}_{n+1}^\circ \hookrightarrow \text{Law}_n) - \text{relative monads} \]

When \(n=0\), this says that algebraic theories are equivalent to \((\text{FinSet} \rightarrow \text{Set})\)-relative monads.
Thm

\( \text{Law}_{n+1} \simeq \text{RMnd}_{+\text{-lin}} (\text{LL}_{n+1}^\circ \text{LL}_{n+1}^\circ) + \text{-linear } (\text{LL}_{n+1}^\circ \text{LL}_{n+1}^\circ \text{relative monads}) \)

\((n+1)^{th}\)-order algebraic theories
Thm

\[ \text{law}_{n+1} \sim \text{RMnd}_{+-\text{lin}}(\text{law}_{n+1}) \]

\[(n+1)^{\text{th}}\text{-order algebraic theories}\]

+linear \( (\text{law}_{n+1} \leftrightarrow \text{law}_n) \)

relative monads

(But what about ordinary monads?)
\[ \mathcal{L}_1 \xrightarrow{L} \mathcal{L} \]

\[ \mathcal{L}_1 \xrightarrow{T_L} \text{Sind}(\mathcal{L}_1^\circ) \]

\[ \text{Sind}(\mathcal{L}_1^\circ) \xrightarrow{\text{Lan}_\mathcal{L} T_L} \mathcal{L}_1^\circ \]

\[ \text{Sind}(\mathcal{L}_1^\circ) \]

\[ T'_L = \text{Lan}_\mathcal{L} T_L \]

---

**Theory**  |  **Relative monad**  |  **Monad**
$\mathbb{L}_1 \xrightarrow{L} \mathbb{L}$

$\mathbb{L}_1 \xrightarrow{T_L} \text{Sind}(\mathbb{L}_1,^\circ) \xleftarrow{\text{Lan} \otimes T_L} \text{Sind}(\mathbb{L}_1,^\circ)$

$T_L' = \text{Lan} \otimes T_L$

$\text{Sind}(\mathbb{L}_1,^\circ) \simeq \text{Set}$

Theory

Relative monad

Monad
\[
\begin{align*}
\mathbb{L}_2 \overset{L}{\longrightarrow} \mathbb{L} \\
\mathbb{L}_1 \overset{L}{\longrightarrow} \mathbb{L} \\
\mathbb{L}_2^0 \overset{T_L}{\longrightarrow} \text{Sind}(\mathbb{L}_2^0) \\
\mathbb{L}_1^0 \overset{T_L}{\longrightarrow} \text{Sind}(\mathbb{L}_1^0) \\
T_L' = \text{Lan}_\ast T_L \\
\text{Sind}(\mathbb{L}_1^0) \simeq \text{Set}
\end{align*}
\]
\[ \begin{align*}
\mathcal{L}_2 & \xrightarrow{L} \mathcal{L} \\
\mathcal{L}_1 & \xrightarrow{L} \mathcal{L} \\
\end{align*} \]

\[ \begin{align*}
\text{Sind}(\mathcal{L}_2) & \xrightarrow{\text{Lan}_{\mathcal{L}} \mathcal{T}_L} \text{Sind}(\mathcal{L}_2) \\
\text{Sind}(\mathcal{L}_1) & \xrightarrow{\text{Lan}_{\mathcal{L}} \mathcal{T}_L} \text{Sind}(\mathcal{L}_1) \\
\end{align*} \]

\[ T'_L = \text{Lan}_{\mathcal{L}} \mathcal{T}_L \]

\[ \text{Sind}(\mathcal{L}_1) \simeq \text{Set} \]
\[
\begin{align*}
\text{Theory} & \quad \text{Relative monad} & \quad \text{Monad} \\
\mathcal{L}_2 \xrightarrow{L} \mathcal{L} & \quad \mathcal{L}_2 \xrightarrow{\mathcal{T}_L} \text{Sind}(\mathcal{L}_2) & \quad \mathcal{T}'_L = \text{Lan}_\mathcal{T}_L \\
\mathcal{L}_1 \xrightarrow{L} \mathcal{L} & \quad \mathcal{L}_1 \xrightarrow{\mathcal{T}_L} \text{Sind}(\mathcal{L}_1) & \quad \text{Sind}(\mathcal{L}_1) \approx \text{Set}
\end{align*}
\]
Thm

\( \text{Law}_{n+1} \cong \text{RMnd}_{+,-\text{lin}}(\text{k}_{\text{Law}_{n+1}}^\circ) \)

\(+\text{-linear (}\text{Law}_{n+1} \hookrightarrow \text{Law}_n)\) - relative monads

\(\cong \text{Mnd}_{+,-\text{lin, sf}}(\text{Law}_n)\)

sifted - cocontinuous + linear monads on \(\text{Law}_n\)

\((n+1)\text{th-order algebraic theories}\)
Idea

Variable-binding structure is algebraic over algebraic structure.
Coreflections

Thm

There is a coreflection of categories:

\[
\begin{array}{ccc}
\text{inclusion of presentations} & \text{discard } (n+1)^{th} - \text{order terms} \\
\downarrow & \uparrow \\
\text{Law}_n & \text{Law}_{n+1}
\end{array}
\]
Coreflections

There is a chain of coreflections,

$$\text{Law}_1 \leftrightarrow \text{Law}_2 \leftrightarrow \cdots \leftrightarrow \text{Law}_\omega$$

allowing us to freely extend or restrict the order of a higher-order algebraic theory.
Prop. Let $L : \mathbb{L}^{n+1} \to \mathbb{L}$ be an $(n+1)^{\text{th}}$-order algebraic theory. The corresponding monad is given by

$$T_L(X) \equiv \mathbb{L} \uplus \left[ X \right]$$
Prop. Let $L : \ll_{n+1} \rightarrow L$ be an $(n+1)^{th}$-order algebraic theory. The corresponding monad is given by
\[
T_L(X) \cong L + [X]
\]
When $n=0$, this says that $T_L$ takes a set of constants, freely adds them to $L$, then extracts the new constants formed from those in $X$ under the operations of $L$. 
Summary

- Higher-order algebraic theories generalise algebraic theories by (higher-order) variable binding operators.

- There are coreflections $\text{Law}_n \xleftrightarrow{} \text{Law}_{n+1}$.

- $\text{Law}_n \simeq \text{Sind}(\mathcal{L}^\text{op}_{n+1})$

- $\text{Law}_{n+1} \simeq \text{Mnds}_{\text{sf},+-\text{lin}}(\text{Law}_n)$
Algebras

Let $L: 1^{(n+1)} \to L$ be an $(n+1)^{th}$-order algebraic theory, and let $T_L : \text{Law}_n \to \text{Law}_n$ be the corresponding monad.

$$T_L - \text{Alg} \simeq \text{Cart}(L, \text{Set})$$