A Linear-time Algorithm for Testing the Truth of Certain Quantified Boolean Formulas

by

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Presentation of

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The Problem

Evaluate the formula:

$$Q_1 x_1 Q_2 x_2 \cdots Q_n x_n \quad C$$

where each Q_i is either $\forall$ or $\exists$

and C is in conjunctive normal form with at most two literals per clause.

For example

$$\exists a \exists b \forall c \exists d (a \vee b) \wedge (b \vee \bar{c}) \wedge (\bar{b} \vee \bar{d}) \wedge (b \vee d) \wedge (d \vee a)$$

Note

$a \vee b$	is equivalent to	$\bar{a} \longrightarrow b$
	or	$\bar{b} \longrightarrow a$
$\bar{a} \vee b$	is equivalent to	$a \longrightarrow b$
	or	$\bar{b} \longrightarrow \bar{a}$
$a \vee \bar{b}$	is equivalent to	$\bar{a} \longrightarrow \bar{b}$
	or	$b \longrightarrow a$
$\bar{a} \vee \bar{b}$	is equivalent to	$a \longrightarrow \bar{b}$
	or	$b \longrightarrow \bar{a}$

Graph Formation

Without loss of generality assume that all clauses of C have two literals, since $(x) = (x \vee x)$.

Assuming the formula contains n variables $v_1, \ldots, v_n$, create $2n$ vertices named $v_1, \ldots, v_n$ and $\bar{v_1}, \ldots, \bar{v_n}$.

For each clause of the form: $(a \vee b)$, add edges $\bar{a} \rightarrow b$ and $\bar{b} \rightarrow a$.

Add appropriate edges for clauses of the form:

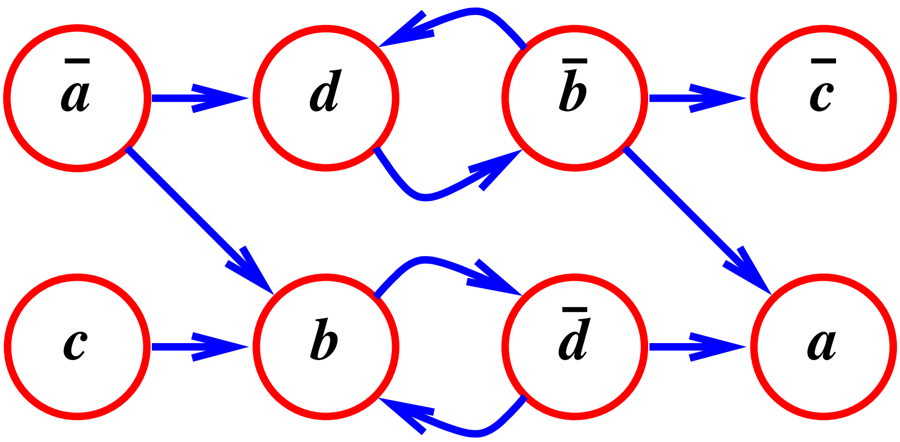
$$(a \vee \bar{b}),\ (\bar{a} \vee b) \text{ and } (\bar{a} \vee \bar{b}).$$

Example

If C is

$$(a \vee b) \wedge (b \vee \bar{c}) \wedge (\bar{b} \vee \bar{d}) \wedge (b \vee d) \wedge (d \vee a)$$

The graph is:



Duality Property

The constructed graph is isomorphic to one obtained by reversing all edges and complementing the names of the vertices.

Strongly Connected Components

A strongly connected component is a maximal subset of the vertices for which paths exists from any vertex to any other vertex.

All vertices in a strongly connected component must be assigned the same truth value
($x \longrightarrow y \longrightarrow \cdots \longrightarrow x$ implies $x = y$).

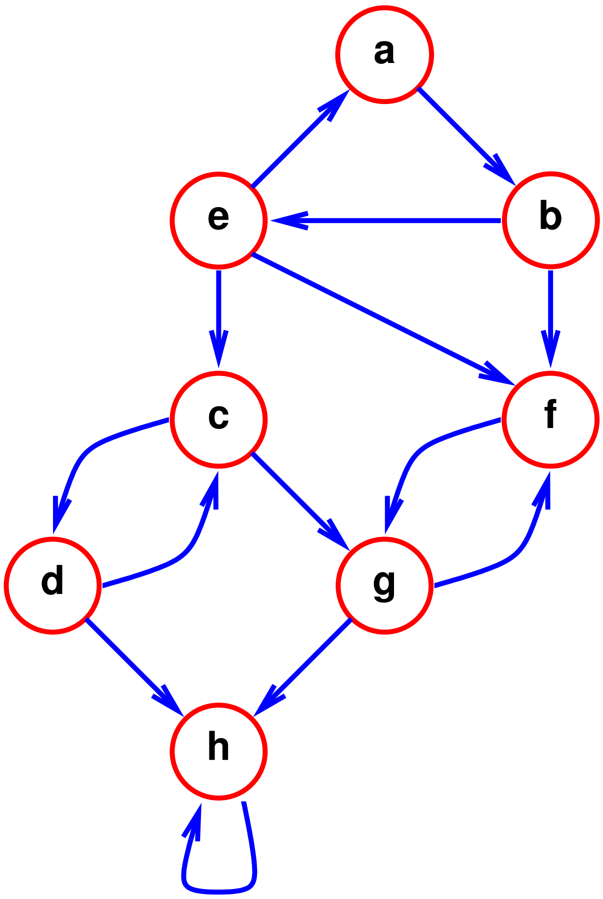
If a strongly connected component contain a variable x and its complement $\bar{x}$ then there is an inconsistency.

All the strongly connected components can be found in linear time.

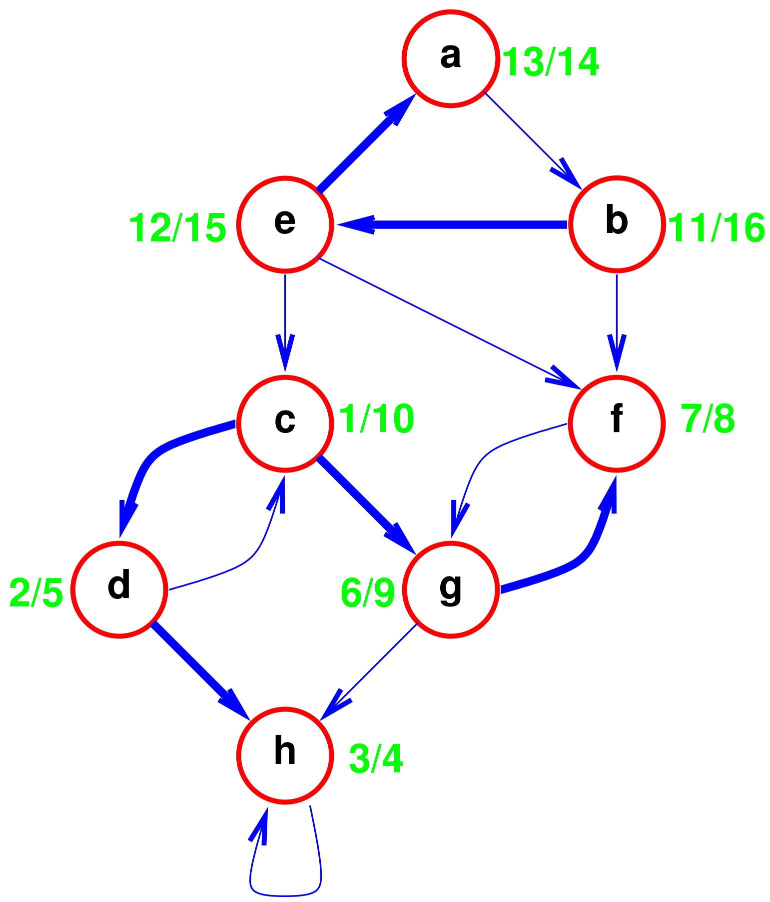
Components Algorithm

1) Perform depth first search on graph G(V,E), attaching the discovery time ($d[u]$) and finishing time ($f[u]$) to every vertex (u) of G.

For example, consider



After DFS



Forefather

Define $\phi(u) = w$ (the forefather of u)

where $w \epsilon V$

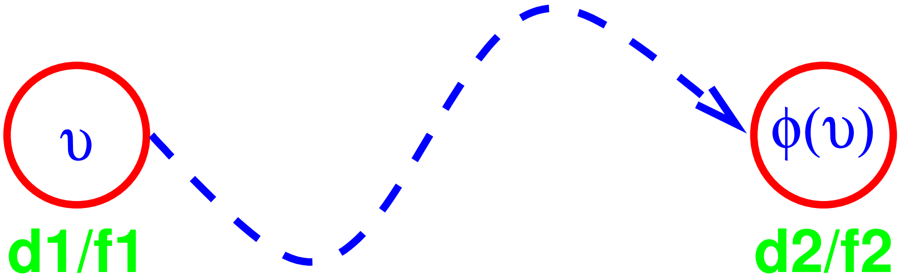
and $u \longrightarrow \cdots \longrightarrow w$

and $\forall w' (u \longrightarrow \cdots \longrightarrow w' \Rightarrow f[w'] \leq f[w])$

Clearly, since $u \longrightarrow \cdots \longrightarrow u$

$$f[u] \leq f[\phi(u)] \qquad (1)$$

Forefather Property



Either $u = \phi(u)$

or $u \neq \phi(u)$

if $d2 < f2 < d1 < f1$

or $d1 < d2 < f2 < f1$

contradicts of (1)

if $d1 < f1 < d2 < f2$

contradicts $u \longrightarrow \cdots \longrightarrow \phi(u)$

so $d2 < d1 < f1 < f2$

ie $\phi(u) \longrightarrow \cdots \longrightarrow u$

so u and $\phi(u)$ are in the same strongly connected component.

Algorithm Continued

2) Find the vertex r with largest $f[r]$ that is not in any strongly connected component so far identified. Note that r is a forefather.

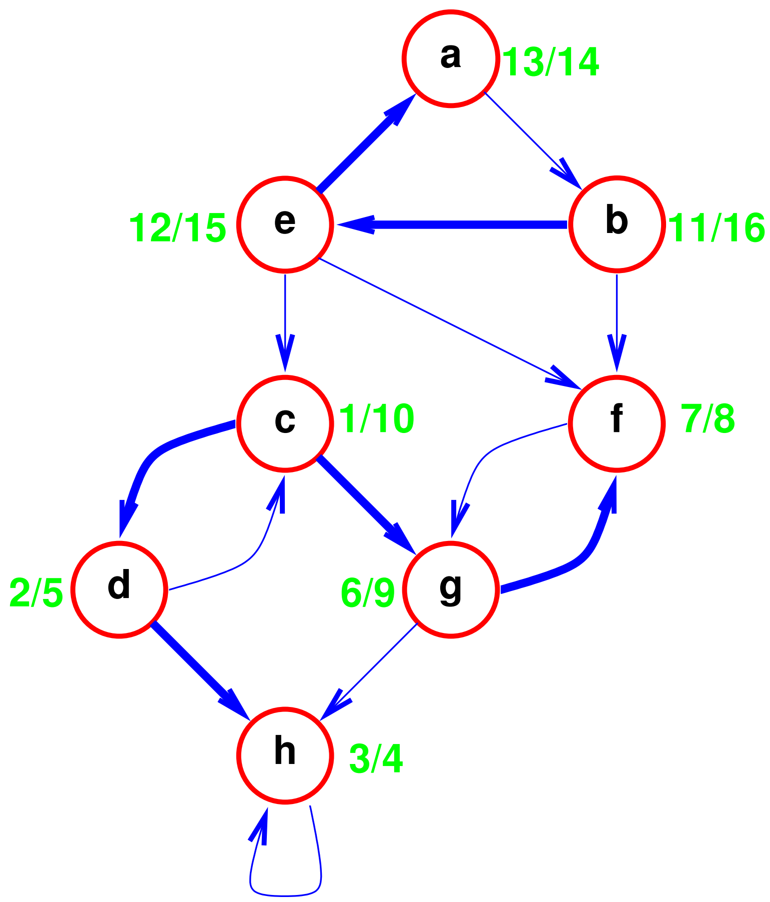
3) Form the set of vertices $\{u|\phi(u) = r\}$ – i.e. the strongly connected component containing r. This set is the same as $\{u|u \longrightarrow \cdots \longrightarrow r\}$

   This set is the set of vertices reachable from r in the graph $G^T = G$ with all its edges reversed. This set can be found using DFS on G^T.

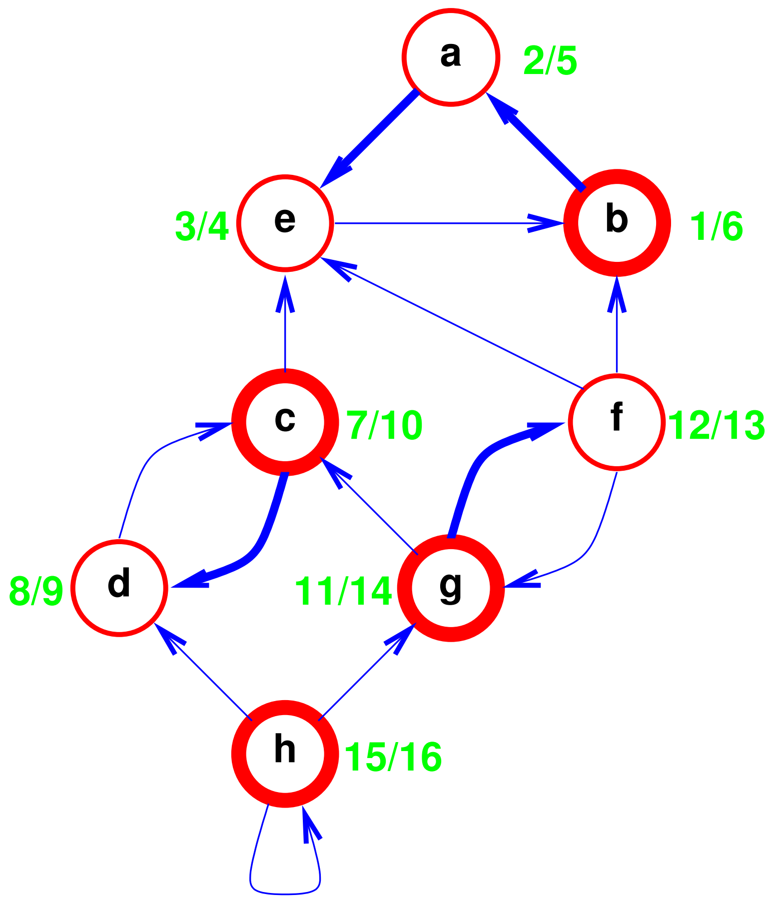
4) Repeat from (2) until all components have been found.

The complexity is $O(|V| + |E|)$.

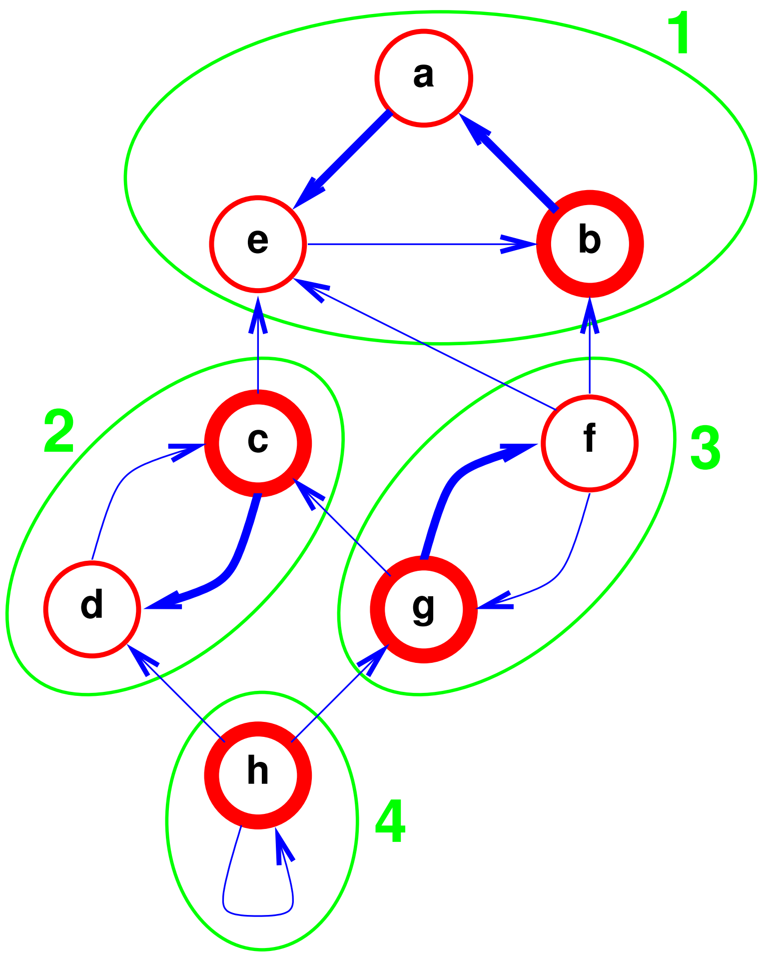
Recall



After DFS on G^T



Components



Satisfiability

If all the quantifiers are $\exists$, then we just have to determine whether it is possible to assign truth values to the vertices with the following properties:

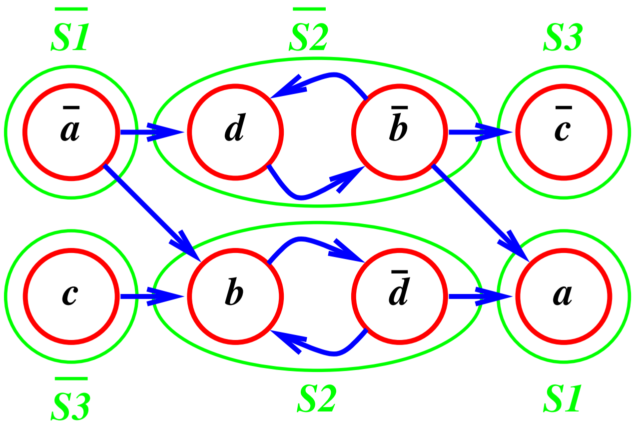
- Complementary vertices must be assigned compementary truth values,
- no edge $u \rightarrow v$ can have u assigned true and v assigned false.

Example (again)

If C is

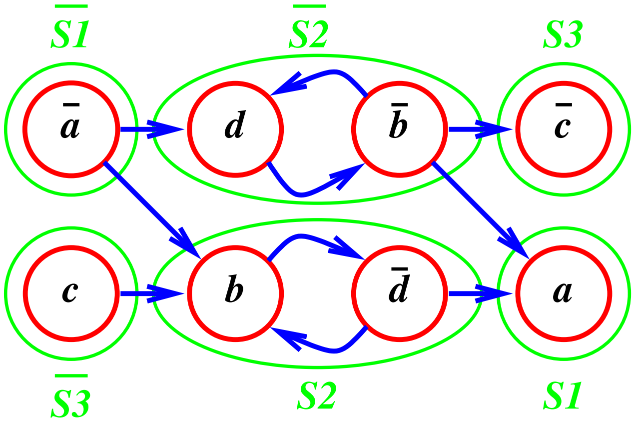
$$(a \vee b) \wedge (b \vee \bar{c}) \wedge (\bar{b} \vee \bar{d}) \wedge (b \vee d) \wedge (d \vee a)$$

The graph is:



C is satisfiable if and only if no vertex x is in the same strong component as $\bar{x}$.

Assignment Algorithm



Consider strongly connected components in reverse topological order:

$S1$	Mark $S1$ true and $\bar{S}1$ false,
$S2$	Mark $S2$ true and $\bar{S}2$ false,
$S3$	Mark $S3$ true and $\bar{S}3$ false,
$\bar{S}2$	Already marked,
$\bar{S}1$	Already marked,
$\bar{S}3$	Already marked.

General Case

We call a vertex universal if the corresponding variable is universally qualified, and existential otherwise.

A formula F is true if and only if none of the following conditions holds:

1. A vertex x is in the same strongly connected component as its complement $\bar{x}$.
2. An existential vertex x occurs in the same strongly connected component as a universally declared vertex y, with x declared before y. For example: $\cdots \exists x \cdots \forall y \cdots \quad C$.
3. There is a path from a universal vertex x to another universal vertex y. For example: $\cdots \forall x \cdots \forall y \cdots \quad C$.

The Algorithm

Initially all the strongly connected components are unmarked, but become marked true, false or contingent as the algorithm proceeds.

1) Let S be the next unmarked strongly connected component, chosen is reverse topological order. If there is no such S return "F is true". If S contains a variable x and its complement $\bar{x}$, return "F is false".

2) If S has some false or contingent successor,

   2.1) If S contains at least one universal vertex, return "F is false".

   2.2) If $\bar{S} \rightarrow \cdots \rightarrow S$, return "F is false". Mark S false and $\bar{S}$ true then goto (1).

3) // All successors, if any, are marked true. If S contains two or more universal vertices, return "F is false".

The Algorithm (cont.)

4) // All successors, if any, are marked true and
   // S contains less than two universal vertices.
   If S contains no universal vertices, mark S true and $\bar{S}$ false then goto (1).

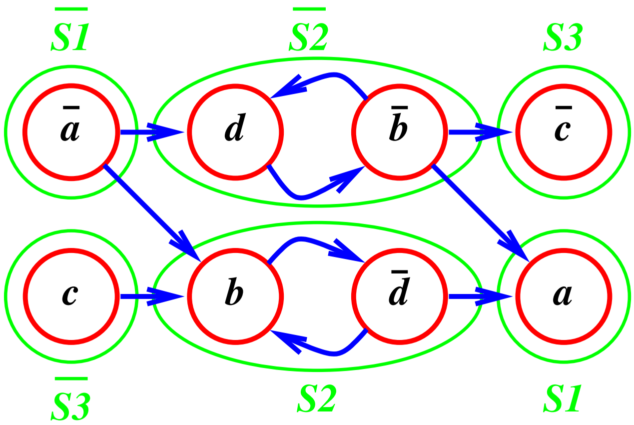
5) // S has just one universal vertex, y say.
   If S also contains an existential vertex x declared before the universal vertex y,
   (i.e. $\cdots \exists x \cdots \forall y \cdots C$), return "F is false".
   If $\bar{S} \rightarrow \cdots \rightarrow S$, return "F is false".
   Mark S and $\bar{S}$ contingent and goto (1).

Example (again)

If the formula is

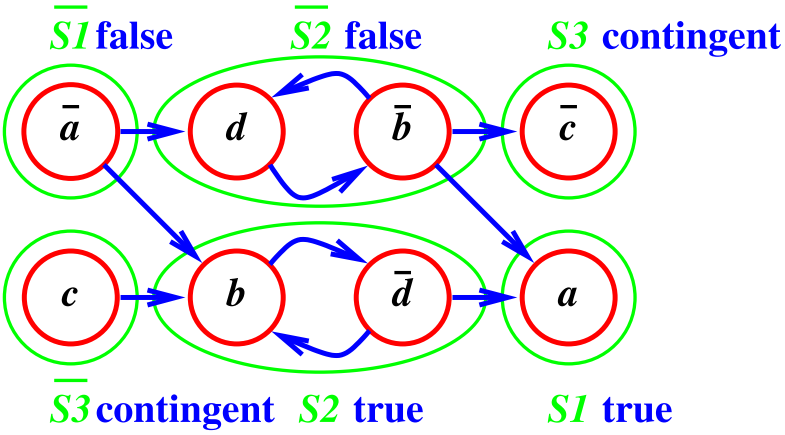
$$\exists a \exists b \forall c \exists d (a \vee b) \wedge (b \vee \bar{c}) \wedge (\bar{b} \vee \bar{d}) \wedge (b \vee d) \wedge (d \vee a)$$

The graph is:



Algorithm Trace

Formula: $\exists a \exists b \forall c \exists d \quad C$



Consider strongly connected components in reverse topological order:

$S1$	Mark $S1$ true and $\bar{S}1$ false,
$S2$	Mark $S2$ true and $\bar{S}2$ false,
$S3$	Mark $S3$ and $\bar{S}3$ contingent,
$\bar{S}2$, $\bar{S}1$, $\bar{S}3$	Already marked.

So the formula is true.