A Mathematical Theory of Substitution an its Applications to Syntax and Semantics

Marcelo Fiore

Computer Laboratory
University of Cambridge

ICMS

26.V.2007

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(tu·to·ri·al)

a class in which a tutor gives *intensive instruction* in some subject to an individual student or a small group of students

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Aim

To give an elementary introductory presentation of the basic ideas underlying a mathematical theory of algebraic models for languages with variable binding and substitution.

Specification of first-order syntax

Signature: A set of *operators* equipped with *arities*.

$$|-| : \Sigma \longrightarrow \mathbb{N}$$

Specification of first-order syntax

Signature: A set of *operators* equipped with *arities*.

$$|-| : \Sigma \longrightarrow \mathbb{N}$$

Example: The signature of monoids has operators $\{\mathsf{e}, \mathsf{m}\}$ with arities $|\mathsf{e}| = 0$ and $|\mathsf{m}| = 2$.

Inductive definition of abstract syntax

$$\text{(Variables)} \quad \frac{x \in X}{\lfloor x \rfloor \in \Sigma^\star(X)}$$

$$\text{(Operators)} \quad \frac{t_1, \ldots, t_n \in \Sigma^\star(X)}{o(t_1, \ldots, t_n) \in \Sigma^\star(X)} \quad (\,|o| = n\,)$$

- Induction principle.
- Structural recursive definitions.

Analysis of abstract syntax

Σ-algebras:

$$A\ ,\quad \{\alpha_o : A^{|o|} \longrightarrow A\}_{o\in\Sigma}$$

Homomorphisms:

$$\begin{array}{ccc} A^{|o|} & \xrightarrow{h^{|o|}} & B^{|o|} \\ {\scriptstyle\alpha_o}\downarrow & & \downarrow{\scriptstyle\beta_o} \\ A & \xrightarrow[h]{} & B \end{array} \qquad\qquad (\,o \in \Sigma\,)$$

$$h\big(\alpha_o(a_1,\ldots,a_n)\big) = \beta_o\big(h(a_1),\ldots,h(a_n)\big)$$

- Definition in a category with finite products.

The structure of abstract syntax

1. $\Sigma^\star(X)$ is a Σ-algebra.

$$\begin{array}{rcl} (\Sigma^\star X)^{|o|} & \longrightarrow & \Sigma^\star X \\ t_1, \ldots, t_n & \longmapsto & o(t_1, \ldots, t_n) \end{array}$$

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$$\frac{X \vdash t_1 \;,\; \ldots \;,\; X \vdash t_n}{X \vdash o(t_1, \ldots, t_n)}$$

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2. $\lfloor - \rfloor : X \longrightarrow \Sigma^\star(X)$ is a free Σ-algebra on X.

Universal definition of abstract syntax

Definition: $X \longrightarrow \Sigma^{\star}(X)$ is the free Σ-algebra on X.

ABSTRACT SYNTAX = FREE ALGEBRAS

- Induction principle.
- Structural recursive definitions.
- Abstraction.
- Generality.

Σ-algebras revisited

$$\frac{\{A^{|o|} \xrightarrow{\alpha_o} A\}_{o \in \Sigma}}{\underbrace{\left(\coprod_{o \in \Sigma} A^{|o|}\right)}_{\Sigma(A)} \xrightarrow[\alpha = [\alpha_o]_{o \in \Sigma}]{} A}$$

- Definition in a category with finite products and coproducts

Σ-algebras revisited

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$$\begin{array}{ccccc} A & & \Sigma(A) & & o(a_1,\ldots,a_n) \\ {\scriptstyle\rho}\big\downarrow & \longmapsto & \big\downarrow{\scriptstyle -[\rho]} & & \big\downarrow \\ B & & \Sigma(B) & & o\big(\rho(a_1),\ldots,\rho(a_n)\big) \end{array}$$

- Definition in a category with finite products and coproducts

Abstract signatures

Algebras for an endofunctor:

$$A\ , \quad \alpha : SA \to A$$

Homomorphisms:

$$\begin{array}{ccc} SA & \xrightarrow{Sh} & SB \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \beta} \\ A & \xrightarrow[h]{} & B \end{array}$$

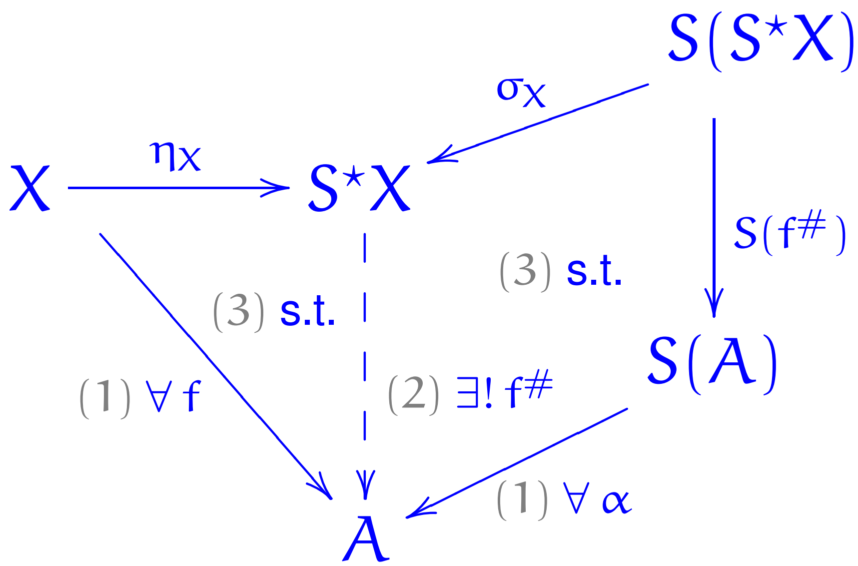
Abstract syntax

Free S-algebras: $S^\star X \cong X + S(S^\star X)$

$$X \xrightarrow{\eta_X} S^\star X \xleftarrow{\sigma_X} S(S^\star X)$$

Abstract syntax

Free S-algebras: $S^{\star}X \cong X + S(S^{\star}X)$



Abstract syntax

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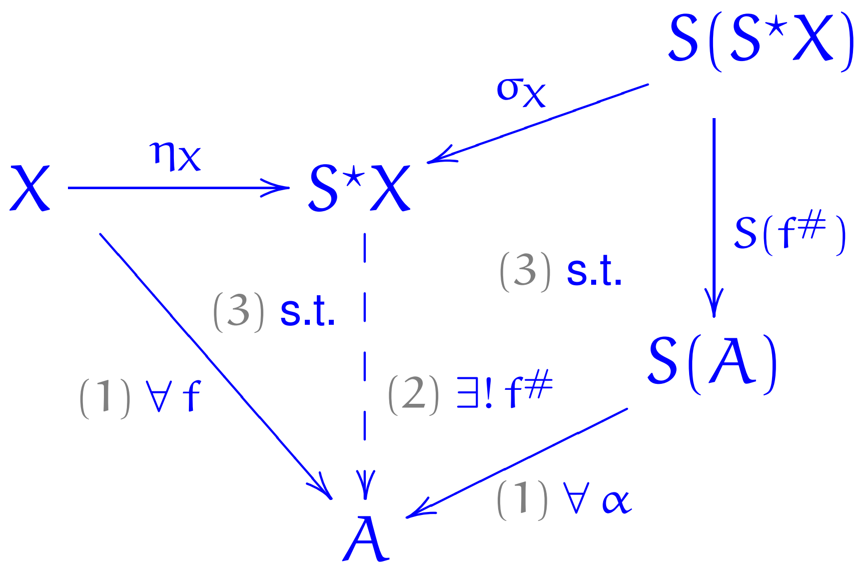

- *Initial algebras are free algebras*: $S^\star 0$ is an initial S-algebra.
- *Free algebras are initial algebras*: $S^\star X = \left(X + S(-)\right)^\star 0$.

Initial-algebras

- $\Sigma^{\star}0 = \coprod_{n \in \mathbb{N}} \Sigma^{n} 0$

Initial-algebras

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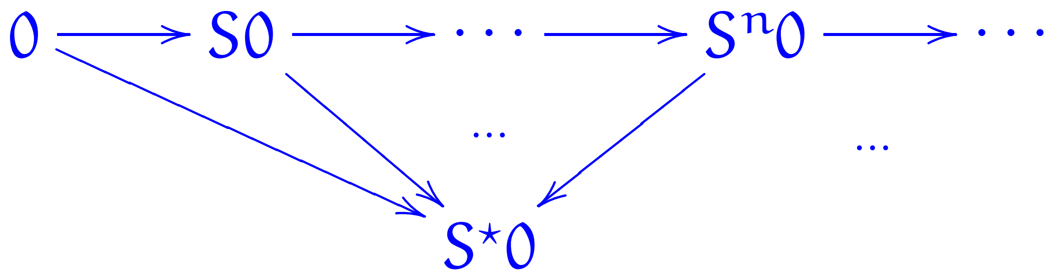
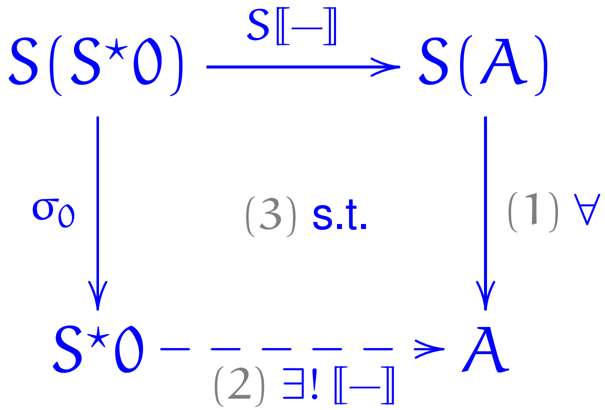
- For S *finitary*:



Initial-algebra semantics

Compositionality:



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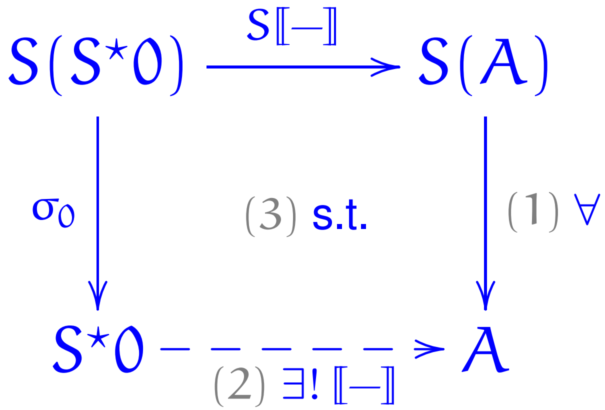
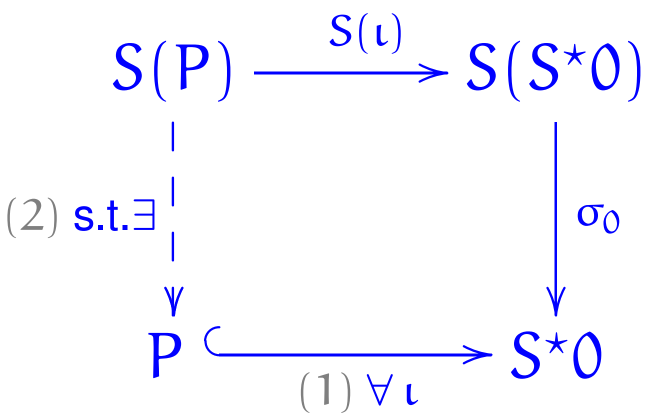


Induction Principle:



$$\Longrightarrow \quad \iota : P \cong S^{\star}0$$

Renaming

Renaming:

$$\begin{array}{ccc} X & & S^{\star}X \\ {\scriptstyle\rho}\big\downarrow & \longmapsto & \big\downarrow{\scriptstyle(-)[\rho]} \\ Y & & S^{\star}Y \end{array}$$

Renaming

Renaming:

$$
\begin{array}{ccc}
X & \xrightarrow{\eta_X} & S^{\star}X \\
{\scriptstyle \rho}\big\downarrow & \longmapsto & \big\downarrow{\scriptstyle (-)[\rho] = (\eta_Y\, \rho)^{\#}} \\
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\end{array}
$$

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Functorial laws:

$$\begin{cases} t[\mathrm{id}] = t \\ t[\rho][\tau] = t[\tau\rho] \end{cases}$$

Parameterised structural recursion

Cartesian strength:

$$S(X) \times P \xrightarrow{\ell} S(X \times P) \ : \big(o(x_1, \ldots, x_n), p\big) \longmapsto o\big((x_1, p), \ldots, (x_n, p)\big)$$

Parameterised structural recursion

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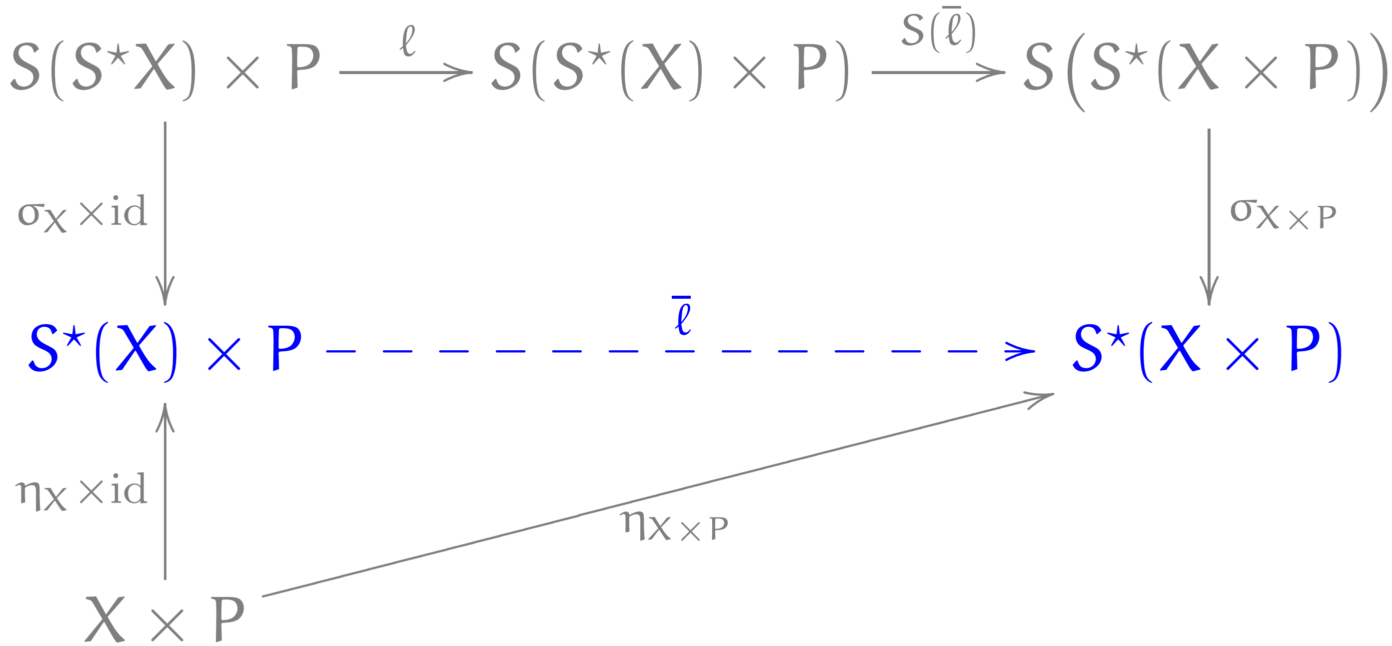

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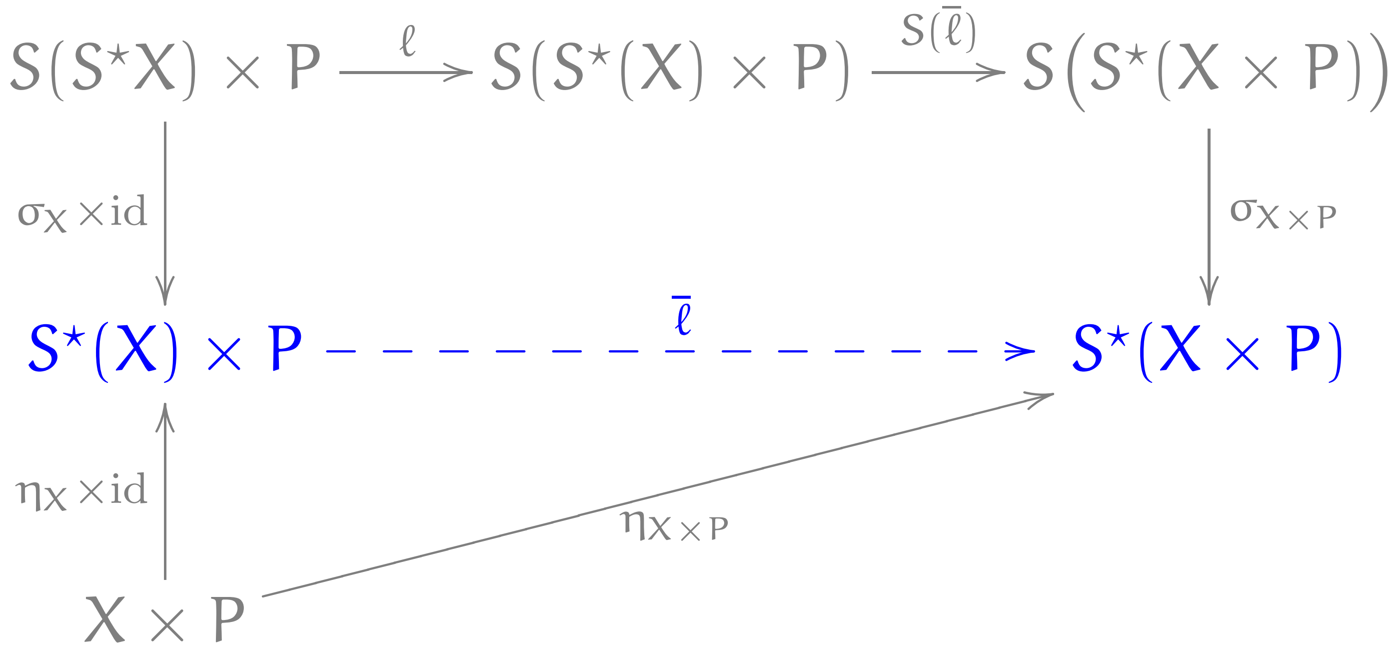


NB: The discussion to follow on renaming and substitution applies more generally to a **tensorial strength** $S(X) \otimes P \longrightarrow S(X \otimes P)$ with respect to a closed tensor product.

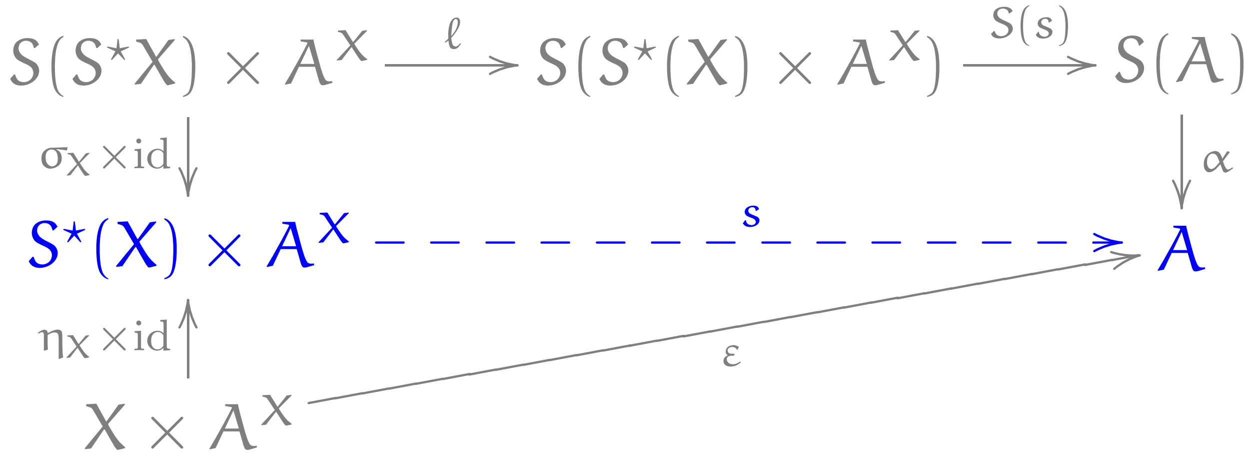
Renaming

Definition by parameterised structural recursion:

$$
\begin{array}{ccccc}
S(S^{\star}X) \times Y^{X} & \xrightarrow{\ \ell\ } & S(S^{\star}(X) \times Y^{X}) & \xrightarrow{\ S(-[=])\ } & S(S^{\star}Y) \\
\Big\downarrow{\scriptstyle \sigma_X \times \mathrm{id}} & & & & \Big\downarrow{\scriptstyle \sigma_Y} \\
S^{\star}(X) \times Y^{X} & & \xrightarrow{\quad -[=] \quad} & & S^{\star}Y \\
\Big\uparrow{\scriptstyle \eta_X \times \mathrm{id}} & & & & \Big\uparrow{\scriptstyle \eta_Y} \\
X \times Y^{X} & & \xrightarrow[\ \varepsilon\ ]{} & & Y
\end{array}
$$

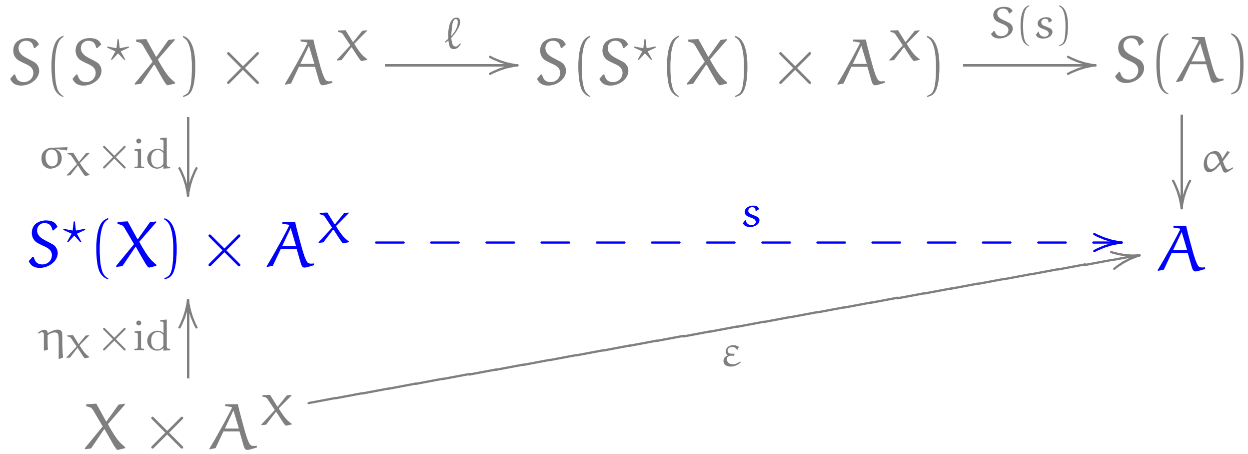
Substitution

Definition by structural recursion:

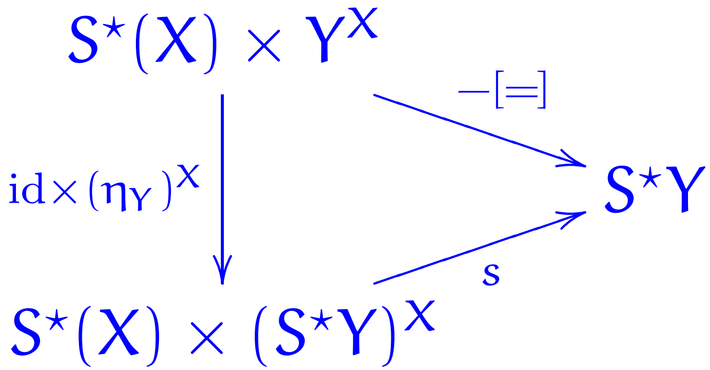


Substitution

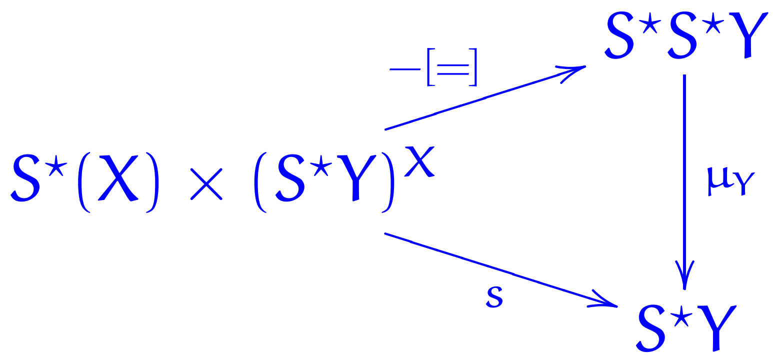
Definition by structural recursion:



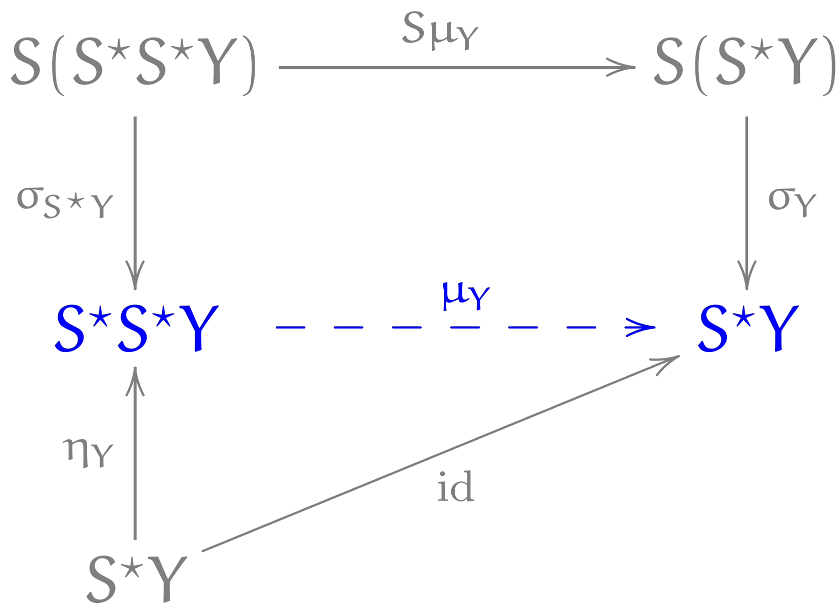
Obs: Substitution generalises renaming.



Obs: Renaming yields substitution.



where



Substitution structure

The structure

$$S^\star(X) \times (S^\star Y)^X \xrightarrow{s_{X,Y}} S^\star(Y) \xleftarrow{\eta_Y} Y$$

is a *substitution structure* for $S^\star$

Substitution structure

The structure

$$S^\star(X) \times (S^\star Y)^X \xrightarrow{s_{X,Y}} S^\star(Y) \xleftarrow{\eta_Y} Y$$

is a *substitution structure* for $S^\star$ in the sense of satisfying the following axioms:

1. Projection.

$$x_j[{}^{t_i}/_{x_i}] = t_j$$

$$\begin{array}{ccc} X \times (S^\star Y)^X & \xrightarrow{\eta_X \times \mathrm{id}} & S^\star(X) \times (S^\star Y)^X \\ & \searrow^{\varepsilon} & \downarrow s \\ & & S^\star(Y) \end{array}$$

2. Extensionality.

$$t[{}^{x_i}/_{x_i}] = t$$

$$\begin{array}{ccc} S^\star(X) \times 1 & \xrightarrow{\mathrm{id} \times \lceil \eta_X \rceil} & S^\star(X) \times (S^\star X)^X \\ & \underset{\cong}{\searrow} & \downarrow s \\ & & S^\star(X) \end{array}$$

3. **Associativity.**

$$\big(t[{}^{u_i}/_{x_i}]\big)[{}^{v_j}/_{y_j}] = t[{}^{u_i[{}^{v_j}/_{y_j}]}/_{x_i}]$$

$$\begin{array}{ccccc}
S^\star(X)\times(S^\star Y)^X\times(S^\star Z)^Y & & \longrightarrow & & S^\star(X)\times\big(S^\star(Y)\times(S^\star Z)^Y\big)^X \\
\Big\downarrow{\scriptstyle s\times \mathrm{id}} & & & & \Big\downarrow{\scriptstyle \mathrm{id}\times s^X} \\
S^\star(Y)\times(S^\star Z)^Y & \xrightarrow[s]{} & S^\star Z & \xleftarrow[s]{} & S^\star(X)\times(S^\star Z)^X
\end{array}$$

4. Compatibility with renaming.

(a)

$$x_j[x_i \longmapsto y_{\rho i}] = y_{\rho j}$$

$$\begin{array}{ccc} X \times Y^X & \xrightarrow{\eta_X \times \mathrm{id}} & S^\star(X) \times Y^X \\ \big\downarrow{\scriptstyle \varepsilon} & & \big\downarrow{\scriptstyle -[=]} \\ Y & \xrightarrow[\eta_Y]{} & S^\star Y \end{array}$$

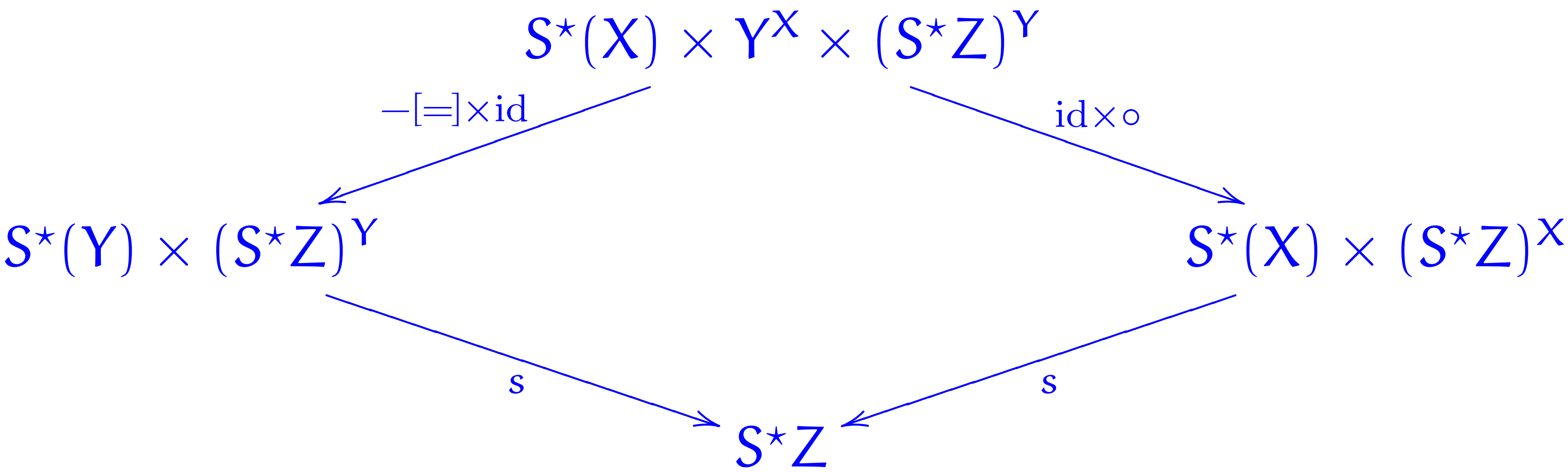
(b)

$$\left(t[{}^{t_i}/_{x_i}]\right)[y_j \longmapsto z_{\rho j}] = t[{}^{t_i[y_j \longmapsto z_{\rho j}]}/_{x_i}]$$

$$\begin{array}{ccccc}
S^\star(X) \times (S^\star Y)^X \times Z^Y & & \longrightarrow & & S^\star(X) \times \left(S^\star(Y) \times Z^Y\right)^X \\
\downarrow{\scriptstyle s\times \mathrm{id}} & & & & \downarrow{\scriptstyle \mathrm{id}\times(-[=])^X} \\
S^\star(Y) \times Z^Y & \xrightarrow[-[=]]{} & S^\star Z & \xleftarrow[s]{} & S^\star(X) \times (S^\star Z)^X
\end{array}$$

(c)

$$\big(t[x_i \longmapsto y_{\rho i}]\big)[{}^{t_j}/_{y_j}] = t[{}^{t_{\rho i}}/_{x_i}]$$



Example:

Substitution structure on the clone of operations:

- $\langle D, D\rangle(X) = [D^X, D]$
- $X \longrightarrow [D^X, D] : x \longmapsto \lambda v : D^X.\, v(x)$
- $[D^X, D] \times [D^Y, D]^X \longrightarrow [D^Y, D] : \tau, f \longmapsto \lambda v : D^Y.\, \tau\big(\lambda x : X.\, f(x)(v)\big)$

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NB: The structure

$$
\begin{array}{ccccc}
(S^\star X)^X \times (S^\star X)^X & \longrightarrow & (S^\star X)^X & \longleftarrow & 1 \\
(g, f) & \longmapsto & \lambda x : X.\, s(f(x), g) \qquad \lambda x : X.\, \eta_X(x) & \longleftarrow\!\shortmid & ()
\end{array}
$$

is a monoid.

Synthesis of substitution structure

substitution structure $T(X) \times (TY)^X \longrightarrow TY \longleftarrow Y$

Synthesis of substitution structure

substitution structure $\mathsf{T}(X) \times (\mathsf{T}Y)^X \longrightarrow \mathsf{T}Y \longleftarrow Y$

$\equiv$

substitution structure $\left(\coprod_X \mathsf{T}(X) \times (\mathsf{T}Y)^X\right) \longrightarrow \mathsf{T}Y \longleftarrow Y$

Synthesis of substitution structure

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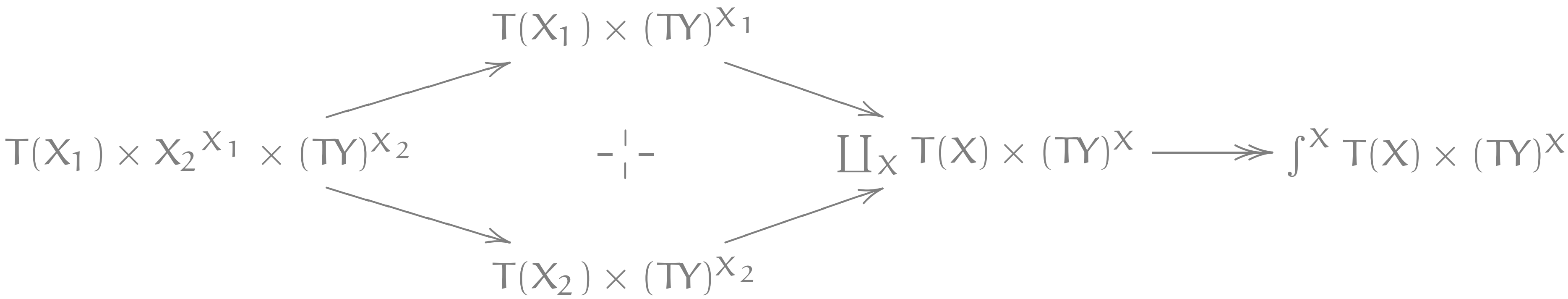
$\equiv$

substitution structure $\left(\coprod_X T(X) \times (TY)^X\right) \longrightarrow TY \longleftarrow Y$

$\equiv$

$\left(\int^X T(X) \times (TY)^X\right) \longrightarrow TY \longleftarrow Y$ satisfying (1–3) and (4a&b)



Synthesis of substitution structure = monoid structure

substitution structure $\mathsf{T}(\mathsf{X}) \times (\mathsf{T}\mathsf{Y})^{\mathsf{X}} \longrightarrow \mathsf{T}\mathsf{Y} \longleftarrow \mathsf{Y}$

$\equiv$

$$\underbrace{\left(\int^{\mathsf{X}} \mathsf{T}(\mathsf{X}) \times (\mathsf{T}\mathsf{Y})^{\mathsf{X}} \right)}_{(\mathsf{T} \bullet \mathsf{T})(\mathsf{Y})} \longrightarrow \mathsf{T}\mathsf{Y} \longleftarrow \mathsf{Y}$$ satisfying (1–3) and (4a&b)

$\equiv$

$\mathsf{T} \bullet \mathsf{T} \longrightarrow \mathsf{T} \longleftarrow \mathrm{Id}$ satisfying monoid laws [cf. (1–3)]

Synthesis of substitution structure = monoid structure

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$\equiv$

$\mathsf{T}\,\mathsf{T} \longrightarrow \mathsf{T} \longleftarrow \mathrm{Id}$ satisfying monad laws

- $S^{\star}$ is the free monad on S

The finitary aspect of syntax

$$\Sigma^\star(X) = \bigcup_{C \subseteq_{\text{fin}} X} \Sigma^\star(C)$$

The finitary aspect of syntax

The finitary condition

$$\Sigma^\star(X) = \bigcup_{C \subseteq_{\mathrm{fin}} X} \Sigma^\star(C)$$

amounts to consider $\Sigma^\star$ as a *variable set*

$$\mathbb{F} \longrightarrow \mathbf{Set}$$

for $\mathbb{F}$ a category of *contexts* given by finite sets of variables and renamings between them.

The mathematical universe of variable sets $\mathbf{Set}^{\mathbb{F}}$ is a natural and convenient setting for syntax and semantics.

Variable sets

Variable sets: $P \in \mathbf{Set}^{\mathbb{F}}$

$$\left\{ \begin{array}{l} \{\, P(C) \,\}_{C \in \mathbb{F}} \\ -[=] : P(C) \times \mathbb{F}[C, D] \longrightarrow P(D) \\ \qquad \text{s.t. } p[\mathrm{id}] = p \text{ and } (p[\rho])[\tau] = p[\tau\rho] \\ \qquad \text{for all } p \in P(C) \text{ and } C \xrightarrow{\rho} D \xrightarrow{\tau} E \end{array} \right.$$

$[p \in P(C) \leftrightsquigarrow C \vdash p : P]$

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Examples:

- $\Sigma^{\star}$
- $\langle D, D \rangle$
- V, with $V(C) = C$ [the type of variables]

Variable functions

Variable functions: $f : P \longrightarrow Q$ in $\mathbf{Set}^{\mathbb{F}}$

$$\left\{ \begin{array}{l} \{ f_C : P(C) \longrightarrow Q(C) \}_{C \in \mathbb{F}} \\ \qquad \text{s.t. } \big(f_C(p)\big)[\rho] = f_D(p[\rho]) \\ \qquad \text{for all } \rho : C \longrightarrow D \end{array} \right.$$

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Examples:

- $[\![-]\!] : \Sigma^* \longrightarrow \langle \mathcal{D}, \mathcal{D} \rangle \; : (C \vdash t) \longmapsto [\![t]\!] \in [\mathcal{D}^C, \mathcal{D}]$

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- $f : V \longrightarrow P$

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- $f : V \longrightarrow P$ amounts to $x \vdash p : P$ $\qquad \big(V \cong \mathbb{F}[\{x\}, -]\big)$

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Examples:

- $[\![-]\!] : \Sigma^* \longrightarrow \langle \mathcal{D}, \mathcal{D} \rangle \;\; : (C \vdash t) \longmapsto [\![t]\!] \in [\mathcal{D}^C, \mathcal{D}]$
- $f : V \longrightarrow P$ amounts to $x \vdash p : P$ $\quad \big(V \cong \mathbb{F}[\{x\}, -]\big)$
- $f : \mathbf{1} \longrightarrow P$ amounts to $\vdash p : P$ $\quad \big(\mathbf{1} \cong \mathbb{F}[\emptyset, -]\big)$

The universe of variable sets

Substitution tensor product: $P \bullet Q$, with

$$(P \bullet Q)(C) = \int^{D \in \mathbb{F}} P(D) \times (QC)^D$$

Laws:

- $$(P \bullet Q) \bullet R \cong P \bullet (Q \bullet R)$$
- $$V \bullet P \cong P \qquad\qquad P \cong P \bullet V$$
- The substitution tensor product is closed.

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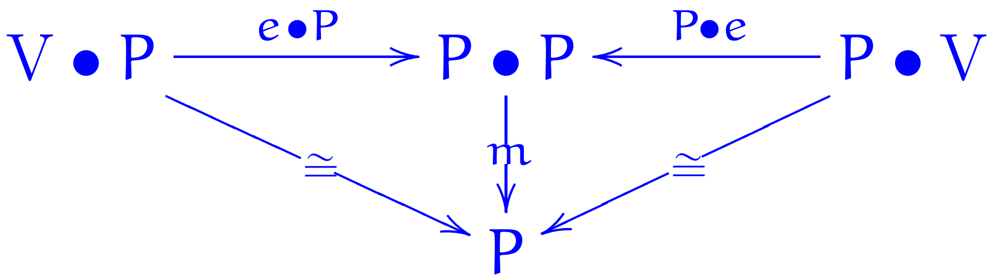
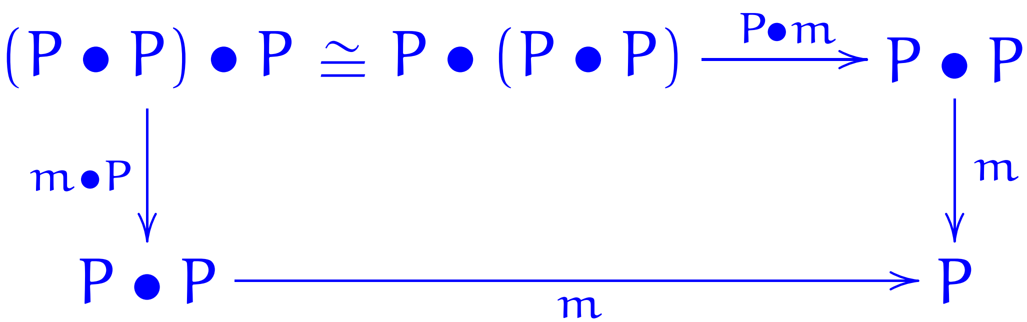
Laws:

- $$(P \bullet Q) \bullet R \quad \cong \quad P \bullet (Q \bullet R)$$
  $$\big(p\langle x_i \mapsto q_i\rangle_i\big)\langle y_j \mapsto r_j\rangle_j \;\longmapsto\; p\big\langle x_i \mapsto q_i\langle y_j \mapsto r_j\rangle_j\big\rangle_i$$
  $$\big(p\langle x_i \mapsto q_i\rangle_i\big)\langle y_j^{(i)} \mapsto r_j^{(i)}\rangle_{i,j} \;\longleftarrow\; p\big\langle x_i \mapsto q_i\langle y_j^{(i)} \mapsto r_j^{(i)}\rangle_{i,j}\big\rangle_i$$
- $$V \bullet P \;\cong\; P \qquad\qquad P \;\cong\; P \bullet V$$
  $$x_j\langle x_i \mapsto p_i\rangle \;\longmapsto\; p_j \qquad\qquad p \;\longmapsto\; p\langle x_i \mapsto x_i\rangle$$
  $$x\langle x \mapsto p\rangle \;\longleftarrow\; p \qquad\qquad p[x_i \mapsto y_{\rho i}] \;\longleftarrow\; p\langle x_i \mapsto y_{\rho i}\rangle$$
- The substitution tensor product is closed.

- Substitution structure = monoid structure with respect to $(V, \bullet)$:



$$x_j[{}^{p_i}/_{x_i}] = p_j \qquad\qquad p[{}^{x_i}/_{x_i}] = p$$



$$\left(p[{}^{q_i}/_{x_i}]\right)[{}^{r_j}/_{y_j}] \;=\; p[{}^{q_i[{}^{r_j}/_{y_j}]}/_{x_i}]$$

Finite products:

- $\mathbf{1}$, with $\mathbf{1}(C) = 1$
- $P \times Q$, with $(P \times Q)(C) = P(C) \times Q(C)$

Example:

- $V^n(-) \cong \mathbb{F}[\{x_1, \ldots, x_n\}, -]$

Finite products:

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Finite sums:

- $\mathbf{0}$, with $\mathbf{0}(C) = \emptyset$
- $P + Q$, with $(P + Q)(C) = P(C) \uplus Q(C)$

Exponentials:

P^Q, with

$$P^Q(C)$$

$$= \left\{ f \in \prod_{\rho : C \to D \text{ in } \mathbb{F}} [QD, PD] \;\middle|\; \begin{array}{c} \begin{array}{ccc} QD & \xrightarrow{f_\rho} & PD \\ {\scriptstyle -[=]}\big\downarrow & & \big\downarrow{\scriptstyle -[=]} \\ QE & \xrightarrow[f_{\tau\rho}]{} & PE \end{array} \\ \text{for all } C \xrightarrow{\rho} D \xrightarrow{\tau} E \end{array} \right\}$$

Example: P^V

$$P^V(C) = \left\{ f \in \prod_{\rho: C \to D \text{ in } \mathbb{F}} [D, PD] \;\middle|\; \begin{array}{c} \begin{array}{ccc} D & \xrightarrow{f_\rho} & PD \\ {\scriptstyle \rho}\downarrow & & \downarrow{\scriptstyle -[=]} \\ E & \xrightarrow[f_{\tau\rho}]{} & PE \end{array} \\ \text{for all } C \xrightarrow{\rho} D \xrightarrow{\tau} E \end{array} \right\}$$

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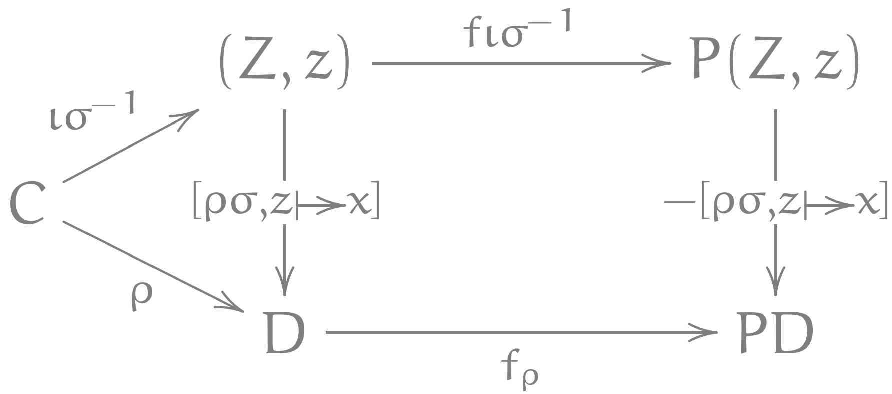
For $\rho : C \longrightarrow D$ in $\mathbb{F}$, $x \in D$, $\sigma : Z \cong C$, $z \notin Z$ we have



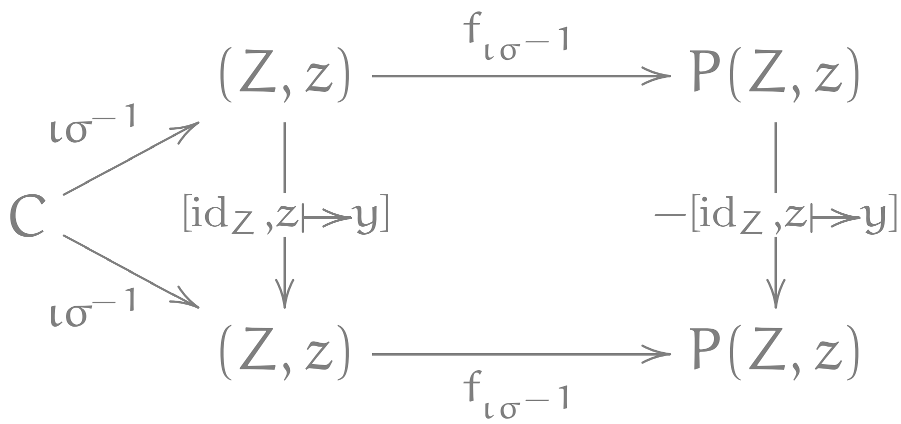
and hence that $f_\rho(x) = \big(f_{\iota\sigma^{-1}}(z)\big)[\rho\sigma, z \mapsto x]$.

Moreover, for all $y \in Z$, we have

$$
\begin{array}{ccccc}
 & & (Z, z) & \xrightarrow{f_{\iota\sigma^{-1}}} & P(Z, z) \\
 & \overset{\iota\sigma^{-1}}{\nearrow} & \Big\downarrow {\scriptstyle [\mathrm{id}_Z, z \mapsto y]} & & \Big\downarrow {\scriptstyle -[\mathrm{id}_Z, z \mapsto y]} \\
C & & & & \\
 & \underset{\iota\sigma^{-1}}{\searrow} & & & \\
 & & (Z, z) & \xrightarrow[f_{\iota\sigma^{-1}}]{} & P(Z, z)
\end{array}
$$

and hence that $f_{\iota\sigma^{-1}}(y) = \big(f_{\iota\sigma^{-1}}(z)\big)[\mathrm{id}_Z, z \mapsto x]$.

Moreover, for all $y \in Z$, we have



and hence that $f_{\iota\sigma^{-1}}(y) = \big(f_{\iota\sigma^{-1}}(z)\big)[\mathrm{id}_Z, z \mapsto x]$.

Thus, f is completely determined by $f_{\iota\sigma^{-1}}(z) \in P(Z, z)$.

$$
\begin{aligned}
&P^V(C) \\
&\quad\cong \left\{ p \in \prod_{\sigma : Z \cong C} \prod_{z \notin Z} P(Z, z) \;\middle|\; p_{\sigma_1, z_1}[\sigma_2^{-1}\sigma_1, z_1 \mapsto z_2] = p_{\sigma_2, z_2} \right\} \\
&\quad\cong \left\{ (Z, \sigma, z, p) \;\middle|\; \sigma : Z \cong C,\ z \notin Z,\ (Z, z \vdash p : P) \right\}_{/\sim}
\end{aligned}
$$

$$
\begin{aligned}
& P^{V}(C) \\
\cong \quad & \left\{ \, p \in \textstyle\prod_{\sigma : Z \cong C} \prod_{z \notin Z} P(Z, z) \;\middle|\; p_{\sigma_1, z_1}[\sigma_2^{-1}\sigma_1, z_1 \mapsto z_2] = p_{\sigma_2, z_2} \, \right\} \\
\cong \quad & \left\{ \, p \in \textstyle\prod_{z \notin C} P(C, z) \;\middle|\; p_x[\mathrm{id}_C, x \mapsto y] = p_y \, \right\} \\
\cong \quad & \left\{ \, (x)p \;\middle|\; x \notin C,\ (C, x \vdash p : P) \, \right\} / \sim
\end{aligned}
$$

$$
\begin{aligned}
& P^{V}(C) \\
& \cong \left\{ p \in \prod_{\sigma : Z \cong C} \prod_{z \notin Z} P(Z, z) \;\middle|\; p_{\sigma_1, z_1}[\sigma_2^{-1}\sigma_1, z_1 \mapsto z_2] = p_{\sigma_2, z_2} \right\} \\
& \cong \left\{ p \in \prod_{z \notin C} P(C, z) \;\middle|\; p_x[\mathrm{id}_C, x \mapsto y] = p_y \right\} \\
& \cong \left\{ (x)p \;\middle|\; x \notin C,\ (C, x \vdash p : P) \right\} / \sim \\
& \cong \left\{ p \;\middle|\; C, \nu_C \vdash p : P \right\}
\end{aligned}
$$

$$
\begin{aligned}
& P^V(C) \\
\cong\ & \left\{\, p \in \prod_{\sigma : Z \cong C} \prod_{z \notin Z} P(Z, z) \;\middle|\; p_{\sigma_1, z_1}[\sigma_2^{-1}\sigma_1, z_1 \mapsto z_2] = p_{\sigma_2, z_2} \,\right\} \\
\cong\ & \left\{\, p \in \prod_{z \notin C} P(C, z) \;\middle|\; p_x[\mathrm{id}_C, x \mapsto y] = p_y \,\right\} \\
\cong\ & \left\{\, (x)p \;\middle|\; x \notin C,\ (C, x \vdash p : P) \,\right\} /_{\sim} \\
\cong\ & \left\{\, p \;\middle|\; C, \nu_C \vdash p : P \,\right\} \\
\cong\ & P(C + 1)
\end{aligned}
$$

The arity of variable binding

Operator	Arity	Operation
app	$2 = (0, 0)$	$P^2 \longrightarrow P$
lam	(1)	$P^V \longrightarrow P$
let	$(0, 1)$	$P \times P^V \longrightarrow P$

$$f : P^V \longrightarrow P \quad \leftrightsquigarrow \quad \frac{C, x \vdash p : P}{C \vdash f\big((x)p\big) : P}$$

λ-calculus syntax

The abstract syntax of the λ-calculus (up to α-equivalence) Λ is algebraically described as the free Σ_λ-algebra on V for Σ_λ the binding signature $\{\mathsf{app} : (0, 0), \mathsf{lam} : (1)\}$ interpreted in $\mathbf{Set}^{\mathbb{F}}$.

$$\left\{ \begin{array}{ll} \mathsf{var} : V \longrightarrow \Lambda & \dfrac{}{\mathsf{var}(x) : \Lambda} \\[2ex] \mathsf{app} : \Lambda^2 \longrightarrow \Lambda & \dfrac{C \vdash t_1 : \Lambda \qquad C \vdash t_2 : \Lambda}{C \vdash \mathsf{app}(t_1, t_2) : \Lambda} \\[2ex] \mathsf{lam} : \Lambda^V \longrightarrow \Lambda & \dfrac{C, x \vdash t : \Lambda}{C \vdash \mathsf{lam}\big((x)t\big) : \Lambda} \end{array} \right.$$

$$\Lambda = \bigcup_{n\in\mathbb{N}} S_\lambda{}^n 0 \quad \text{for} \quad S_\lambda = V + (-)^2 + (-)^V$$

	0	V	$S_\lambda(V)$	$S_\lambda{}^2(V)$	$\cdots$
$\{\}$					
$\{x\}$					
$\{x, y\}$					
$\{x, y, z\}$					
$\vdots$					

$\Lambda = \bigcup_{n \in \mathbb{N}} S_\lambda{}^n 0$ for $S_\lambda = V + (-)^2 + (-)^V$

	0	V	$S_\lambda(V)$	$S_\lambda{}^2(V)$	$\ldots$
$\{\}$					
$\{x\}$		var(x)			
$\{x, y\}$		var(x) var(y)			
$\{x, y, z\}$		var(z) $\ldots$			
$\vdots$					

$$\Lambda = \bigcup_{n \in \mathbb{N}} S_\lambda{}^n 0 \text{ for } S_\lambda = V + (-)^2 + (-)^V$$

	0	V	$S_\lambda(V)$	$S_\lambda{}^2(V)$	$\cdots$
$\{\}$					
$\{x\}$		var(x)	var(x)		
$\{x, y\}$		var(x) var(y)	var(x), var(y)		
$\{x, y, z\}$		var(z) $\cdots$	var(z) $\cdots$		
$\vdots$					

$$\Lambda = \bigcup_{n \in \mathbb{N}} {S_\lambda}^n \emptyset \text{ for } S_\lambda = V + (-)^2 + (-)^V$$

	$\emptyset$	V	$S_\lambda(V)$	${S_\lambda}^2(V)$	$\cdots$
$\{\}$					
$\{x\}$		var(x)	var(x) app(var(x), var(x))		
$\{x, y\}$		var(x) var(y)	var(x), var(y) app(var(x), var(y)) $\cdots$		
$\{x, y, z\}$		var(z) $\cdots$	var(z) $\cdots$		
$\vdots$					

$$\Lambda = \bigcup_{n\in\mathbb{N}} S_\lambda{}^n 0 \text{ for } S_\lambda = V + (-)^2 + (-)^V$$

	0	V	$S_\lambda(V)$	$S_\lambda{}^2(V)$	$\cdots$
$\{\}$			$\mathsf{lam}((x)\mathsf{var}(x))$		
$\{x\}$		$\mathsf{var}(x)$	$\mathsf{var}(x)$ $\mathsf{app}(\mathsf{var}(x), \mathsf{var}(x))$ $\cdots$ $\mathsf{lam}((y)\mathsf{var}(x))$		
$\{x, y\}$		$\mathsf{var}(x)$ $\mathsf{var}(y)$	$\mathsf{var}(x)$, $\mathsf{var}(y)$ $\mathsf{app}(\mathsf{var}(x), \mathsf{var}(y))$ $\cdots$ $\mathsf{lam}((z)\mathsf{var}(z))$		
$\{x, y, z\}$		$\mathsf{var}(z)$ $\cdots$	$\mathsf{var}(z)$ $\cdots$		
$\vdots$					

$$\Lambda = \bigcup_{n \in \mathbb{N}} S_\lambda{}^n \emptyset \text{ for } S_\lambda = V + (-)^2 + (-)^V$$

	$\emptyset$	V	$S_\lambda(V)$	$S_\lambda{}^2(V)$	$\cdots$
$\{\}$			lam$((x)$var$(x))$	$\cdots$ lam$((x)$lam$((y)$var$(x)))$	
$\{x\}$		var(x)	var(x) app(var(x), var$(x))$ $\cdots$ lam$((y)$var$(x))$	$\cdots$ lam$((y)$app(var(x), var$(y)))$ lam$((y)$lam$((z)$var$(z)))$	
$\{x, y\}$		var(x) var(y)	var(x), var(y) app(var(x), var$(y))$ $\cdots$ lam$((z)$var$(z))$	$\cdots$	
$\{x, y, z\}$		var(z) $\cdots$	var(z) $\cdots$	$\cdots$	
$\vdots$					

Internal structural induction principle

$$
\begin{aligned}
&\Vdash \forall P \subseteq \Lambda. \\
&\qquad \big[\ (\forall v \in V.\, \mathsf{var}(v) \in P) \\
&\qquad\quad \wedge\ (\forall t_1, t_2 \in \Lambda.\, t_1, t_2 \in P \Longrightarrow \mathsf{app}(t_1, t_2) \in P) \\
&\qquad\quad \wedge\ (\forall f \in \Lambda^V.\, (\forall v \in V.\, f(v) \in P) \Longrightarrow \mathsf{lam}(f) \in P)\ \big] \\
&\qquad \Longrightarrow\ \Lambda \subseteq P
\end{aligned}
$$

External structural induction principle

For $\{ P(C) \subseteq \Lambda(C) \}_{C \in \mathbb{F}}$ such that

$$\forall t \in \Lambda(C). \forall \rho : C \longrightarrow D \text{ in } \mathbb{F}. t \in P(C) \Longrightarrow t[\rho] \in P(D)$$

if

$$\begin{aligned}
&\forall C \in \mathbb{F}. \\
&\quad \big(\forall x \in C. \mathsf{var}(x) \in P(C)\big) \\
&\wedge \\
&\quad \big(\forall t_1, t_2 \in \Lambda(C). t_1, t_2 \in P(C) \Longrightarrow \mathsf{app}(t_1, t_2) \in P(C)\big) \\
&\wedge \\
&\quad \big(\forall x \in \mathcal{V} \setminus C, t \in \Lambda(C, x). t \in P(C, x) \Longrightarrow \mathsf{lam}\big((x)t\big) \in P(C)\big)
\end{aligned}$$

then

$$\forall C \in \mathbb{F}. \forall t \in \Lambda(C). t \in P(C)$$

- Rule induction on the derivation of terms

External structural induction principle

For $\{ P(C) \subseteq \Lambda(C) \}_{C \in \mathbb{F}}$ such that

$$\forall t \in \Lambda(C). \forall \rho : C \longrightarrow D \text{ in } \mathbb{F}. t \in P(C) \Longrightarrow t[\rho] \in P(D)$$

if

$$\begin{array}{l} \forall C \in \mathbb{F}. \\ \quad \big(\forall x \in C. \mathsf{var}(x) \in P(C)\big) \\ \wedge \\ \quad \big(\forall t_1, t_2 \in \Lambda(C). t_1, t_2 \in P(C) \Longrightarrow \mathsf{app}(t_1, t_2) \in P(C)\big) \\ \wedge \\ \quad \big(\forall x \in \mathcal{V} \setminus C, t \in \Lambda(C, x). t \in P(C, x) \Longrightarrow \mathsf{lam}\big((x)t\big) \in P(C)\big) \\ \quad \big(\forall t \in \Lambda(C+1). t \in P(C+1) \Longrightarrow \mathsf{lam}\big(t\big) \in P(C)\big) \end{array}$$

then

$$\forall C \in \mathbb{F}. \forall t \in \Lambda(C). t \in P(C)$$

- Rule induction on the derivation of terms

Substitution tensorial strength

- $P \bullet (-)$:

$$\big(P \bullet (Q)\big) \bullet R \xrightarrow{\cong} P \bullet (Q \bullet R)$$

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- $(-) + (=)$:

$$\begin{array}{rcl} (P + Q) \bullet R & \longrightarrow & (P \bullet R) + (Q \bullet R) \\ \imath(p)\langle x_i \mapsto r_i \rangle & \longmapsto & \imath\big(p\langle x_i \mapsto r_i \rangle\big) \\ \jmath(q)\langle x_i \mapsto r_i \rangle & \longmapsto & \jmath\big(q\langle x_i \mapsto r_i \rangle\big) \end{array}$$

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- $(-) \times (=)$:

$$\begin{array}{rcl} (P \times Q) \bullet R & \longrightarrow & (P \bullet R) \times (Q \bullet R) \\ (p, q)\langle x_i \mapsto r_i \rangle & \longmapsto & \big(p\langle x_i \mapsto r_i \rangle, q\langle x_i \mapsto r_i \rangle\big) \end{array}$$

- $(-)^V$:

  Every $v : V \longrightarrow R$, induces

$$(P^V) \bullet R \quad \longrightarrow \quad (P \bullet R)^V$$

$$C \vdash \quad \big((x)p\big)\langle x_i \mapsto r_i \rangle \quad \longmapsto \quad (y)\big(p\langle x_i \mapsto r_i[\iota], x \longmapsto r\rangle\big)$$

for $\iota : C \hookrightarrow (C, y)$

and $r = v_{(C,y)}(y)$

- $(-)^V$:

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for $\iota : C \hookrightarrow (C, y)$

and $r = v_{(C,y)}(y)$

Pointed tensorial strength:

$$\ell_{X,(p:I \to P)} : S(X) \otimes P \longrightarrow S(X \otimes P)$$

Algebras with substitution

For an endofunctor S with a pointed tensorial strength ℓ on a monoidal category $(I, \otimes)$, define $S\text{-}\mathbf{Mon}$ as the category with objects X equipped with an S-algebra structure $\sigma : SX \longrightarrow X$ and a monoid structure $X \otimes X \xrightarrow{m} X \xleftarrow{e} I$ that are compatible in the sense that

$$
\begin{array}{ccccc}
S(X) \otimes X & \xrightarrow{\ell_{X,e}} & S(X \otimes X) & \xrightarrow{Sm} & SX \\
\Big\downarrow{\scriptstyle \sigma \otimes X} & & & & \Big\downarrow{\scriptstyle \sigma} \\
X \otimes X & & \xrightarrow[m]{\hspace{6em}} & & X
\end{array}
$$

Morphisms are both S-algebra and monoid homomorphisms.

Example: For $\mathcal{D}^{\mathcal{D}} \triangleleft \mathcal{D}$, the canonical monoid structure on $\langle \mathcal{D}, \mathcal{D} \rangle$ with respect to the substitution tensor product has a Σ_λ-**Mon** algebra structure as follows:

- Application.

$$\langle \mathcal{D}, \mathcal{D} \rangle \times \langle \mathcal{D}, \mathcal{D} \rangle \longrightarrow \langle \mathcal{D}, \mathcal{D}^{\mathcal{D}} \rangle \times \langle \mathcal{D}, \mathcal{D} \rangle \cong \langle \mathcal{D}, \mathcal{D}^{\mathcal{D}} \times \mathcal{D} \rangle \longrightarrow \langle \mathcal{D}, \mathcal{D} \rangle$$

- Abstraction.

$$\langle \mathcal{D}, \mathcal{D} \rangle^{\mathrm{V}} \cong \langle \mathcal{D}, \mathcal{D}^{\mathcal{D}} \rangle \longrightarrow \langle \mathcal{D}, \mathcal{D} \rangle$$

Initial algebra semantics with substitution

If the tensor product is closed, then the free S-algebra on I is an initial S-$\mathbf{Mon}$ algebra.

Initial algebra semantics with substitution

If the tensor product is closed, then the free S-algebra on I is an initial S-**Mon** algebra.

- The monoid structure on $S^{\star}\mathrm{I}$ is induced by parameterised structural recursion and amounts to substitution.

Example: $\Lambda = (\Sigma_\lambda)^{\star} V$

$$C \vdash s\big(\mathsf{var}(x_j), \langle x_i \mapsto t_i \rangle\big) = t_j$$

$$C \vdash s\big(\mathsf{app}(t, t'), \langle x_i \mapsto t_i \rangle\big) = \mathsf{app}\big(s(t, \langle x_i \mapsto t_i \rangle), s(t', \langle x_i \mapsto t_i \rangle)\big)$$

$$C \vdash s\big(\mathsf{lam}((x)t), \langle x_i \mapsto t_i \rangle\big) = \mathsf{lam}\Big((y)s\big(t, \langle x_i \mapsto t_i, x \mapsto \mathsf{var}(y) \rangle\big)\Big) \text{ for } y \notin C$$

Example:

For $\mathcal{D}^{\mathcal{D}} \lhd \mathcal{D}$ there exists a unique Σ_λ-$\mathbf{Mon}$ homomorphism $\Lambda \longrightarrow \langle \mathcal{D}, \mathcal{D} \rangle$.

/

Example:

For $\mathcal{D}^{\mathcal{D}} \lhd \mathcal{D}$ there exists a unique Σ_λ-$\mathbf{Mon}$ homomorphism $\Lambda \longrightarrow \langle \mathcal{D}, \mathcal{D} \rangle$.

More generally:

> If the tensor product is closed, then the free S_X-algebra on I for $S_X = \mathrm{X} \otimes (-) + S(-)$ is a free S-$\mathbf{Mon}$ algebra on X.

Obs: The free L_X-algebra on I for $\mathrm{L}_X = \mathrm{X} \otimes (-)$ is a free monoid on X.

Second-order abstract syntax

Term metavariables as variable sets.

- A term metavariable is specified as follows:

$$n \vdash M \qquad (n \in \mathbb{N})$$

Second-order abstract syntax

Term metavariables as variable sets.

- A term metavariable is specified as follows:

$$n \vdash M \qquad (n \in \mathbb{N})$$

- A set of term metavariables X induces the variable set $\overline{X}$ as follows:

$$C \vdash M[\rho_1, \ldots, \rho_n] : \overline{X}$$

for all $(n \vdash M) \in X$ and $\rho : [n] \longrightarrow C$.

Second-order abstract syntax = Free algebras with substitution

Example: The initial $\big(V + \overline{X} \bullet (-) + \Sigma_\lambda(-)\big)$-algebra $\mathcal{M}_\lambda(\overline{X})$ is the free Σ_λ-**Mon** algebra on $\overline{X}$ and can be syntactically presented as follows:

$$\frac{}{C \vdash \mathsf{var}(x) : \mathcal{M}_\lambda(\overline{X})} \quad (x \in C)$$

$$\frac{C \vdash t_1 : \mathcal{M}_\lambda(\overline{X}) \qquad C \vdash t_2 : \mathcal{M}_\lambda(\overline{X})}{C \vdash \mathsf{app}(t_1, t_2) : \mathcal{M}_\lambda(\overline{X})} \qquad \frac{C, x \vdash t : \mathcal{M}_\lambda(\overline{X})}{C \vdash \mathsf{lam}\big((x)t\big) : \mathcal{M}_\lambda(\overline{X})}$$

$$\frac{C \vdash t_1 : \mathcal{M}_\lambda(\overline{X}) \;,\; \ldots \;,\; C \vdash t_n : \mathcal{M}_\lambda(\overline{X})}{C \vdash M[t_1, \ldots, t_n] : \mathcal{M}_\lambda(\overline{X})} \quad \big((n \vdash M) \in X\big)$$

Two sample terms:

$$\mathsf{app}\big(\mathsf{lam}\big((x)M[\mathsf{var}(x)]\big), N[]\big) \qquad\qquad M\big[N[]\big]$$

$$\frac{C \vdash t_1 : \mathcal{M}_\lambda(\overline{X}) \; , \; \ldots \; , \; C \vdash t_n : \mathcal{M}_\lambda(\overline{X})}{C \vdash M[t_1, \ldots, t_n] : \mathcal{M}_\lambda(\overline{X})} \quad \big((n \vdash M) \in X\big)$$

Two sample terms:

$$\mathsf{app}\big(\mathsf{lam}\big((x)M[\mathsf{var}(x)]\big), N[\,]\big) \qquad\qquad M\big[N[\,]\big]$$

- The monoid multiplication

$$\mathcal{M}_\lambda(\overline{X}) \bullet \mathcal{M}_\lambda(\overline{X}) \longrightarrow \mathcal{M}_\lambda(\overline{X})$$

extends the substitution of terms to term metavariables as follows:

$$s\big(M[t_1, \ldots, t_n], \langle x_i \mapsto u_i \rangle\big) = M\big[s\big(t_1, \langle x_i \mapsto u_i \rangle\big), \ldots, s\big(t_n, \langle x_i \mapsto u_i \rangle\big)\big]$$

Second-order substitution

Cartesian strength:

- $P \times (-)$:

$$\big(P \times (Q)\big) \times R \xrightarrow{\cong} P \times (Q \times R)$$

- $(-) + (=)$:

$$(P + Q) \times R \xrightarrow{\cong} (P \times R) + (Q \times R)$$

- $(-)^V$:

$$\begin{array}{rcl} (P^V) \times R & \longrightarrow & (P \times R)^V \\ (f, r) & \longmapsto & \lambda v : V.\, (f(v), r) \end{array}$$

- $P \bullet (-)$:

$$(P \bullet (Q)) \times R \longrightarrow P \bullet (Q \times R)$$

$$(p\langle x_i \mapsto q_i \rangle, r) \longmapsto p\langle x_i \mapsto (q_i, r)\rangle$$

- $P \bullet (-)$:

$$(P \bullet (Q)) \times R \longrightarrow P \bullet (Q \times R)$$
$$(p\langle x_i \mapsto q_i \rangle, r) \longmapsto p\langle x_i \mapsto (q_i, r)\rangle$$

- $(-) \bullet (=)$:

  Every $v : V \longrightarrow Q$ induces

$$(P \bullet Q) \times R \longrightarrow (P \times R) \bullet (Q \times R)$$
$$C = (z_j)_j \vdash (p\langle x_i \longmapsto q_i\rangle_i, r) \longmapsto \big(p[\imath], r[\jmath]\big)\langle x_i \mapsto (q_i, r), z_j \mapsto (v_C(z_j), r)\rangle_{i,j}$$

where $(x_i)_i \xhookrightarrow{\imath} (x_i, z_j)_{i,j} \xhookleftarrow{\jmath} (z_j)_j$

Consider a cartesian closed and monoidal category with a strength

$$(X \otimes Y) \times Z \xrightarrow{\ell_{X,(y:I \to Y),Z}} (X \times Z) \otimes (Y \times Z)$$

and an endofunctor S on it equipped with a cartesian strength.

Let $\mathcal{M}X$ be an initial $\big(I + X \otimes (-) + S(-)\big)$-algebra.

Then, $\mathcal{M}$ admits a renaming structure given as follows:

$$\big(I + X \otimes \mathcal{M}(X) + S(\mathcal{M}X)\big) \times Y^X \longrightarrow I + (X \times Y^X) \otimes (\mathcal{M}(X) \times Y^X) + S(\mathcal{M}(X) \times Y^X)$$

$$I + (X \times Y^X) \otimes (\mathcal{M}(X) \times Y^X) + S(\mathcal{M}(X) \times Y^X) \xrightarrow{id + (\varepsilon \otimes r) + Sr} I + Y \otimes \mathcal{M}(Y) + S(\mathcal{M}Y)$$

$$\big(I + X \otimes \mathcal{M}(X) + S(\mathcal{M}X)\big) \times Y^X \xrightarrow{[e_X, a_X, \sigma_X] \times id} \mathcal{M}(X) \times Y^X$$

$$I + Y \otimes \mathcal{M}(Y) + S(\mathcal{M}Y) \xrightarrow{[e_Y, a_Y, \sigma_Y]} \mathcal{M}Y$$

$$\mathcal{M}(X) \times Y^X \xrightarrow{r} \mathcal{M}Y$$

Furthermore, if the tensor product is closed, for S equipped with a pointed tensorial strength, $\mathcal{M}X$ is a free S-**Mon** algebra on X and $\mathcal{M}$ admits a (meta) substitution structure given as follows:

$$\begin{array}{ccc} & I + \left(X \times (\mathcal{M}Y)^X\right) \otimes \left(\mathcal{M}(X) \times (\mathcal{M}Y)^X\right) + S\left(\mathcal{M}(X) \times (\mathcal{M}Y)^X\right) & \\ \nearrow & & \searrow {\scriptstyle \mathrm{id}+(\varepsilon\otimes m)+Sm} \\ \left(I + X \otimes \mathcal{M}(X) + S(\mathcal{M}X)\right) \times (\mathcal{M}Y)^X & & I + \mathcal{M}(Y) \otimes \mathcal{M}(Y) + S(\mathcal{M}Y) \\ \downarrow {\scriptstyle [e_X, a_X, \sigma_X]\times \mathrm{id}} & & \downarrow {\scriptstyle [e_Y, s_Y, \sigma_Y]} \\ \mathcal{M}(X) \times (\mathcal{M}Y)^X & \xrightarrow[m]{} & \mathcal{M}Y \end{array}$$

Example: For

$$X = \{\, n_i \vdash M_i \,\}_i$$

consider

$$x_1, \ldots, x_n \vdash t : \mathcal{M}_\lambda(\overline{X})$$

and

$$\left\{\, x_1, \ldots, x_n, x_1^{(i)}, \ldots, x_{n_i}^{(i)} \vdash t_i : \mathcal{M}_\lambda(\overline{Y}) \,\right\}_i$$

and let

$$\Theta = \left\{\, M_i(x_1^{(i)}, \ldots, x_{n_i}^{(i)}) := t_i \,\right\}_i$$

Then, we have

$$x_1, \ldots, x_n \vdash m(t, \Theta) : \mathcal{M}_\lambda(\overline{Y})$$

given as follows:

- $\mathrm{m}(\mathsf{var}(x), \Theta) = \mathsf{var}(x)$
- $\mathrm{m}(\mathsf{app}(t_1, t_2), \Theta) = \mathsf{app}(\mathrm{m}(t_1, \Theta), \mathrm{m}(t_2, \Theta))$
- $\mathrm{m}(\mathsf{lam}((x)t), \Theta) = \mathsf{lam}((x)\mathrm{m}(t, \Theta))$
- $\mathrm{m}(M_i[u_1, \ldots, u_{n_i}], \Theta) = s(t_i, \langle\, x_j^{(i)} \mapsto \mathrm{m}(u_j, \Theta), x_k \mapsto \mathsf{var}(x_k)\,\rangle_{j,k})$

given as follows:

- $m(\mathsf{var}(x), \Theta) = \mathsf{var}(x)$
- $m(\mathsf{app}(t_1, t_2), \Theta) = \mathsf{app}(m(t_1, \Theta), m(t_2, \Theta))$
- $m(\mathsf{lam}((x)t), \Theta) = \mathsf{lam}((x)m(t, \Theta))$
- $m(M_i[u_1, \ldots, u_{n_i}], \Theta) = s(t_i, \langle x_j^{(i)} \mapsto m(u_j, \Theta), x_k \mapsto \mathsf{var}(x_k) \rangle_{j,k})$

For instance,

$$m(\mathsf{app}(\mathsf{lam}((x)M[\mathsf{var}(x)]), N[]), \{M(z) := t, N() := u\}) = \mathsf{app}(\mathsf{lam}((x)s(t, \langle z \mapsto \mathsf{var}(x)\rangle)), u)$$

and

$$m(M[N[]], \{M(z) := t, N() := u\}) = s(t, \langle z \mapsto u\rangle)$$

Developments and programme

- ⋆ Mathematical theory of substitution
  - · single variable and simultaneous substitution
  - · homogeneous and heterogeneous substitution
  - · specification and programs for substitution
  - · cartesian, linear, mixed, *etc.* substitution
- ⋆ Reduction of type theory to algebra
  - · algebraic syntax and semantics
  - · admissibility of cut
  - · NBE (Normalisation By Evaluation)
  - · dependent sorts
  - · higher-order types

- ⋆ Equational and inequational theories
  - rewriting
  - modularity
- ⋆ Structural combinatorics
  - species of structures

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