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An Axiomatics and a Combinatorial Model of Creation/Annihilation Operators and Differential Structure

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August 2007

SPIRIT OF THE TALK

One should allow oneself to be led in the direction which the mathematics suggests... one must follow up a mathematical idea and see what its consequences are, even though one gets led to a domain which is completely foreign to what one started with.... Mathematics can lead us in a direction we would not take if we only followed up physical ideas by themselves.

Paul Dirac

THE TALK

Axiomatic theory of quantum structures inspired by a combinatorial model

PLAN OF THE TALK

creation/annihilation operators (I)

Axiomatic theory of quantum structures inspired by a <u>combinatorial model</u> (II)

and differential structure (III)

(I) AXIOMATICS

$\mathcal{S}$ – spaces and linear maps

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S — spaces and linear maps

biproducts: $0, *$

6/5/14

(I) AXIOMATICS

S — spaces and linear maps

— biproducts: $0, *$

a symmetric monoidal structure equipped with commutative bialgebra structure

$$\begin{tikzcd} A*A \arrow[dr] & & A*A \\ & A \arrow[ur] \arrow[dr] & \\ 0 \arrow[ur] & & 0 \end{tikzcd}$$

local additive structure (by convolution):

- $$\begin{tikzcd} A \arrow[rr, "0"] \arrow[dr] & & B \\ & 0 \arrow[ur] & \end{tikzcd}$$

- $$\frac{A \xrightarrow{f,g} B}{\begin{tikzcd} A \arrow[r, "f+g"] \arrow[d] & B \\ A*A \arrow[r, "f*g"'] & B*B \arrow[u] \end{tikzcd}}$$

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symmetric monoidal structure : $I, \otimes$

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local additive structure (by convolution):

- $$\begin{tikzcd} A \arrow[rr, "\emptyset"] \arrow[dr] & & B \\ & 0 \arrow[ur] & \end{tikzcd}$$
- $$\frac{A \overset{f,g}{\rightrightarrows} B}{\begin{tikzcd} A \arrow[r, "f+g"] \arrow[d] & B \\ A*A \arrow[r, "f*g"'] & B*B \arrow[u] \end{tikzcd}}$$

— symmetric monoidal structure: $I, \otimes$

scaler multiplication:

$$\frac{I \xrightarrow{\lambda} I, \quad A \xrightarrow{f} B}{\begin{tikzcd} A \arrow[r, "\lambda\cdot f"] \arrow[d, phantom, "\cong" rotate=90] & B \arrow[d, phantom, "\cong" rotate=90] \\ I\otimes A \arrow[r, "\lambda\otimes f"'] & I\otimes B \end{tikzcd}}$$

(I) AXIOMATICS

$\mathcal{S}$ — spaces and linear maps

biproducts: $0, *$

$$\begin{tikzcd}
 & 0\otimes C \arrow[dr] \arrow[d] & \\
A\otimes C \arrow[ur] \arrow[dr] & 0 \arrow[r] & A\otimes C \\
 & &
\end{tikzcd}$$

$\otimes$ is locally bilinear — $\boxed{\text{compatible}}$

$$\begin{tikzcd}
 & (A*A)\otimes C \arrow[dr] \arrow[d] & \\
A\otimes C \arrow[ur] \arrow[r] & (A\otimes C)*(A\otimes C) \arrow[r] & A\otimes C
\end{tikzcd}$$

symmetric monoidal structure: $I, \otimes$

$$\emptyset\otimes f = \emptyset \qquad f\otimes\emptyset = \emptyset$$

$$g\otimes(f+f') = (g\otimes f) + (g\otimes f')$$

$$(g+g')\otimes f = (g\otimes f) + (g'\otimes f)$$

(I) AXIOMATICS

spaces and linear maps

Bosonic Fock space $\rightsquigarrow\ !\ \circlearrowleft\ S$

strong symmetric monoidal endofunctor:

$$!0 \cong I$$

$$!(A * B) \cong\ !A \otimes\ !B$$

biproducts: $0, *$

$\boxed{\text{compatible}}$

symmetric monoidal structure : $I, \otimes$

COMMUTATIVE BIALGEBRA STRUCTURE

$$\begin{tikzcd}
A*A \arrow[dr, "\nabla"] & & A*A \\
& A \arrow[ur, "\Delta"] \arrow[dr, "n"] & \\
0 \arrow[ur, "u"] & & 0
\end{tikzcd}$$

A SAMPLE LAW

$$\begin{tikzcd}
A*A \arrow[r, "\nabla"'] \arrow[d, "\Delta*\Delta"'] & A \arrow[r, "\Delta"'] & A*A \\
A*A*A*A \arrow[rr, "1*\gamma*1"'] & & A*A*A*A \arrow[u, "\nabla*\nabla"']
\end{tikzcd}$$

EXPONENTIAL
COMMUTATIVE BIALGEBRA STRUCTURE

$$
\begin{tikzcd}
{!A \otimes !A \cong !(A * A)} \arrow[dr, "\nabla"] \arrow[dr, bend right, "m"'] & & {!(A * A) \cong !A \otimes !A} \\
& {!A} \arrow[ur, "\Delta"] \arrow[ur, bend right, "d"'] \arrow[dr, "n"'] \arrow[dr, bend left, "e"] & \\
{I \cong !0} \arrow[ur, "u"'] \arrow[ur, bend left, "\iota"] & & {!0 \cong I}
\end{tikzcd}
$$

A SAMPLE LAW

$$
\begin{tikzcd}
A * A \arrow[r, "\nabla"'] \arrow[d, "\Delta * \Delta"'] & A \arrow[r, "\Delta"'] & A * A \\
A * A * A * A \arrow[rr, "1 * \gamma * 1"'] & & A * A * A * A \arrow[u, "\nabla * \nabla"']
\end{tikzcd}
$$

12/11/10

EXPONENTIAL COMMUTATIVE BIALGEBRA STRUCTURE

$$
\begin{tikzcd}
!A \otimes !A \cong !(A * A) \arrow[rd, "\nabla"] \arrow[rd, bend right, "m"'] & & !(A * A) \cong !A \otimes !A \\
& !A \arrow[ru, "\Delta"] \arrow[ru, bend right, "d"'] \arrow[rd, "n"] \arrow[rd, bend left, "e"] & \\
I \cong !0 \arrow[ru, "u"'] \arrow[ru, bend left, "i"] & & !0 \cong I
\end{tikzcd}
$$

A SAMPLE LAW

$$
\begin{tikzcd}
!A \otimes !A \arrow[r, phantom, "\cong"] \arrow[d, "!\Delta \otimes !\Delta"] \arrow[dd, bend right=60, "d \otimes d"'] \arrow[rr, bend left, "m"] & !(A * A) \arrow[r, "!\nabla"'] \arrow[d, "!(\Delta * \Delta)"] & !A \arrow[r, "!\Delta"'] \arrow[rr, bend left, "d"] & !(A * A) \arrow[r, phantom, "\cong"] & !A \otimes !A \\
!(A * A) \otimes !(A * A) \arrow[r, phantom, "\cong"] \arrow[d, phantom, "\cong" rotate=90] & !(A * A * A * A) \arrow[rr, "!(1 * \gamma * 1)"'] & & !(A * A * A * A) \arrow[u, "!(\nabla * \nabla)"'] \arrow[r, phantom, "\cong"] & !(A * A) \otimes !(A * A) \arrow[u, "!\nabla \otimes !\nabla"'] \arrow[d, phantom, "\cong" rotate=90] \\
!A \otimes !A \otimes !A \otimes !A \arrow[rrrr, "1 \otimes \gamma \otimes 1"'] & & & & !A \otimes !A \otimes !A \otimes !A \arrow[uu, bend right=60, "m \otimes m"']
\end{tikzcd}
$$

CREATION/ANNIHILATION OPERATORS

For natural maps

$$A \xrightarrow{\eta} !A \xrightarrow{\epsilon} A$$

define their associated

creation (raising) / annihilation (lowering) operators

as follows:

$$\begin{tikzcd} A\otimes !A \arrow[rr, "\bar{\eta}"] \arrow[dr, "\eta\otimes 1"'] & & !A \\ & !A\otimes !A \arrow[ur, "m"'] & \end{tikzcd}$$

$$\begin{tikzcd} !A \arrow[rr, "\underline{\epsilon}"] \arrow[dr, "d"'] & & A\otimes !A \\ & !A\otimes !A \arrow[ur, "\epsilon\otimes 1"'] & \end{tikzcd}$$

COMMUTATION RELATION

The following commutation relation holds:

$$
\begin{aligned}
\underline{\epsilon}\ \bar{\eta} : A\otimes !A &\longrightarrow A \otimes !A \\
&\| \\
&\left(A\otimes !A \xrightarrow{\eta\otimes 1} !A\otimes !A \xrightarrow{\epsilon\otimes 1} A\otimes !A \right) \\
+ &\left(A\otimes !A \xrightarrow{1\otimes \underline{\epsilon}} A\otimes A\otimes !A \underset{\gamma\otimes 1}{\cong} A\otimes A\otimes !A \xrightarrow{1\otimes \bar{\eta}} A\otimes !A \right)
\end{aligned}
$$

COMMUTATION RELATION

The following commutation relation holds:

$$\underline{\epsilon}\ \bar{\eta} : A \otimes !A \longrightarrow A \otimes !A$$

$$\|$$

$$\left(A \otimes !A \xrightarrow{\eta \otimes 1} !A \otimes !A \xrightarrow{\epsilon \otimes 1} A \otimes !A \right)$$

$$+$$

$$\left(A \otimes !A \xrightarrow{1 \otimes \underline{\epsilon}} A \otimes A \otimes !A \underset{\gamma \otimes 1}{\cong} A \otimes A \otimes !A \xrightarrow{1 \otimes \bar{\eta}} A \otimes !A \right)$$

Hence, for all $v : I \to A$ and $w : A \to I$,

$$(\underline{\epsilon})w\ (\bar{\eta})v$$

$$= (w\,\epsilon\,\eta\,v) \cdot 1 \ + (\bar{\eta})v\,(\underline{\epsilon})w$$

where $(\bar{\eta})v = \left(!A \cong I \otimes !A \xrightarrow{v \otimes !} A \otimes !A \xrightarrow{\bar{\eta}} !A \right)$

$(\underline{\epsilon})w = \left(!A \xrightarrow{\underline{\epsilon}} A \otimes !A \xrightarrow{w \otimes !} I \otimes !A \cong !A \right)$

<u>Obs</u>: In models, η / ϵ arise:

- by linear adjointness

$$\eta_A = (\epsilon_A)^{\dagger} \qquad\qquad [\text{Vicary}]$$

- by duality

$$\eta_A = (\epsilon_{A^*})^* \qquad \text{where } !(A^*) \cong (!A)^*$$

- by categorical adjointness

$$\eta_A \dashv \epsilon_A$$

In all cases,

$$\epsilon \eta = 1$$

and hence

$$(\underline{\epsilon})_w \, (\bar{\eta})_v = (wv) \cdot 1 + (\bar{\eta})_v \, (\underline{\epsilon})_w$$

A combinatorial model

Objects: [small] categories (of base vectors, dimensions, coordinates, ...)

Maps: profunctors (bimodules, distributors, ...)
$\quad\quad\ \ \|$
set-valued matrices, relations, ...

[Set-enriched cat. theory]

$$P : \mathbb{A} \nrightarrow \mathbb{B} \quad \equiv \quad P : \mathbb{A} \times \mathbb{B}^{o} \to \underline{Set}$$

$P(a,b)$ = set of witnesses, proofs, ... of the P-relationship between a and b.

Notation: $a \overset{p}{\nleftarrow} b \quad \equiv \quad p \in P(a,b)$

Then:
$$\frac{a' \xleftarrow{f} a \overset{p}{\nleftarrow} b \xleftarrow{g} b'}{a' \overset{f \cdot p \cdot g}{\nleftarrow} b'}$$

$f \cdot p \cdot g = P(f,g)(p)$

Profunctor composition

$$\frac{\mathbb{A} \overset{P}{\nrightarrow} \mathbb{B} \overset{Q}{\nrightarrow} \mathbb{C}}{\mathbb{A} \overset{Q\cdot P}{\nrightarrow} \mathbb{C}}$$

$(Q\cdot P)(a,c)$ = paths $a \overset{p}{\nleftarrow} b \overset{q}{\nleftarrow} c$ up to homotopy, deformation, ...

$$\begin{tikzcd} & b \arrow[dl, "p"'] & \\ a \arrow[rr, phantom, "2"] & & c \arrow[ul, "q"'] \arrow[dl, "q'"] \\ & b' \arrow[ul, "p'"] & \end{tikzcd}
\quad \text{iff} \quad
\begin{tikzcd} & b \arrow[dl, "p"'] & \\ a & & c \arrow[ul, "q"'] \arrow[dl, "q'"] \\ & b' \arrow[ul, "p'"] \arrow[uu, dotted, "\exists"'] & \end{tikzcd}$$

$$(Q\cdot P)(a,c) = \sum_{b\in\mathbb{B}} P(a,b)\,.\,Q(b,c) \Big/ \approx$$

$$= \int^{b\in\mathbb{B}} P(a,b)\,.\,Q(b,c)$$

Profunctors as linear maps

spaces: presheaf categories $\widehat{\mathbb{X}} = \underline{\underline{\mathrm{Set}}}^{\mathbb{X}^{op}}$

- Basis embedding:

$$\begin{aligned} \mathbb{X} &\hookrightarrow \widehat{\mathbb{X}} \\ x &\longmapsto \vec{x} \quad \text{where } (\vec{x})_{x'} = \mathbb{X}(x', x) \end{aligned}$$

Yoneda

- Every presheaf is a linear combination of the basis vectors:

$$X \cong \int^{x \in \mathbb{X}} X_x . \vec{x}$$

Profunctors as linear maps

spaces: presheaf categories $\hat{\mathbb{X}} = \underline{\underline{Set}}^{\mathbb{X}^{op}}$

- Basis embedding:

$$\mathbb{X} \hookrightarrow \hat{\mathbb{X}}$$
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- Every presheaf is a linear combination of the basis vectors:

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linear maps: cocontinuous functors

$F : \hat{\mathbb{X}} \to \hat{\mathbb{Y}}$

$$F(X) = F\left(\int^x X_x . \vec{x}\right)$$
$$= \int^x X_x . F(\vec{x}) \qquad \text{LINEARITY}$$
$$= \int^x X_x . \int^y F(\vec{x})_y . \vec{y}$$
$$= \int^y \left[\int^x X_x . \underbrace{F(\vec{x})_y}_{\text{the profunctor associated to } F}\right] . \vec{y}$$

Linear structure

- $\underline{\text{Prof}}$ is a compact closed bicategory with biproducts:

$$\mathbb{A} \otimes \mathbb{B} = \mathbb{A} \times \mathbb{B}$$

$$\mathbb{A}^{*} = \mathbb{A}^{\circ}$$

$$\mathbb{A} \ast \mathbb{B} = \mathbb{A} + \mathbb{B}$$

- $\underline{\text{Prof}}_g$ admits $\dagger$-structure:

restricted to groupoids

$$\frac{\mathbb{A} \overset{P}{\longrightarrow\!\!\!\!\!\!\!+\;\;} \mathbb{B}}{\mathbb{A} \underset{P^{\dagger}}{\;\;+\!\!\!\!\!\!\!\longleftarrow} \mathbb{B}}$$

$$P^{\dagger}(b,a) = P(a,b)$$ permutation

$$f \cdot_{P^{\dagger}} p \cdot_{P^{\dagger}} g = g^{-1} \cdot_{P} x \cdot_{P} f^{-1}$$ conjugation

Fock space

free symm. mon. completion

$!\mathbb{A} =_{def}$

objects: $|!\mathbb{A}| = \Sigma_n \, |\mathbb{A}|^n$

morphisms: $!\mathbb{A}(\langle x_1 \ldots x_m \rangle, \langle y_1 \ldots y_m \rangle)$

$$= \sum_{\sigma \in \underline{Bij}(m,n)} \prod_{i=1}^{m} \mathbb{A}(x_i, y_{\sigma i})$$

$$\frac{\mathbb{A} \xrightarrow{P} \mathbb{B}}{!\mathbb{A} \xrightarrow[!P]{} !\mathbb{B}}$$

$(!P)(\langle a_1 \ldots a_m \rangle, \langle b_1 \ldots b_n \rangle)$

$$= \sum_{\sigma \in \underline{Bij}(m,n)} \prod_{i=1}^{m} P(a_i, b_{\sigma i})$$

Categorical structure

symm. promonoidal structure [Day]

I. $\underline{\mathrm{CommMon}}(\underline{\underline{\mathrm{Prof}}}) \cong \underline{\mathrm{CommCoMon}}(\underline{\underline{\mathrm{Prof}}})^{\circ}$ by duality

$$\begin{tikzcd} \underline{\mathrm{CommMon}}(\underline{\underline{\mathrm{Prof}}}) \arrow[d, shift left] \arrow[d, phantom, "\dashv"] \\ ! \circlearrowright \underline{\underline{\mathrm{Prof}}} \arrow[u, shift left] \end{tikzcd}$$

II.

$$\begin{tikzcd} \underline{\mathrm{CommCoMon}}(\underline{\underline{\mathrm{Prof}}}) \arrow[d, shift left] \arrow[d, phantom, "\vdash"] \\ ! \circlearrowright \underline{\underline{\mathrm{Prof}}} \arrow[u, shift left] \end{tikzcd}$$

$$(!A)^* \cong !(A^*)$$

III.

$$\begin{tikzcd} !!A \arrow[dr, "\mu"'] & & !!A \\ & !A \arrow[ur, "\delta"] \arrow[dr, "\epsilon"] & \\ A \arrow[ur, "\eta"] & & A \end{tikzcd}$$

Canonically induce creation/annihilation operators

Formal/combinatorial content of the comm. relation

creation: for $P : \mathbb{I} \nrightarrow \mathbb{A}$ $(\equiv P \in \widehat{\mathbb{A}})$

$$\partial_P : !\mathbb{A} \nrightarrow !\mathbb{A}$$

$$\partial_P(A, A') = \int^a P(a) \cdot [A', A \oplus \langle a \rangle]$$

annihilation: for $Q : \mathbb{A} \nrightarrow \mathbb{I}$ $(\equiv Q \in \widehat{\mathbb{A}^\circ})$

$$\alpha_Q : !\mathbb{A} \nrightarrow !\mathbb{A}$$

$$\alpha_Q(A, A') = \int^a Q(a) \cdot [A' \oplus \langle a \rangle, A]$$

Formal/combinatorial content of the comm. relation

creation: for $P : \mathbb{I} \nrightarrow \mathbb{A}$ ($\equiv P \in \widehat{\mathbb{A}}$)

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$$\partial_P(A, A') = \int^a P(a) \cdot [A', A \oplus \langle a \rangle]$$

annihilation: for $Q : \mathbb{A} \nrightarrow \mathbb{I}$ ($\equiv Q \in \widehat{\mathbb{A}^\circ}$)

$$\alpha_Q : !\mathbb{A} \nrightarrow !\mathbb{A}$$

$$\alpha_Q(A, A') = \int^a Q(a) \cdot [A' \oplus \langle a \rangle, A]$$

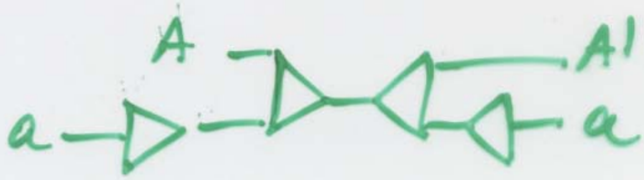
commutation relation:

$$(\alpha_Q \cdot \partial_P)(A, A')$$

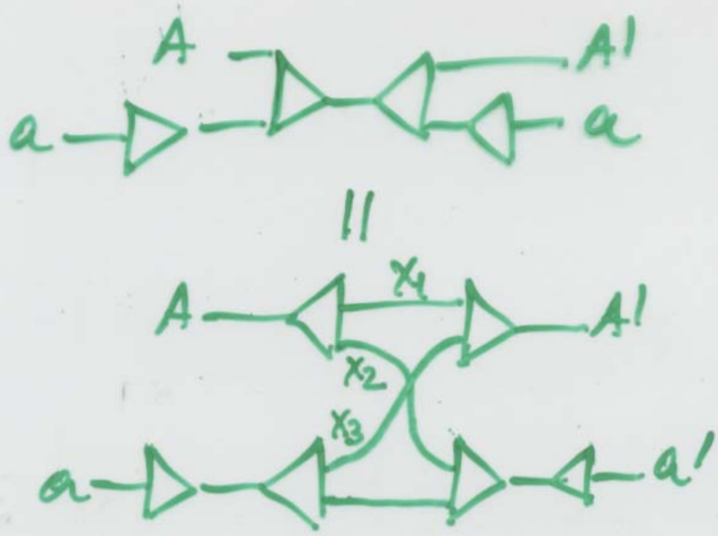
$$= \int^X \int^{a'} Q(a') \cdot [A' \oplus \langle a' \rangle, X] \int^a P(a) \cdot [X, A \oplus \langle a \rangle]$$

$$= \int^{a', a} Q(a') \cdot P(a) \cdot [A' \oplus \langle a' \rangle, A \oplus \langle a \rangle]$$

$[A' \oplus \langle a' \rangle, A \oplus \langle a \rangle]$

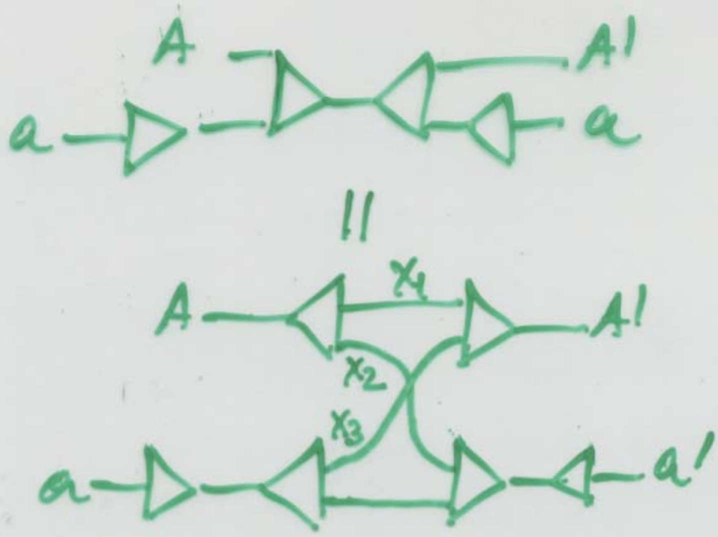


$[A' \oplus \langle a' \rangle, A \oplus \langle a \rangle]$

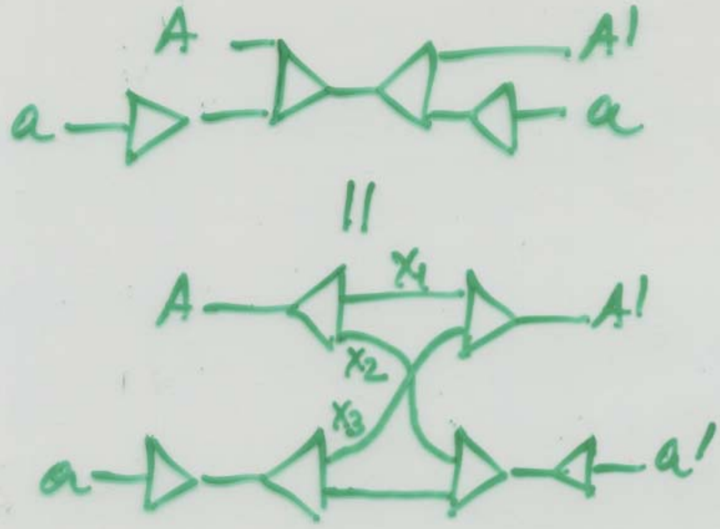


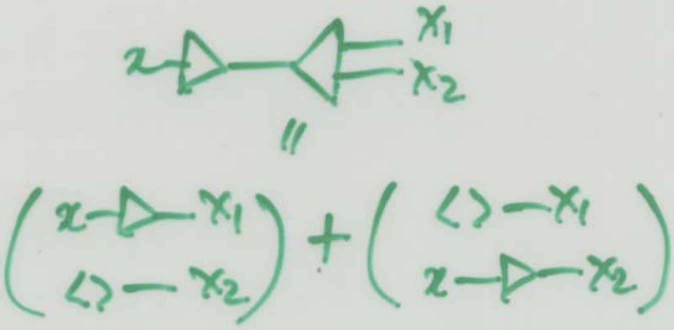
$$[A' \oplus \langle a' \rangle, A \oplus \langle a \rangle]$$

$$= \int^{x_1, x_2, x_3, x_4} [A', x_1 \oplus x_3]\,[x_1 \oplus x_2, A]\,[\langle a' \rangle, x_2 \oplus x_4]\,[x_3 \oplus x_4, \langle a \rangle]$$



$$
\begin{aligned}
&[A' \oplus \langle a' \rangle, A \oplus \langle a \rangle] \\
&= \int^{x_1, x_2, x_3, x_4} [A', x_1 \oplus x_3] \\
&\qquad [x_1 \oplus x_2, A] \\
&\qquad [\langle a' \rangle, x_2 \oplus x_4] \\
&\qquad [x_3 \oplus x_4, \langle a \rangle]
\end{aligned}
$$

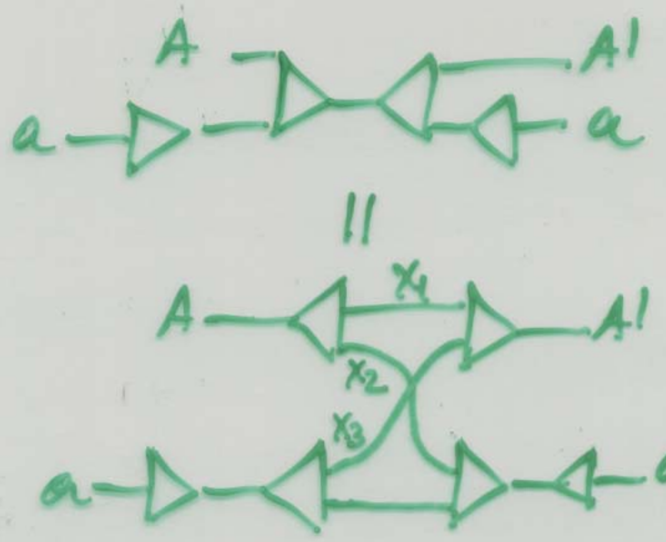


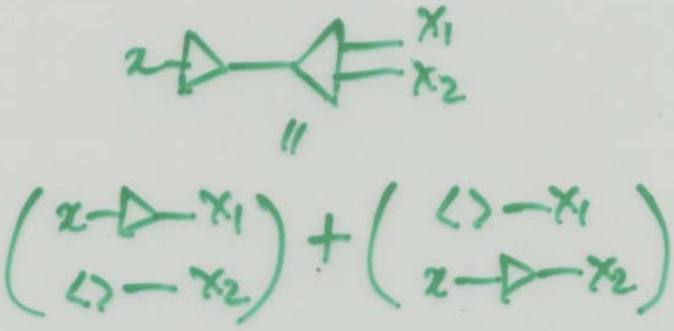


$$[A' \oplus \langle a' \rangle, A \oplus \langle a \rangle]$$

$$= \int^{x_1, x_2, x_3, x_4} [A', x_1 \oplus x_3] \; [x_1 \oplus x_2, A] \; [\langle a' \rangle, x_2 \oplus x_4] \; [x_3 \oplus x_4, \langle a \rangle]$$

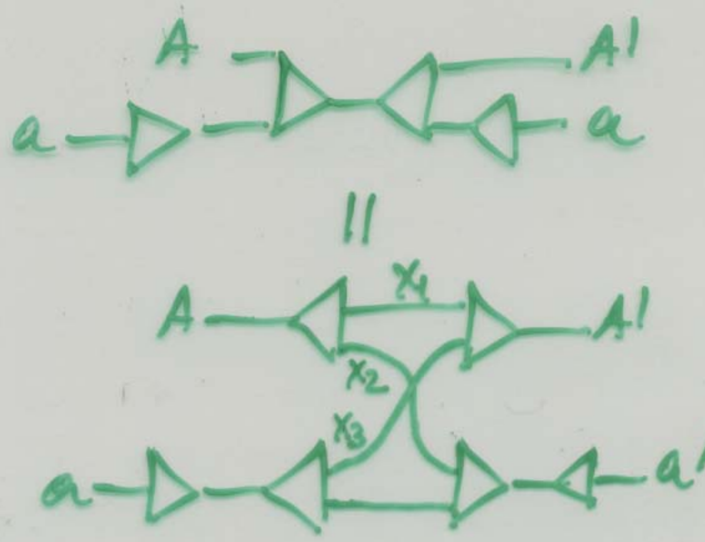
$$= \int^{x_1, x_2, x_3, x_4} [A', x_1 \oplus x_3] \cdot [x_1 \oplus x_2, A] \cdot \left(\begin{matrix} [\langle a' \rangle, x_2] \cdot [\langle\rangle, x_4] \\ + \\ [\langle\rangle, x_2] \cdot [\langle a' \rangle, x_4] \end{matrix} \right) \left(\begin{matrix} [x_3, \langle a \rangle] \cdot [x_4, \langle\rangle] \\ + \\ [x_3, \langle\rangle] \cdot [x_4, \langle a \rangle] \end{matrix} \right)$$



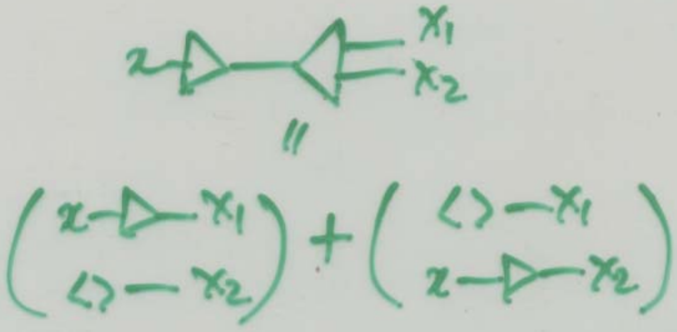


$$[A' \oplus \langle a' \rangle, A \oplus \langle a \rangle]$$

$$= \int^{x_1, x_2, x_3, x_4} [A', x_1 \oplus x_3] \; [x_1 \oplus x_2, A] \; [\langle a' \rangle, x_2 \oplus x_4] \; [x_3 \oplus x_4, \langle a \rangle]$$



$$= \int^{x_1, x_2, x_3, x_4} [A', x_1 \oplus x_3] \cdot [x_1 \oplus x_2, A] \cdot \begin{pmatrix} [\langle a' \rangle, x_2] \cdot [\langle \rangle, x_4] \\ + \\ [\langle \rangle, x_2] \cdot [\langle a' \rangle, x_4] \end{pmatrix} \begin{pmatrix} [x_3, \langle a \rangle] \cdot [x_4, \langle \rangle] \\ + \\ [x_3, \langle \rangle] \cdot [x_4, \langle a \rangle] \end{pmatrix}$$



$$= [a', a] \cdot [A', A]$$

$$+ \int^{x} [A', \langle a \rangle \oplus x] \cdot [x \oplus \langle a' \rangle, A]$$

$$(\alpha_Q \cdot \partial_P)(A,A')$$

$$= \int^{a,a'} P(a) \cdot Q(a') \cdot \left(\begin{array}{l} [a',a] \cdot [A',A] \\ + \displaystyle\int^X [A', \langle a \rangle \oplus X] \cdot [X \oplus \langle a' \rangle, A] \end{array} \right)$$

$$= \int^{a,a'} P(a) \cdot Q(a') \cdot [a',a] \cdot [A',A]$$

$$+ \int^X \underbrace{\int^a P(a)\, [A', \langle a \rangle \oplus X]}_{\partial_P(X,A')} \underbrace{\int^{a'} Q(a') \cdot [X \oplus \langle a' \rangle, A]}_{\alpha_Q(A,X)}$$

$$= \left(\int^a P(a)\, Q(a) \right) \cdot [A',A]$$

$$+ \quad (\partial_P \cdot \alpha_Q)(A,A')$$

Coherence states

Def: For $P : !\mathbb{X} \nrightarrow \mathbb{A}$,

$$
\begin{tikzcd}
!\mathbb{X} \arrow[rr, "\underline{\mathrm{Coh}}(P)", "\shortmid" marking] \arrow[dr, "\delta_{\mathbb{X}}"', "\shortmid" marking] & \text{def} & !\mathbb{A} \\
& !!\mathbb{X} \arrow[ur, "!P"', "\shortmid" marking] &
\end{tikzcd}
$$

Kleisli extension

Example: $\mathbb{X} = \mathbb{O} \Rightarrow !\mathbb{X} = \mathbb{1}$

For $P \in \hat{\mathbb{A}}$, $\underline{\mathrm{Coh}}(P) \in \widehat{!\mathbb{A}}$

$$\underline{\mathrm{Coh}}\left(\int^{a \in \mathbb{A}} P_a \cdot \vec{a} \right)$$

$$= \int^{A \in !\mathbb{A}} \left(\prod_{a \in A} Pa \right) \cdot \vec{A}$$

Exponentials

__Def__: For $P : !\mathbb{X} \nrightarrow \mathbb{A}$ and $\mathbb{A}$ with a symmetric promonoidal structure (= commutative monoid structure in __Prof__) $M = (M^{(n)} : \mathbb{A}^n \nrightarrow \mathbb{A})$

$$\begin{tikzcd}
!\mathbb{X} \arrow[rr, "\underline{\exp}_M(P)", "+"{marking}] \arrow[dr, "\underline{\mathrm{Coh}}(P)"'] & \arrow[d, phantom, "\mathrm{def}"] & \mathbb{A} \\
& !\mathbb{A} \arrow[ur, "M^{\#}"'] &
\end{tikzcd}$$

__Examples__:

I. For $P \in \widehat{\mathbb{A}}$, $\underline{\exp}_M(P) \in \widehat{\mathbb{A}}$: $(\mathbb{X} = 0)$

$$(\underline{\exp}\, P)_a = \int^n \int^{a_1 \dots a_n} \left(\Pi_i\, P a_i\right) M^{(n)}(a_1 \dots a_n; a)$$

II. For $P \in \widehat{\mathbb{A}}$, $\underline{\exp}_{\mu_{\mathbb{A}}}(\partial p) : !\mathbb{A} \nrightarrow !\mathbb{A}$: $(\mathbb{X} = \mathbb{A})$

$$\begin{aligned}
&\left(\underline{\exp}_{\mu_{\mathbb{A}}}(\partial p)\right)(A, A') \\
&= \int^n \int^{x_1 \dots x_n} \int^{a_1 \dots a_n} \Pi_i\, P(a_i) \cdot \left[\oplus_i x_i, A\right] \cdot \left[A', \oplus_i (x_i \otimes \langle a_i \rangle)\right]
\end{aligned}$$

Application: An axiomatic fact:

$$\underline{\mathrm{Coh}}(P) = \underline{\exp}_{\mu_A}(\partial_P)\cdot \iota_A$$

$$\left(\underline{\exp}_{\mu_A}(\partial_P)\right)(\langle\rangle, A')$$

$$= \int^n \int^{a_1 - a_n} \left(\textstyle\prod_i P(a_i)\right) \cdot \left[A', \textstyle\bigoplus_i \langle a_i\rangle\right]$$

$$= \int^n \int^{a_1 - a_n} \left(\textstyle\prod_i P(a_i)\right) \sum_{\sigma\in \underline{\mathrm{Bij}}(n,k)} \textstyle\prod_i \left[a'_{\sigma i}, a_i\right]$$

for $A' = \langle a'_1, \ldots, a'_k\rangle$

$$= \int^n \sum_{\sigma\in\underline{\mathrm{Bij}}(n,k)} \textstyle\prod_i P(a'_{\sigma i})$$

$$= \textstyle\prod_j P(a'_j)$$

$$= \underline{\mathrm{Coh}}(P)_{A'}$$

Differential structure [TLCA 2007]

cf [Blute Cockett Seely]

$$\begin{tikzcd} S_! \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathcal{G} \arrow[l, shift left] \arrow[loop below, "!"] \end{tikzcd}$$

spaces and smooth maps (KOKLEISLI CAT.)

cartesian closed

linear exponential comonad (MODEL OF LL)

$$\frac{!X \xrightarrow{f} A}{\begin{tikzcd} X \otimes !X \arrow[rr, "Diff(f)"] \arrow[dr, "\bar{\eta}"'] & \arrow[d, phantom, "def"] & A \\ & !X \arrow[ur, "f"'] & \end{tikzcd}}$$

- For η satisfying 3 AXIOMS, Diff satisfies all the laws of differentiation (chain rule, ...)

Axioms

$$\begin{tikzcd}
& !A \arrow[dr, "\varepsilon"] & \\
A \arrow[ur, "\eta"] \arrow[rr, "1"'] & & A
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes !B \arrow[r, "\eta \otimes 1"] \arrow[dr, "1 \otimes \varepsilon"'] & !A \otimes !B \arrow[r, "\varphi"] & !(A \otimes B) \\
& A \otimes B \arrow[ur, "\eta"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r, "\eta"] \arrow[d, phantom, "\cong" rotate=90] & !A \arrow[r, "\delta"] & !!A \\
A \otimes I \arrow[d, "\eta \otimes n"'] & & \\
!A \otimes !A \arrow[rr, "\eta \otimes \delta"'] & & !!A \otimes !!A \arrow[uu, "m"']
\end{tikzcd}$$