Adjoints and Fock space in the context of profunctors

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A bicategorical model of some quantum stuff

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- My original interest:
  Models of computational and combinatorial structures [FOSSACS 2005]

- Other interests:
  Higher-dimensional algebra
  Linear logic

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(also modules, distributors)

Profunctors

= generalised relations, metrices, ...

enriched category theory

Def: Profunctors between categories

$P : \mathbb{A} \nrightarrow \mathbb{B} \quad \equiv \quad P : \mathbb{A} \times \mathbb{B}^{o} \to \underline{\underline{Set}}$

$P(a,b)$ = set of witnesses, proofs, ... of the relationship aPb

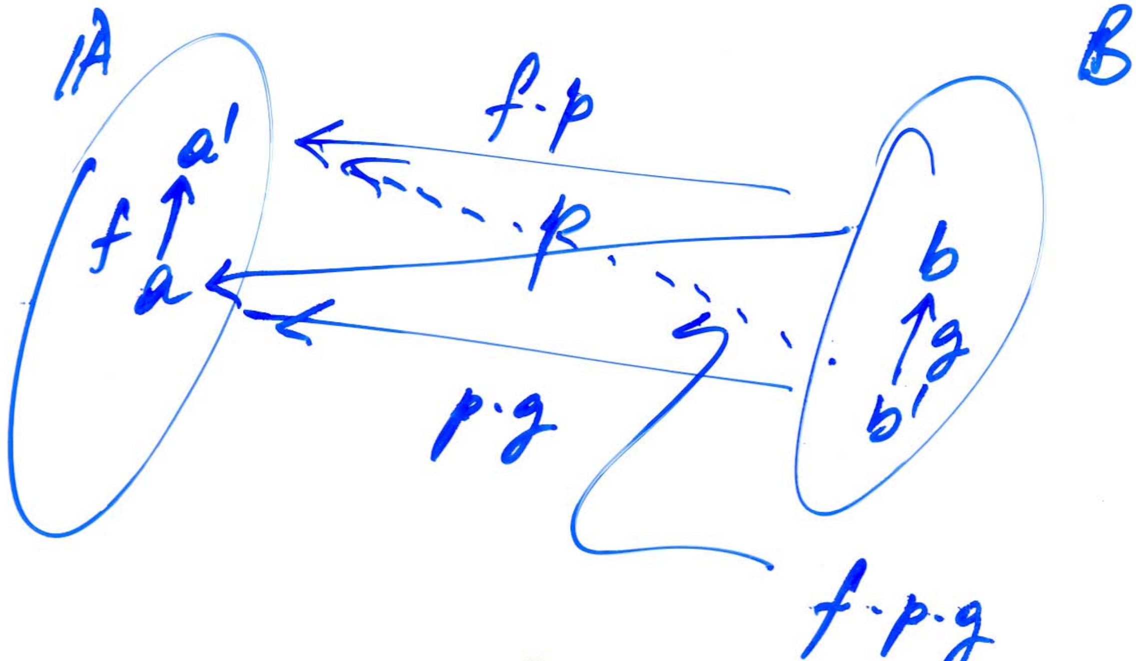
Ex: Identity profunctor

$I_{\mathbb{A}} : \mathbb{A} \nrightarrow \mathbb{A}$

$I_{\mathbb{A}}(a, a') = \mathbb{A}(a', a)$

Profunctors pictorially

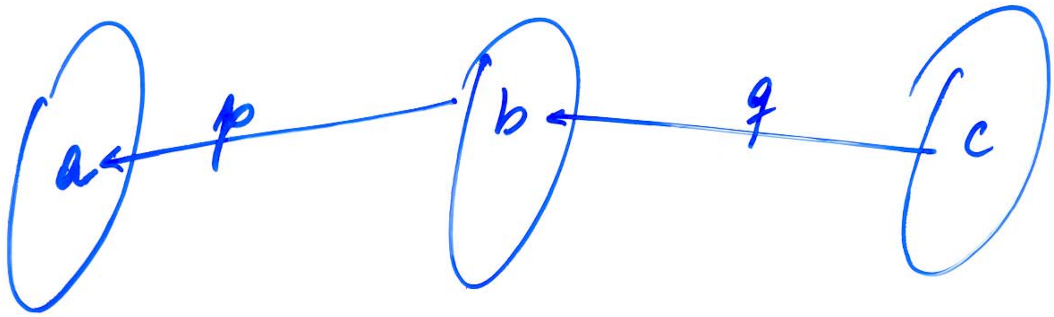
$$\mathbb{A} \overset{P}{-\!\!\!\mapstochar\!\!\!\rightarrow} \mathbb{B}$$



$p \in P(a,b)$

Profunctor composition

$$\begin{tikzcd} \mathbb{A} \arrow[r, "P", "|" marking] & \mathbb{B} \arrow[r, "Q", "|" marking] & \mathbb{C} \end{tikzcd}$$



$(Q\cdot P)(a,c) =$ profs $a \leftarrow b \leftarrow c$
up to homotopy, deformation, ...

$$\begin{tikzcd}[row sep=small] & b \arrow[dl, "p"'] & \\ a & \sim & c \arrow[ul, "q"'] \arrow[dl, "q'"] \\ & b' \arrow[ul, "p'"] & \end{tikzcd} \quad \text{iff} \quad \begin{tikzcd}[row sep=small] & b \arrow[dl, "p"'] & \\ a & & c \arrow[ul, "q"'] \arrow[dl, "q'"] \\ & b' \arrow[ul, "p'"] \arrow[uu, dotted, "\sim"] & \end{tikzcd}$$

$$(Q\cdot P)(a,c) = \sum_{b\in\mathbb{B}} P(a,b)\cdot Q(b,c) \Big/ \approx$$

$$= \int^{b\in\mathbb{B}} P(a,b)\cdot Q(b,c)$$

8/16

Ex:

$$
\begin{tikzcd}
A \arrow[r, "I_A", "\shortmid"{marking}] & A \arrow[r, "P", "\shortmid"{marking}] & B
\end{tikzcd}
$$

$$
\begin{tikzcd}
a & a' \arrow[l, "\text{in } A", "f"'] \arrow[d, dashed, "f"] & b \arrow[l, "\text{in } P", "p"'] \arrow[dl, "f \cdot p"] \\
 & a \arrow[ul, "id"] &
\end{tikzcd}
$$

$$
\begin{array}{c}
(P \cdot I_A)(a,b) \\
\cong \\
P(a,b)
\end{array}
$$

8/7/16

Ex:

$$\begin{tikzcd} \mathbb{A} \arrow[r, "I_{\mathbb{A}}", "|" marking] & \mathbb{A} \arrow[r, "P", "|" marking] & \mathbb{B} \end{tikzcd}$$

$$\begin{tikzcd} a & a' \arrow[l, "\text{in } \mathbb{A}", "f"'] \arrow[d, dashed, "f"] & b \arrow[l, "\text{in } P", "p"'] \arrow[dl, "f \cdot p"] \\ & a \arrow[ul, "id"] & \end{tikzcd}$$

$(P \cdot I_{\mathbb{A}})(a,b)$

$\cong$

$P(a,b)$

$P \cdot I_{\mathbb{A}} \cong P$

Formally:

$$(P \cdot I_{\mathbb{A}})(a,b) = \int^{a'} I_{\mathbb{A}}(a,a') \,.\, P(a',b)$$

$$= \int^{a'} \mathbb{A}(a',a) \,.\, P(a',b)$$

$$\cong P(a,b)$$

Density formula
Yoneda Lemma

Ex:

$$\begin{tikzcd} \mathbb{A} \arrow[r, "I_{\mathbb{A}}", "|" marking] & \mathbb{A} \arrow[r, "P", "|" marking] & \mathbb{B} \end{tikzcd}$$

$$\begin{tikzcd} a & a' \arrow[l, "\text{in } \mathbb{A}"', "f"] \arrow[d, dashed, "f"] & b \arrow[l, "\text{in } P"', "p"] \arrow[dl, "f \cdot p"] \\ & a \arrow[ul, "id"] & \end{tikzcd}$$

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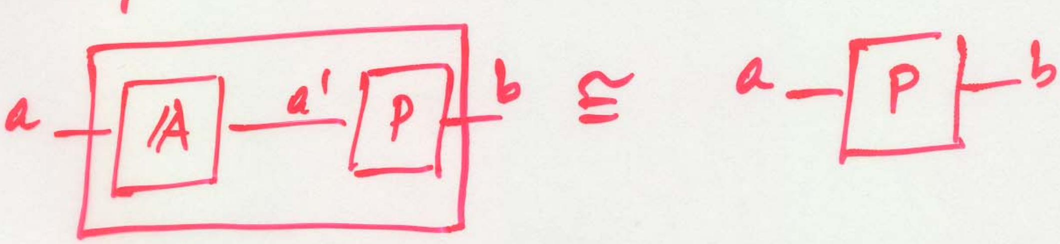
Density formula
Yoneda Lemma

Graphically:



Bicategorical structure

$$P \cdot I_A \cong P \cong P \cdot I_B$$

$$(P \cdot Q) \cdot R \cong P \cdot (Q \cdot R)$$

Two aspects:

I. Mathematical / Model-Theoretic

II. Logical / Formal / Proof-Theoretic

Bicategorical structure

$$P \cdot I_A \cong P \cong P \cdot I_B$$

$$(P \cdot Q) \cdot R \cong P \cdot (Q \cdot R)$$

Two aspects:

I. Mathematical / Model-Theoretic

II. Logical / Formal / Proof-Theoretic

Example:

$$((P \cdot Q) \cdot R)(a,d)$$

$$= \int^{c} (P \cdot Q)(a,c) \cdot R(c,d)$$

$$= \int^{c} \left(\int^{b} P(a,b) \cdot Q(b,c) \right) \cdot R(c,d)$$

$$\cong \int^{b} P(a,b) \cdot \left(\int^{c} Q(b,c) \cdot R(c,d) \right)$$

$$= \int^{b} P(a,b) \cdot (Q \cdot R)(b,d)$$

$$= (P \cdot (Q \cdot R))(b,d)$$

▶ The bicategory of profunctors is compact closed

(e.g [Day & Street])

- Symmetric monoidal structure

unit : $\mathbb{1}$ tensor : $\mathbb{A} \times \mathbb{B}$

- Duality

$$\mathbb{A}^* = \mathbb{A}^\circ$$

$$\frac{\mathbb{A} \overset{P}{\longrightarrow\!\!\!\!\!\!\!+\;\;} \mathbb{B}}{\mathbb{A}^* \underset{P^*}{\;\;+\!\!\!\!\!\!\!\longleftarrow} \mathbb{B}^*}$$

$$P^*(b,a) = P(a,b)$$

- Unit / Counit

$\eta_{\mathbb{A}} : \mathbb{1} \;+\!\!\!\!\!\!\!\longrightarrow \mathbb{A}^\circ \times \mathbb{A}$, $\varepsilon_{\mathbb{A}} : \mathbb{A} \times \mathbb{A}^\circ \;+\!\!\!\!\!\!\!\longrightarrow \mathbb{1}$

$\eta_{\mathbb{A}}(*, (a,a')) = \mathbb{A}(a',a)$ $\varepsilon_{\mathbb{A}}((a,a'),*) = \mathbb{A}(a',a)$

$\boxed{?}$ What about adjoints ?

$$\frac{\mathbb{A} \overset{P}{\nrightarrow} \mathbb{B}}{\mathbb{A} \underset{p^{+}}{\nleftarrow} \mathbb{B}}$$

$p^{+}(b,a) =$ ... with covariant action in b
and contravariant action in a !

? What about adjoints ?

$$\frac{\mathbb{A} \xrightarrow{\;P\;} \mathbb{B}}{\mathbb{A} \xleftarrow[\;P^{+}\;]{} \mathbb{B}}$$

$P^{+}(b,a) = \ldots$ with covariant action in b
and contravariant action in a !

! Need to add symmetries !

- The bicategory of profunctors between groupoids is strongly compact closed

categories all of whose morphisms are invertible

[Abramsky & Coecke]

$$P^{+}(b,a) = P(a,b)$$

permutation

$$f \cdot_{P^{+}} x \cdot_{P^{+}} g = g^{-1} \cdot_{P} x \cdot_{P} f^{-1}$$

conjugation

[Notation: $P^{+}(b,a) = P(\bar{a},\bar{b})$]

- $(\cdot)^\dagger$ is an involutive, identity on objects, monoidal $\underbrace{\text{pseudo-functor}}_{\text{map of bicategories}}$

- $P^{\dagger\dagger} = P$
- $I_{\mathbb{G}}^{\dagger} \cong I_{\mathbb{G}}$

$$\begin{aligned} I_{\mathbb{G}}^{\dagger}(g, g') &= \mathbb{G}(\bar{g}, \bar{g}') \\ &\cong \mathbb{G}(g', g) \\ &= I_{\mathbb{G}}(g, g') \end{aligned}$$

- $(P \cdot Q)^\dagger \cong Q^\dagger \cdot P^\dagger$

- strong compactness [Selinger]

$$\begin{tikzcd} & \mathbb{G} \times \mathbb{G}^{\circ} \arrow[d, "\sigma"] \\ \mathbb{I} \arrow[ur, "\varepsilon^\dagger", "\shortmid" marking] \arrow[dr, "\eta"', "\shortmid" marking] \arrow[r, phantom, "\cong"] & \mathbb{G}^{\circ} \times \mathbb{G} \end{tikzcd}$$

Fock space

$$!X = \bigoplus_{n\in\mathbb{N}} X^{\otimes_s n}$$

Fock space on categories

$$!\mathbb{X} = \sum_{n\in\mathbb{N}} \mathbb{X}^{\otimes n}$$

where

$\mathbb{X}^{\otimes n}$ = objects : $(x_1, \ldots, x_n)$

morphisms: $(x_i)_i \xrightarrow[(\sigma, f)]{} (y_j)_j$

with $\sigma \in \mathfrak{S}_n$

$$f = \{ f_i : x_i \to y_{\sigma i} \}_i$$

- $!\mathbb{X}$ is the free symmetric monoidal category on $\mathbb{X}$

[N.B. For $\mathbb{X}$ a groupoid, $!\mathbb{X}$ is also a groupoid]

A context for higher-order operators

$$\begin{tikzcd} \underline{\mathrm{Cat}} \arrow[loop above, "!"] \end{tikzcd}$$

monoidal monad

$\mathbb{A} \to !\mathbb{A}$

$!!\mathbb{A} \to !\mathbb{A}$

$!(\mathbb{A}+\mathbb{B}) \cong !\mathbb{A} \times !\mathbb{B}$

$!\mathbb{O} \cong \mathbb{1}$

$$\begin{tikzcd} \underline{\mathrm{Prof}} \arrow[loop above, "!"] \end{tikzcd}$$

monoidal comonad

$!\mathbb{A} \nrightarrow \mathbb{A}$

$!\mathbb{A} \nrightarrow !!\mathbb{A}$

$!(\mathbb{A}+\mathbb{B}) \cong !\mathbb{A} \times !\mathbb{B}$

$!\mathbb{O} \cong \mathbb{1}$

2-dimensional model of linear logic

A context for higher-order operators

$\underline{\underline{\mathrm{Cat}}} \circlearrowleft\, !$ monoidal monad	$\underline{\underline{\mathrm{Prof}}} \circlearrowleft\, !$ monoidal comonad
$\mathbb{A} \to\, !\mathbb{A}$	$!\mathbb{A} \nrightarrow \mathbb{A}$
$!!\mathbb{A} \to\, !\mathbb{A}$	$!\mathbb{A} \nrightarrow\, !!\mathbb{A}$
$!(\mathbb{A}+\mathbb{B}) \cong\, !\mathbb{A} \times !\mathbb{B}$	$!(\mathbb{A}+\mathbb{B}) \cong\, !\mathbb{A} \times !\mathbb{B}$
$!\mathbb{0} \cong \mathbb{1}$	$!\mathbb{0} \cong \mathbb{1}$

2-dimensional model of linear logic

The Kleisli bicategory $\underline{\underline{\mathrm{Prof}}}_!$ is cartesian closed

[Fiore & Gambino & Hyland & Winskel]

objects: $\mathbb{A}$

morphisms:
$$\frac{\frac{\mathbb{A} \mathrel{!\!\!\to} \mathbb{B}}{!\mathbb{A} \nrightarrow \mathbb{B}}}{!\mathbb{A} \times \mathbb{B}^{\circ} \to \mathrm{Set}}$$

$$\mathbb{A} \sqcap \mathbb{B} =_{\mathrm{def}} \mathbb{A} + \mathbb{B}$$

$$\mathbb{B}^{\mathbb{A}} =_{\mathrm{def}}\, !\mathbb{A}^{\circ} \times \mathbb{B}$$

Generalised $\underbrace{\text{species of structures}}_{\text{[Joyal]}: \mathbb{1} \mathrel{!\!\!\to} \mathbb{1}}$

The calculus of generalised species

- Composition
  - given by substitution [MFPS 2006] (generalising that of [symmetric] operads)
- Cartesian closed structure
  - $\underline{\text{hom}}(\mathbb{A}, \mathbb{B}) = !\mathbb{A}^{\circ} \times \mathbb{B}$
- Linear structure
  - $\underline{\text{lin}}(\mathbb{A}, \mathbb{B}) = \mathbb{A}^{\circ} \times \mathbb{B}$
  - the linear structure embeds in the closed structure
- Addition and multiplication
  - semiring structure compatible with composition

- Differential structure
  - partial derivatives (satisfying Leibniz rule, the chain rule, ...)
  - differential application or Jacobian
  - differentiation operator

    $\underline{\hom}(\mathbb{A},\mathbb{B}) \mapsto \underline{\hom}(\mathbb{A}, \underline{\mathrm{lin}}(\mathbb{A},\mathbb{B}))$

    (constant on linear maps, ...)

Creation and Annihilation operators

$$\gamma_v, \alpha_u \in \underline{\text{Lin}}(!A, !A)$$

$$\gamma_v(A, A') = [A', A \oplus \langle v \rangle]$$

$$\alpha_u(A, A') = [A' \oplus \langle u \rangle, A]$$

<u>Sanity check</u>: In the calculus of generalised species, respectively correspond to

$$(-)\cdot X_a \quad \text{and} \quad \frac{\partial}{\partial a}(-)\,.$$

<u>Further</u>:

$$\alpha_u \cdot \gamma_v \cong [u,v]\, I + \gamma_v \cdot \alpha_u$$

$$(\alpha_u \cdot \gamma_v)(A, A')$$

$$= \int^X \gamma_v(A, X) \cdot \alpha_u(X, A')$$

$$= \int^X [X, A \oplus \langle v \rangle] \cdot [A' \oplus \langle u \rangle, X]$$

$$\cong [A' \oplus \langle u \rangle, A \oplus \langle v \rangle]$$

$$\overset{(*)}{\cong} [u, v] \cdot [A', A] + \int^X [X \oplus \langle u \rangle, A] \cdot [A', X \oplus \langle v \rangle]$$

$$= [u, v] \cdot I(A, A') + \int^X \alpha_u(A, X) \cdot \gamma_v(X, A')$$

$$= [u, v] \cdot I(A, A') + (\gamma_v \cdot \alpha_u)(A, A')$$

$$= ([u, v] I + \gamma_v \cdot \alpha_u)(A, A')$$

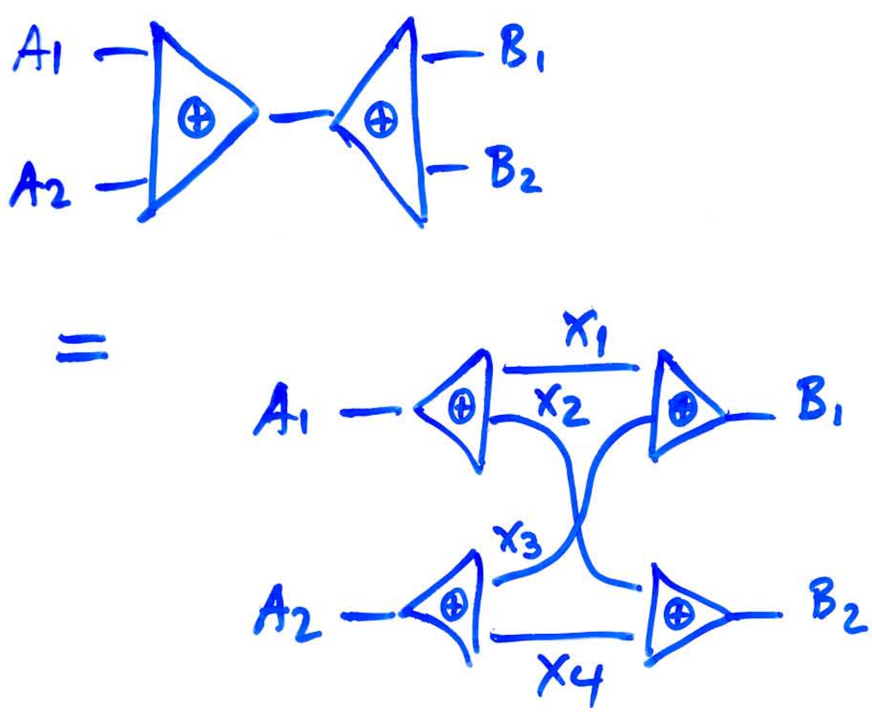
(*) is derivable from

$$!\mathbb{A}[A_1 \oplus A_2, B_1 \oplus B_2]$$

$$\cong \int^{x_1, x_2, x_3, x_4} [A_1, x_1 \oplus x_2] \cdot [A_2, x_3 \oplus x_4] \cdot [x_1 \oplus x_3, B_1] \cdot [x_2 \oplus x_4, B_2]$$

(∗) is derivable from

$$
\begin{aligned}
&!\mathbb{A}[A_1 \oplus A_2,\, B_1 \oplus B_2] \\
&\cong \int^{x_1, x_2, x_3, x_4} [A_1,\, x_1 \oplus x_2] \\
&\qquad \cdot [A_2,\, x_3 \oplus x_4] \\
&\qquad \cdot [x_1 \oplus x_3,\, B_1] \\
&\qquad \cdot [x_2 \oplus x_4,\, B_2]
\end{aligned}
$$



etc. etc.

$$!\mathbb{A}[\langle u\rangle, \langle v\rangle] \cong \mathbb{A}[u, v]$$

$$!\mathbb{A}[\langle u\rangle, A_1 \oplus A_2]$$

$$\cong \; !\mathbb{A}[\langle u\rangle, A_1] \,.\, !\mathbb{A}[\langle\rangle, A_2]$$

$$+ \; !\mathbb{A}[\langle\rangle, A_1] \,.\, !\mathbb{A}[\langle u\rangle, A_2]$$

Intensional & Extensional Theories

$\mathbb{A}$ (basis) $\longleftrightarrow$ $\hat{\mathbb{A}} = [\mathbb{A}^{o}, \mathrm{Set}]$ (space)

profunctors $\longleftrightarrow$ cocontinuous functors

generalised species $\longleftrightarrow$ analytic functors $\equiv$ power series

N.B. the 2-dimensional theory requires restriction to groupoids