Differential Structure in Models of Multiplicative Biadditive Intuitionistic Linear Logic

(Extended Abstract)

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Abstract. In the first part of the paper I investigate categorical models of multiplicative biadditive intuitionistic linear logic, and note that in them some surprising coherence laws arise. The thesis for the second part of the paper is that these models provide the right framework for investigating differential structure in the context of linear logic. Consequently, within this setting, I introduce a notion of creation operator (as considered by physicists for bosonic Fock space in the context of quantum field theory), provide an equivalent description of creation operators in terms of creation maps, and show that they induce a differential operator satisfying all the basic laws of differentiation (the product and chain rules, the commutation relations, *etc.*).

1 Introduction

Recent developments in the model theory of linear logic, starting with the work of Ehrhard [6, 7], have uncovered a variety of models with differential structure. Examples include Köthe sequence spaces [6], finiteness spaces [7], the relational model, generalised species of structures [11, 12], interaction systems [15], and complete semilattices [4]. This differential structure manifests itself as differential operators. In this context, a differential operator is a natural linear map

$$!A \multimap B \longrightarrow\, !A \multimap A \multimap B \tag{1}$$

that, when embedded as a map

$$A \Rightarrow B \longrightarrow A \Rightarrow (A \multimap B) \tag{2}$$

in the !-Kleisli category, enjoys the properties and satisfies the laws of differentiation. Intuitively, such an operator D provides a linear approximation $D[f]x : A \multimap B$ for every function $f : A \Rightarrow B$ at any point $x : A$.

The algebra underlying these models has also been investigated recently. Ehrhard and Regnier [8], isolated local-additive and commutative bialgebraic-exponential structure and explained, amongst other things, how they support

Addendum: The *Strength Law* (14) in Definition 4.2(1) is redundant.

the product rule. Blute, Cockett, and Seely [4], considered local-additive and exponential structure further supporting the chain rule. A common feature of these two approaches is that they take the local-additive structure, which allows morphisms to be added and is the minimal expression of linear-algebraic structure, as primitive. However, since local-additive structure in the presence of product structure is equivalent to biproduct structure, one may instead take as primitive the latter; which, furthermore, has the added bonus of inducing commutative bialgebraic-exponential structure. This is the viewpoint advocated here. It leads to the consideration of models of multiplicative biadditive intuitionistic linear logic, in which the additive structures (given by product and coproduct) coincide (as a biproduct), and to the thesis that these provide the right framework for investigating differential structure in the context of linear logic.

The present work is close in spirit to that of Blute, Cockett, and Seely [4] on differential categories, especially their Section 4. For latter comparison, I now highlight the relevant parts of their development. A notion of differential operator essentially as in (1) is introduced (see [4, Definition 2.3]). This is so that, for instance, the induced differential operator as in (2) satisfies the usual product and chain rules. Differential operators are shown to be in correspondence with so-called deriving transformations of the form

$$\partial : \; A \otimes {!A} \longrightarrow {!A} \tag{3}$$

(see [4, Definition 2.5 and Proposition 2.6]). Moreover, these are further seen to correspond to certain natural maps

$$\eta : \; A \longrightarrow {!A} \tag{4}$$

(see [4, Definition 4.11 and Theorem 4.12]).

Without loss of generality, my analysis of differential structure starts with the consideration of operators as in (3). These I call creation operators; as, interpreting the exponential as the bosonic Fock space construction [5], that models quantum systems of many identical non-interacting particles, they intuitively correspond to operators modelling particle creation. Indeed, categorical models of multiplicative intuitionistic linear logic come equipped with a canonical notion of annihilation operator

$$\propto : \; {!A} \longrightarrow A \otimes {!A}$$

with respect to which creation operators are shown to satisfy the commutation relations (see *e.g.* [14]). The above forms for creation and annihilation operators is non-standard; the standard forms are derivable.

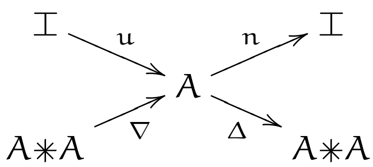
The concept of creation operator given in this paper is novel and differs from that of deriving transformation mentioned above. This is clearly seen by comparing Theorem 4.1 below, which establishes a bijective correspondence between creation operators and certain natural maps as in (4), that I call creation maps, and [4, Corollary 4.13], which provides the corresponding result for deriving transformations. A crucial difference between the axiomatisations is that the one provided here, besides being sharper, involves an axiom describing the interaction between the differential structure and the monoidal strength of the exponential. The present axiomatisation of creation maps has been directly influenced by and developed through a thorough analysis of the differential structure of generalised species of structures [10, 11], which is a bicategorical generalisation of that of the relational model of linear logic.

Organisation and contribution of the paper. Section 2 provides basic background on biproduct structure. The emphasis there is on giving an algebraic presentation, analysing some of its consequences (importantly commutative bialgebraic structure), and then characterising it in terms of enrichment. I guess that these results are folklore. However, I do not know references for them. In Section 3, I define categorical models of multiplicative biadditive intuitionistic linear logic to be models of multiplicative intuitionistic linear logic, as have been considered in the literature, equipped with biproduct structure compatible with the monoidal structure. This directly induces commutative bialgebraic-exponential structure. More surprisingly, I note that in these models some unexpected coherence laws arise. These are important for the analysis of differential structure carried over in Section 4. As mentioned above, differential structure is first analysed in terms of creation operators, for which the commutation relations with respect to a canonical notion of annihilation operator hold. Subsequently, creation operators are characterised in terms of the simpler notion of creation maps. These are shown to induce differential operators satisfying all the basic laws of differentiation. Finally, Section 5 concludes with general remarks and prospects for further work.

2 Biproduct Structure and Enrichment

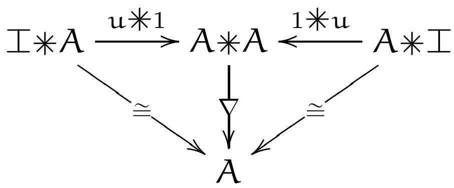
Biproduct structure. I give an algebraic presentation of biproduct structure, both on categories and on monoidal categories. This is the key to the modelling of biadditive structure in models of linear logic.

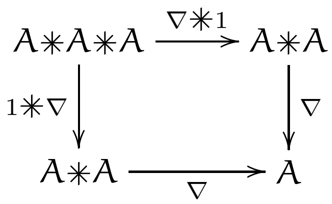
Definition 2.1. *A* biproduct structure *on a category is given by a symmetric monoidal structure* $(\mathbb{I}, *)$ *on it together with natural transformations*

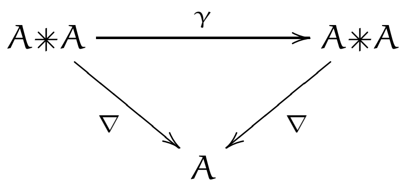


such that:

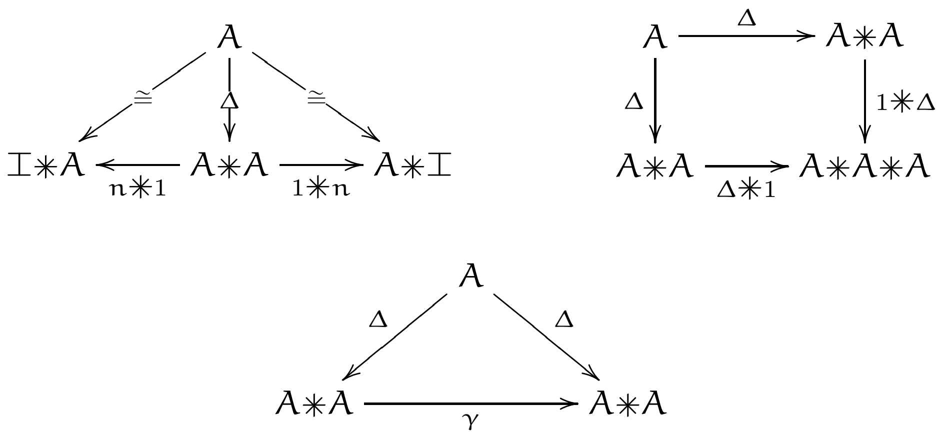
1. (A, u, ∇) *is a commutative monoid.*



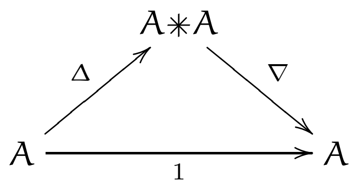




2. (A, n, Δ) *is a commutative comonoid.*



Definition 2.2. *A biproduct structure is* degenerate *whenever the following further law*



is satisfied.

The terminology of Definition 2.1 is justified by the following result.

Proposition 2.1. *In a category with biproduct structure* $(\mathbb{I}, *; u, \nabla; n, \Delta)$ *the following hold.*

1. $\mathbb{I}$ *is a zero object; that is, it is both initial and terminal.*
2. *The diagram*

$$A \xrightarrow{\cong} A * \mathbb{I} \xrightarrow{1 * u} A * B \xleftarrow{u * 1} \mathbb{I} * B \xleftarrow{\cong} B$$

   is a coproduct.

3. *The diagram*

$$A \xleftarrow{\cong} A * \mathbb{I} \xleftarrow{1 * n} A * B \xrightarrow{n * 1} \mathbb{I} * B \xrightarrow{\cong} B$$

   is a product.

That $\mathbb{I}$ is initial and Proposition 2.1(2) follow from Definition 2.1(1); dually, that $\mathbb{I}$ is terminal and Proposition 2.1(3) follow from Definition 2.1(2).

Corollary 2.1. *In a category with biproduct structure* $(\mathrm{I}, \ast; u, \nabla; n, \Delta)$*, the natural transformations* u*,* ∇*,* n*,* Δ *are monoidal; that is,*

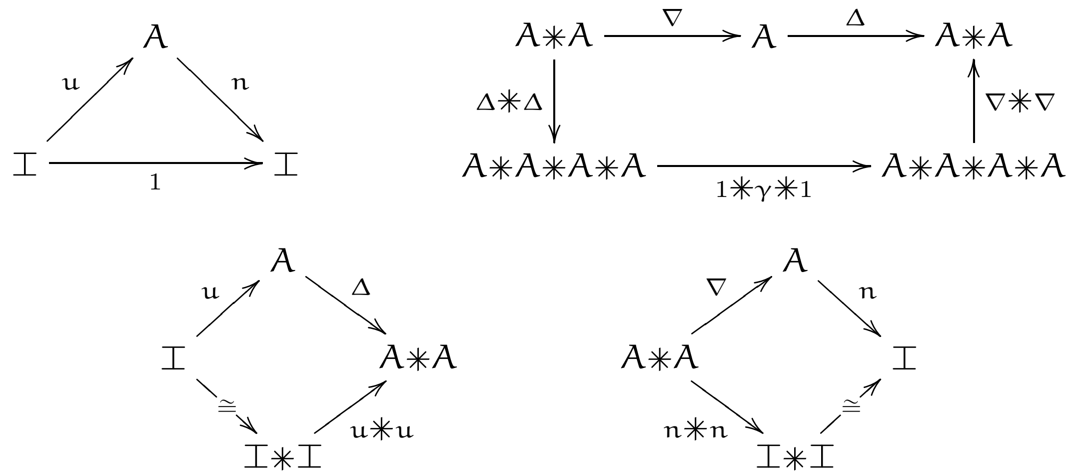
$$
\begin{aligned}
u_{\mathrm{I}} &= n_{\mathrm{I}} = 1_{\mathrm{I}} \\
u_{A\ast B} &= \mathrm{I} \overset{\cong}{\dashrightarrow} \mathrm{I}\ast\mathrm{I} \xrightarrow{u_A \ast u_B} A\ast B \\
n_{A\ast B} &= A\ast B \xrightarrow{n_A \ast n_B} \mathrm{I}\ast\mathrm{I} \overset{\cong}{\dashrightarrow} \mathrm{I}
\end{aligned}
$$

and

$$
\begin{aligned}
\nabla_{\mathrm{I}\ast\mathrm{I}} &= \mathrm{I}\ast\mathrm{I} \overset{\cong}{\dashrightarrow} \mathrm{I} \\
\nabla_{A\ast B} &= A\ast B\ast A\ast B \xrightarrow{1\ast\gamma\ast1} A\ast A\ast B\ast B \xrightarrow{\nabla_A \ast \nabla_B} A\ast B \\
\Delta_{\mathrm{I}\ast\mathrm{I}} &= \mathrm{I} \overset{\cong}{\dashrightarrow} \mathrm{I}\ast\mathrm{I} \\
\Delta_{A\ast B} &= A\ast B \xrightarrow{\Delta_A \ast \Delta_B} A\ast A\ast B\ast B \xrightarrow{1\ast\gamma\ast1} A\ast B\ast A\ast B
\end{aligned}
$$

It is important for our latter development to note that biproduct structure is equivalent to commutative bialgebraic structure.

Proposition 2.2. *In a category with biproduct structure* $(\mathrm{I}, \ast; u, \nabla; n, \Delta)$*, the commutative monoid and comonoid structures* $(u, \nabla; n, \Delta)$ *form a commutative bialgebra; that is,* u *and* ∇ *are comonoid homomorphisms and, equivalently,* n *and* Δ *are monoid homomorphisms.*



Enrichment. I now recall the characterisation of biproduct structure in the context of enrichment.

Let **Mon** (**CMon**) be the symmetric monoidal category of (commutative) monoids with respect to the universal bilinear tensor product. Recall that **Mon**-categories (**CMon**-categories) are categories all of whose homs $[A, B]$ come equipped with a (commutative) monoid structure $(0_{A,B}, +_{A,B})$ such that composition is strict and bilinear; that is,

$$0_{B,C}\, f = 0_{A,C} \qquad \text{and} \qquad f\, 0_{C,A} = 0_{C,B}$$

for all $f : A \longrightarrow B$, and

$$g\,(f + f') = g\,f + g\,f' \qquad \text{and} \qquad (g + g')\,f = g\,f + g'\,f$$

for all $f, f' : A \longrightarrow B$ and $g, g' : B \longrightarrow C$.

Proposition 2.3. *The following are equivalent.*

1. *Categories with biproduct structure.*
2. **Mon**-*categories with* (*necessarily enriched*) *finite products.*
3. **CMon**-*categories with* (*necessarily enriched*) *finite products.*

The enrichment of categories with biproduct structure is given by *convolution* (see *e.g.* [21]) as follows:

$$0 \;=\; (A \xrightarrow{\;n\;} \mathbb{I} \xrightarrow{\;u\;} B)$$

$$f + g \;=\; (A \xrightarrow{\;\Delta\;} A \ast A \xrightarrow{\;f \ast g\;} B \ast B \xrightarrow{\;\nabla\;} B)$$

For **SLat** the symmetric monoidal category of semilattices with respect to the universal bilinear tensor product we have the following result, which justifies the terminology of Definition 2.2.

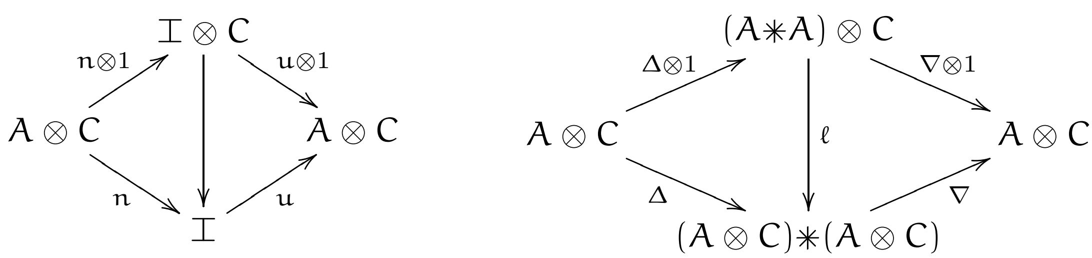
Proposition 2.4. *The following are equivalent.*

1. *Categories with degenerate biproduct structure.*
2. **SLat**-*categories with* (*necessarily enriched*) *finite products.*

Biproduct and monoidal structure. I further consider biproduct structure on symmetric monoidal categories. To this end, note that in a monoidal category with tensor $\otimes$ and binary products $\times$ there is a natural distributive law as follows:

$$\ell = \langle \pi_1 \otimes 1, \pi_2 \otimes 1 \rangle : (A \times B) \otimes C \longrightarrow (A \otimes C) \times (B \otimes C)$$

Definition 2.3. *A symmetric monoidal structure* $(\mathrm{I}, \otimes)$ *and a biproduct structure* $(\mathbb{I}, \ast; u, \nabla; n, \Delta)$ *on a category are* compatible *whenever the following hold:*



Recall that a **Mon**-*enriched* (*symmetric*) *monoidal category* is a (symmetric) monoidal category with a **Mon**-enrichment for which the tensor is strict and bilinear; that is, such that

$$0_{X,Y} \otimes f = 0_{X \otimes A, Y \otimes B} \qquad \text{and} \qquad f \otimes 0_{X,Y} = 0_{A \otimes X, B \otimes Y}$$

for all $f : A \longrightarrow B$, and

$$g \otimes (f + f') = g \otimes f + g \otimes f' \qquad \text{and} \qquad (g + g') \otimes f = g \otimes f + g' \otimes f$$

for all $f, f' : A \longrightarrow B$ and $g, g' : X \longrightarrow Y$.

Propositions 2.3 and 2.4 extend to the symmetric monoidal setting.

Proposition 2.5. *The following are equivalent.*

1. *Categories with compatible symmetric monoidal and biproduct structures.*
2. **Mon**-*enriched symmetric monoidal categories with (necessarily enriched) finite products.*
3. **CMon**-*enriched symmetric monoidal categories with (necessarily enriched) finite products.*

Corollary 2.2. *The following are equivalent.*

1. *Categories with compatible symmetric monoidal and degenerate biproduct structures.*
2. **SLat**-*enriched symmetric monoidal categories with (necessarily enriched) finite products.*

3 Models of Multiplicative Biadditive Intuitionistic Linear Logic

Models of multiplicative intuitionistic linear logic. I recall the definition of categorical model of multiplicative intuitionistic linear logic as it has been developed in the literature, see *e.g.* [17, 20, 2, 3, 1, 18, 19].

Definition 3.1. *An* $\mathcal{L}^!_\otimes$-model *is given by a category equipped with*

1. *a symmetric monoidal structure* $(\mathrm{I}, \otimes)$*;*
2. *a symmetric monoidal endofunctor* $\big(!, \varphi_\mathrm{I} : \mathrm{I} \longrightarrow !\mathrm{I}, \varphi : !A \otimes !B \longrightarrow !(A \otimes B)\big)$*;*
3. *a monoidal comonad structure* $A \xleftarrow{\epsilon} !A \xrightarrow{\delta} !!A$*;*
4. *a monoidal commutative comonoid structure* $\mathrm{I} \xleftarrow{e} !A \xrightarrow{d} !A \otimes !A$

subject to the following compatibility laws:

$$
\begin{array}{ccc}
!A & \xrightarrow{\delta} & !!A \\
{\scriptstyle e}\downarrow & & \downarrow{\scriptstyle !e} \\
\mathrm{I} & \xrightarrow[\varphi_\mathrm{I}]{} & !\mathrm{I}
\end{array}
$$

$$
\begin{array}{ccccc}
!A & & \xrightarrow{\delta} & & !!A \\
{\scriptstyle d}\downarrow & & & {\scriptstyle d_!}\swarrow & \downarrow{\scriptstyle !d} \\
!A \otimes !A & \xrightarrow[\delta\otimes\delta]{} & !!A \otimes !!A & \xrightarrow[\varphi_{!A,!A}]{} & !(!A \otimes !A)
\end{array}
\qquad (5)
$$

Amongst the many coherence conditions imposed by the above definition on $\mathcal{L}^!_\otimes$-models (for which see *e.g.* [3]) note the following two:

$$\begin{array}{ccc} !A \otimes !B & \xrightarrow{\varphi} & !(A \otimes B) \\ {\scriptstyle e\otimes e}\downarrow & & \downarrow{\scriptstyle e_{A\otimes B}} \\ I \otimes I & \xrightarrow{\cong} & I \end{array} \tag{6}$$

$$\begin{array}{ccccc} !A \otimes !B & & \xrightarrow{\qquad\qquad\varphi\qquad\qquad} & & !(A \otimes B) \\ {\scriptstyle d\otimes d}\downarrow & & & & \downarrow{\scriptstyle d_{A\otimes B}} \\ !A \otimes !A \otimes !B \otimes !B & \xrightarrow[1\otimes\gamma\otimes 1]{} & !A \otimes !B \otimes !A \otimes !B & \xrightarrow[\varphi\otimes\varphi]{} & !(A \otimes B) \otimes !(A \otimes B) \end{array} \tag{7}$$

Definition 3.2. *An* $\mathcal{L}^!_{\otimes,\times}$-model *is an* $\mathcal{L}^!_\otimes$*-model on a category with finite products* $(\top, \times)$.

In this context, we obtain the *Seely* monoidal natural isomorphism

$$s : \ !A \otimes !B \xrightarrow{\cong} !(A \times B)$$

given by the composite

$$!A \otimes !B \xrightarrow{\delta\otimes\delta} !!A \otimes !!B \xrightarrow{\varphi} !(!A \otimes !B) \xrightarrow{!\langle\epsilon\otimes e, e\otimes\epsilon\rangle} !\big((A \otimes I) \times (I \otimes B)\big) \xrightarrow{\cong} !(A \times B)$$

with inverse

$$!(A \times B) \xrightarrow{d} !(A \times B) \otimes !(A \times B) \xrightarrow{!(1\times n)\otimes!(n\times 1)} !(A \times \top) \otimes !(\top \times B) \xrightarrow{\cong} !A \otimes !B$$

Also the map $s_I = I \xrightarrow{\varphi_I} !I \xrightarrow{!n} !\top$ is an isomorphism, with inverse $e : !\top \longrightarrow I$. It follows that the diagrams

$$\begin{array}{ccc} & !A & \\ {\scriptstyle d}\swarrow & & \searrow{\scriptstyle !\Delta} \\ !A \otimes !A & \xrightarrow[s_{A,A}]{\cong} & !(A \times A) \end{array} \tag{8}$$

$$\begin{array}{ccc} & !A & \\ {\scriptstyle e}\swarrow & & \searrow{\scriptstyle !n} \\ I & \xrightarrow[s_I]{\cong} & !\top \end{array} \tag{9}$$

commute; so that the *contraction* and *weakening* maps, d and e, arise from the product structure via the Seely isomorphims.

Proposition 3.1. *In an $\mathcal{L}^!_{\otimes,\times}$-model the following coherence law holds.*

$$
\begin{array}{ccccc}
!A \otimes !B & & \xrightarrow[\cong]{s} & & !(A \times B) \\
\Big\downarrow{\scriptstyle \delta\otimes\delta} & & & & \Big\downarrow{\scriptstyle \delta_{A\times B}} \\
!!A \otimes !!B & \xrightarrow[\varphi_{!A,!B}]{} & !(!A \otimes !B) & \xrightarrow[!s]{\cong} & !!(A \times B)
\end{array}
\qquad (10)
$$

The proof uses the definition of s, and the monoidality and associativity laws of δ.

Models of multiplicative biadditive intuitionistic linear logic. I define categorical models of multiplicative biadditive intuitionistic linear logic to be models of multiplicative intuitionistic linear logic equipped with compatible biproduct structure. This is somewhat in the vein of Blute, Cockett, and Seely [4, Section 4].

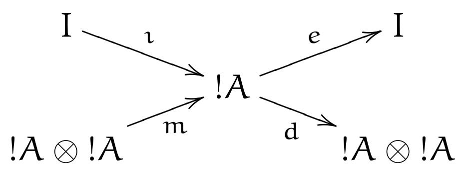
Definition 3.3. *An* $\mathcal{L}^!_{\otimes,*}$-model *is an $\mathcal{L}^!_{\otimes}$-model equipped with a biproduct structure $(\top, *; u, \nabla; n, \Delta)$ compatible with the symmetric monoidal structure $(I, \otimes)$.*

In this context, and via the monoidality of the Seely isomorphisms, the commutative bialgebra structure induced by the biproduct structure yields commutative bialgebraic-exponential structure.

Definition 3.4. *In $\mathcal{L}^!_{\otimes,*}$-models, the* coweakening *and* cocontraction *maps, ι and m, are defined as follows:*

$$
\begin{aligned}
\iota &= I \xrightarrow[\cong]{s_I} !\top \xrightarrow{!u} !A \\
m &= !A \otimes !A \xrightarrow[\cong]{s_{A,A}} !(A * A) \xrightarrow{!\nabla} !A
\end{aligned}
$$

Proposition 3.2. *In an $\mathcal{L}^!_{\otimes,*}$-model, the natural transformations*



form a commutative bialgebra.

More surprisingly, the following result exhibits three coherence laws enjoyed by $\mathcal{L}^!_{\otimes,*}$-models that can respectively be thought of as a kind of unfolding of the coherence conditions (5), (6), (7).

Theorem 3.1. *In an $\mathcal{L}^!_{\otimes,*}$-model the following coherence laws hold.*

1.

$$\begin{array}{ccccc} !A & & \xrightarrow{\quad\delta\quad} & & !!A \\ \uparrow{\scriptstyle m} & & & & \uparrow{\scriptstyle !m} \\ !A\otimes !A & \xrightarrow[\delta\otimes\delta]{} & !!A\otimes !!A & \xrightarrow[\varphi_{!A,!A}]{} & !(!A\otimes !A) \end{array} \tag{11}$$

2.

$$\begin{array}{ccccc} & & !A\otimes !B & \xrightarrow{\varphi} & !(A\otimes B) \\ & \nearrow^{\iota\otimes 1} & & & \uparrow{\scriptstyle \iota_{A\otimes B}} \\ I\otimes !B & & & & \\ & \searrow_{1\otimes e} & & & \\ & & I\otimes I & \xrightarrow{\cong} & I \end{array} \tag{12}$$

3.

$$\begin{array}{ccccc} !A\otimes !B & & \xrightarrow{\quad\varphi\quad} & & !(A\otimes B) \\ \uparrow{\scriptstyle m\otimes 1} & & & & \\ !A\otimes !A\otimes !B & & & & \uparrow{\scriptstyle m} \\ \downarrow{\scriptstyle 1\otimes 1\otimes d} & & & & \\ !A\otimes !A\otimes !B\otimes !B & \xrightarrow[1\otimes\gamma\otimes 1]{} & !A\otimes !B\otimes !A\otimes !B & \xrightarrow[\varphi\otimes\varphi]{} & !(A\otimes B)\otimes !(A\otimes B) \end{array} \tag{13}$$

The proof of Theorem 3.1(1) uses the definition of m and the coherence law (10). The proof of Theorem 3.1(2) uses the coherence law (9), the definitions of s_I and ι, the monoidality of $(!, \varphi_I, \varphi)$, and the strictness of the tensor product to show that both composites are equal to the following one

$$I\otimes !B \overset{\cong}{\dashrightarrow} !B \xrightarrow{!0} !(A\otimes B)$$

Finally, the proof of Theorem 3.1(3) uses the product structure of $(\top, *)$, the bilinearity of the tensor product, the definitions of m and s^{-1}, the coherence law (8), and the monoidality of d.

4 Differential Structure

The analysis of differential structure in $\mathcal{L}^!_{\otimes,*}$-models follows.

Creation operators. The starting point is the definition of annihilation and creation operators; the terminology for which is justified by Proposition 4.1.

Definition 4.1. *In an $\mathcal{L}^!_{\otimes}$-model, the* annihilation operator $\propto : !A \longrightarrow A\otimes !A$ *is the natural transformation given by the following composite*

$$!A \xrightarrow{d} !A\otimes !A \xrightarrow{\epsilon\otimes 1} A\otimes !A$$

Definition 4.2. *A* creation operator *in an $\mathcal{L}^{!}_{\otimes,\ast}$-model is a natural transformation*

$$\partial : A \otimes !A \longrightarrow !A$$

satisfying the following laws.

1. *Strength.*

$$\begin{array}{ccccc} A \otimes !A \otimes !B & \xrightarrow{\partial\otimes 1} & !A \otimes !B & \xrightarrow{\varphi} & !(A \otimes B) \\ \downarrow{\scriptstyle 1\otimes 1\otimes \propto} & & & & \uparrow{\scriptstyle \partial_{A\otimes B}} \\ A \otimes !A \otimes B \otimes !B & \xrightarrow[1\otimes\gamma\otimes 1]{} & A \otimes B \otimes !A \otimes !B & \xrightarrow[1\otimes 1\otimes\varphi]{} & A \otimes B \otimes !(A \otimes B) \end{array} \tag{14}$$

2. *Comonad.*

$$\begin{array}{ccccc} A \otimes !A & \xrightarrow{\partial} & !A & \xrightarrow{\epsilon} & A \\ & \searrow{\scriptstyle 1\otimes e} & & \nearrow{\scriptstyle \cong} & \\ & & A \otimes I & & \end{array} \qquad \begin{array}{ccccc} A \otimes !A & \xrightarrow{\partial} & !A & \xrightarrow{\delta} & !!A \\ \downarrow{\scriptstyle 1\otimes d} & & & & \uparrow{\scriptstyle \partial_{!}} \\ A \otimes !A \otimes !A & & \xrightarrow[\partial\otimes\delta]{} & & !A \otimes !!A \end{array} \tag{15}$$

3. *Multiplication.*

$$\begin{array}{ccccc} A \otimes !A \otimes !A & \xrightarrow{\partial\otimes 1} & !A \otimes !A & \xrightarrow{m} & !A \\ & \searrow{\scriptstyle 1\otimes m} & & \nearrow{\scriptstyle \partial} & \\ & & A \otimes !A & & \end{array} \tag{16}$$

The above form for creation and annihilation operators is non-standard. More commonly, see *e.g.* [14], the literature deals with creation operators $\partial_v : !A \longrightarrow !A$ for vectors $v : I \longrightarrow A$ and annihilation operators $\propto_{v'} : !A \longrightarrow !A$ for covectors $v' : A \longrightarrow I$. In the present setting, these are derived as follows:

$$\begin{aligned} \partial_v &= !A \overset{\cong}{\dashrightarrow} I \otimes !A \xrightarrow{v\otimes 1} A \otimes !A \xrightarrow{\partial} !A \\ \propto_{v'} &= !A \xrightarrow{\propto} A \otimes !A \xrightarrow{v'\otimes 1} I \otimes !A \overset{\cong}{\dashrightarrow} !A \end{aligned}$$

Proposition 4.1. *Creation and annihilation operators in $\mathcal{L}^{!}_{\otimes,\ast}$-models satisfy the following commutation relations:*

1. $\propto\partial = 1 + (1 \otimes \partial)(\gamma \otimes 1)(1 \otimes \propto) \ : A \otimes !A \longrightarrow A \otimes !A$
2. $\partial\,(1 \otimes \partial) = \partial\,(1 \otimes \partial)\,(\gamma \otimes 1) \ : A \otimes A \otimes !A \longrightarrow !A$
3. $(1 \otimes \propto)\propto = (\gamma \otimes 1)(1 \otimes \propto)\propto \ : !A \longrightarrow A \otimes A \otimes !A$

It follows that

$$\begin{aligned} \propto_{v'}\partial_v &= \left(!A \overset{\cong}{\dashrightarrow} I \otimes !A \xrightarrow{(v'v)\otimes 1} I \otimes !A \overset{\cong}{\dashrightarrow} !A\right) + \left(!A \xrightarrow{\partial_v\propto_{v'}} !A\right) \\ \partial_u\partial_v &= \partial_v\partial_u \\ \propto_{u'}\propto_{v'} &= \propto_{v'}\propto_{u'} \end{aligned}$$

for all $u, v : I \longrightarrow A$ and $u', v' : A \longrightarrow I$.

For comparison with the work of Blute, Cockett, and Seely on deriving transformations, note that the laws of (15) are the *linearity* and the *chaining* conditions of [4, Definition 2.5] and that the law of (16) is the *multiplication rule* of [4, Definition 4.10]. The law of (14) is novel, and in its presence the *constant maps* and the *copying* conditions of [4, Definition 2.5] are derivable (see Proposition 4.2 below). Thus, creation operators are deriving transformations satisfying the multiplication rule.

Proposition 4.2. *Every creation operator* ∂ *in an* $\mathcal{L}^{!}_{\otimes,\ast}$*-model is such that*

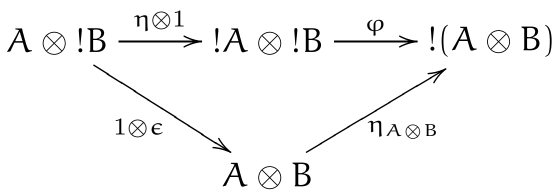
1. $e\,\partial = 0 : A \otimes !A \longrightarrow I$, *and*
2. $d\,\partial = (\partial_1 + \partial_2)\,(1 \otimes d) : A \otimes !A \longrightarrow !A \otimes !A$ *where* $\partial_1 = A \otimes !A \otimes !A \xrightarrow{\partial \otimes 1} !A \otimes !A$ *and* $\partial_2 = A \otimes !A \otimes !A \xrightarrow{\gamma \otimes 1} !A \otimes A \otimes !A \xrightarrow{1 \otimes \partial} !A \otimes !A$.

Propositions 4.1 and 4.2 are better established using the representation of creation operators given in Theorem 4.1 below. The proofs of Propositions 4.1(1) and 4.2(2) use the biproduct structure, the strictness and bilinearity of the tensor product, the coherence of the Seely isomorphisms, and the bialgebraic-exponential structure; the proofs of Propositions 4.1(2*&*3) use the commutative of the bialgebraic-exponential structure; the proof of Proposition 4.2(1) uses the strictness of the tensor product.

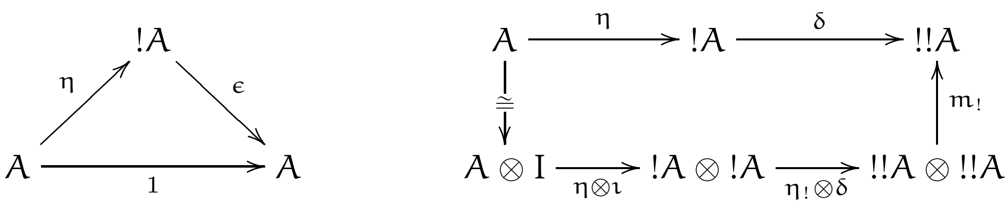
Creation maps. Creation operators have a simpler axiomatisation in terms of creation maps.

Definition 4.3. *A* creation map *in an* $\mathcal{L}^{!}_{\otimes,\ast}$*-model is a natural transformation* $\eta : A \longrightarrow !A$ *satisfying the following laws.*

1. *Strength.*



2. *Comonad.*



As a direct consequence of the strength and first comonad law, creation maps are coherent with respect to the monoidal strength.

Proposition 4.3. *Every creation map* η *in an* $\mathcal{L}^!_{\otimes,\ast}$*-model satisfies the following coherence law:*

$$\begin{array}{ccc} & A \otimes B & \\ \swarrow^{\eta\otimes\eta} & & \searrow^{\eta_{A\otimes B}} \\ !A \otimes !B & \xrightarrow[\varphi]{\hspace{3em}} & !(A \otimes B) \end{array}$$

Theorem 4.1. *The mappings*

$$\partial : A \otimes !A \longrightarrow !A \ \longmapsto\ \eta = \ A \overset{\cong}{\dashrightarrow} A \otimes I \xrightarrow{1\otimes\iota} A \otimes !A \xrightarrow{\partial} !A$$

$$\eta : A \longrightarrow !A \ \longmapsto\ \partial = \ A \otimes !A \xrightarrow{\eta\otimes 1} !A \otimes !A \xrightarrow{m} !A$$

yield a bijection between creation operators and creation maps in $\mathcal{L}^!_{\otimes,\ast}$*-models.*

To show that the map η induced by a creation operator ∂ satisfies the strength law one uses the strength law for ∂ and the coherence law (12); to show that η satisfies the first comonad law one uses the first comonad law for ∂; to show that η satisfies the second comonad law one uses the second comonad law and the multiplication law for ∂. Conversely, to show that the operator ∂ induced by a creation map η satisfies the strength law one uses the strength law for η and the coherence law (13); to show that ∂ satisfies the first comonad law one uses the biproduct structure, the strictness of the tensor product, and the first comonad law for η; to show that ∂ satisfies the second comonad law one uses the strength law and second comonad law for η, the coherence condition (5), the coherence laws (11) and (13), and the comonad laws for $(!, \epsilon, \delta)$; to show that ∂ satisfies the multiplication law one uses the associativity of m.

Differentiation. In the presence of the above differential structure, one obtains a natural *differential operator*

$$\mathrm{D}[-] = [\partial, -] : \ [!A, B] \longrightarrow [A \otimes !A, B]$$

such that the following rules hold.

1. Identity rule.
$$\mathrm{D}[1] \ = \ \partial$$

2. Composition rule.
$$\mathrm{D}[\ell\, f] \ = \ \ell\, \mathrm{D}[f] \qquad (f : !A \longrightarrow B, \ell : B \longrightarrow C)$$

3. Constant rule.
$$\mathrm{D}[e_A] \ = \ 0$$

4. Sum rule.
$$\mathrm{D}[f + g] \ = \ \mathrm{D}[f] + \mathrm{D}[g] \qquad (f, g : !A \longrightarrow B)$$

5. Tensor rule.

$$D[(f \otimes g)\, d] \;=\; \big(D[f] \otimes g + (f \otimes D[g])(\gamma \otimes 1) \big)\,(1 \otimes d) \quad (f : {!A} \longrightarrow B, g : {!A} \longrightarrow C)$$

6. Linearity rule.

$$D[\ell\, \epsilon_A] \;=\; \ell\,(1 \otimes e_A) \qquad (\ell : A \longrightarrow B)$$

7. Chain rule.

$$D[g\, f^\dagger] \;=\; D[g]\,(D[f] \otimes f^\dagger)\,(1 \otimes d) \qquad (f : {!A} \longrightarrow B, g : {!B} \longrightarrow C)$$

where $f^\dagger = (!f)\,\delta : {!A} \longrightarrow {!B}$

Further, for $\mathcal{L}^{!,\multimap}_{\otimes,\ast}$-models, *i.e.* in the presence of closed structure ($\multimap$), one may internalise the differential operator as a *partial derivative operator*

$$D = \lambda u : A.\, \lambda f : {!A} \multimap B.\, \lambda x : {!A}.\, f\big(\partial(u \otimes x)\big) \;: A \multimap ({!A} \multimap B) \multimap {!A} \multimap B$$

for which, moreover, the following rules hold.

1. Symmetry rule.

$$u : A, v : A \;\vdash\; D_u \circ D_v = D_v \circ D_u \;: {!A} \multimap B$$

2. Strength rule.

$$f : !(A \otimes B) \multimap C, u : A, x : {!A}, y : {!B}$$

$$\vdash D_u\big[\lambda x : {!A}.\, f\big(\varphi(x \otimes y)\big)\big]x \;=\; \mathsf{let}\ v \otimes z = \propto(y) : B \otimes {!B}\ \mathsf{in}\ D_{u \otimes v}\big[f\big]\big(\varphi(x \otimes z)\big) \;: C$$

5 Concluding Remarks

The general theme of this paper has been the investigation of categorical models of multiplicative biadditive intuitionistic linear logic, and of differential structure therein. Within each of these two strands, various possibilities for research still remain. I mention a few here.

From the abstract theoretical viewpoint, the consideration of $\mathcal{L}^{!,\multimap}_{\otimes,\ast}$-models equipped with differential structure as categorical models of the differential λ-calculus of Ehrhard and Regnier [9] will be considered in the full version of the paper. A more important next step, however, is to work out the type and proof theory of $\mathcal{L}^{!,\multimap}_{\otimes,\ast}$-models, both as a term assignment system and a graphical calculus, and thereafter extend them to incorporate differential structure. In another direction, the relationship of our axiomatics with the earlier categorical axiomatic investigation of differential structure provided by Synthetic Differential Geometry (see *e.g.* [16, Part I]) should be addressed.

From the model-theoretic viewpoint, the discussion of concrete $\mathcal{L}^{!,\multimap}_{\otimes,\ast}$-models equipped with differential structure will be considered in the full version of the paper, where the diligent, but otherwise evident, verification that the models mentioned at the beginning of the Introduction are examples will be covered. More interestingly, I conjecture that the category of convenient vector spaces and linear maps of Frölicher and Kriegl [13] provides yet another example; as so may be the case, indicated to me by Anders Kock in correspondence, with the category of modules for the ring object of line type in some models of Synthetic Differential Geometry (see *e.g.* [16, Part III]).

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