Second-Order and Dependently-Sorted Abstract Syntax

(Extended Abstract)

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Abstract

The paper develops a mathematical theory in the spirit of categorical algebra that provides a model theory for second-order and dependently-sorted syntax. The theory embodies notions such as α-equivalence, variable binding, capture-avoiding simultaneous substitution, term metavariable, meta-substitution, mono and multi sorting, and sort dependency. As a matter of illustration, a model is used to extract a second-order syntactic theory, which is thus guaranteed to be correct by construction.

Introduction

The algebraic foundations for syntactic structures as needed in computer science are still under development. Such foundations are to provide a mathematical theory in which models are given by algebraic structures, and should reflect the various syntactic notions in a conceptual manner. In particular, free algebraic models are to provide an abstract notion of syntax; so that syntax is formalised in terms of its structure, which, being characterised by a universal property, is thus devoid of inessential details. Such a development is well-known for first-order syntax, and it has more recently been extended to incorporate variable binding [11, 12]. The work presented here goes a step further in this direction. Specifically, I provide algebraic foundations for second-order and dependently-sorted syntax. Thus advancing the research programme of developing algebraic foundations for type theory.

The conceptual framework for the developments of the paper follows.

In the traditional case of mono-sorted first-order syntax, one considers a universe of discourse given by a cartesian category $\mathscr{C}$ on which syntactic structure manifests itself as an endofunctor Σ on $\mathscr{C}$. The associated notion of algebraic structure is given by that of an algebra for an endofunctor. One requires that free Σ-algebras

$$X \longrightarrow \mathcal{S}X \longleftarrow \Sigma(\mathcal{S}X)$$

exist, obtaining a monad of syntax $\mathcal{S}$ on $\mathscr{C}$. The monad structure provides substitution structure.

The category $\mathscr{C}$ is typically cartesian closed, and the endofunctor Σ internalises as a family of maps

$$\Sigma(X) \times Y^X \longrightarrow \Sigma(Y) \text{ in } \mathscr{C}$$

arising from a cartesian strength

$$\Sigma(X) \times Y \stackrel{\cdot}{\longrightarrow} \Sigma(X \times Y) : \mathscr{C} \times \mathscr{C} \longrightarrow \mathscr{C} \ .$$

It follows that the monad $\mathcal{S}$ acquires a cartesian strength

$$\mathcal{S}(X) \times Y \stackrel{\cdot}{\longrightarrow} \mathcal{S}(X \times Y) : \mathscr{C} \times \mathscr{C} \longrightarrow \mathscr{C}$$

and hence that it also internalises. Importantly, this provides an internal substitution operation as a family of maps

$$\mathcal{S}(X) \times (\mathcal{S}Y)^X \longrightarrow \mathcal{S}(Y) \text{ in } \mathscr{C} \ .$$

The generalisation to multi-sorted syntax considers the category $\mathscr{C}^S$, for S a set of sorts, together with a signature endofunctor Σ on it equipped with a strength

$$\Sigma(X) \boxtimes C \stackrel{\cdot}{\longrightarrow} \Sigma(X \boxtimes C) : \mathscr{C}^S \times \mathscr{C} \longrightarrow \mathscr{C}^S \ , \qquad (1)$$

where the action $(-) \boxtimes (=) : \mathscr{C}^S \times \mathscr{C} \longrightarrow \mathscr{C}^S$ is given pointwise by setting $(X \boxtimes C)_s = X_s \times C$ for all $s \in S$. This leads to an internal substitution operation

$$\mathcal{S}(X) \boxtimes [X, \mathcal{S}(Y)] \longrightarrow \mathcal{S}(Y) \text{ in } \mathscr{C}^S \ ,$$

where $\mathscr{C}$ is assumed to have S-indexed products and, for $A, B \in \mathscr{C}^S$, the $\mathscr{C}$-internal hom $[A, B]$ is defined as $\prod_{s \in S} {B_s}^{A_s}$.

The treatment of syntax with variable binding of Fiore, Plotkin and Turi [11] required further considerations. First, the universe of discourse is equipped with a monoidal closed structure $(V, \bullet)$, respectively modelling the types of variables and explicit substitutions. Second, the notion of algebraic structure is generalised to Σ-algebras with substitution structure. These are Σ-algebras equipped with a compatible $(V, \bullet)$-monoid structure (modelling capture-avoiding simultaneous substitution), the definition of which depends on a strength

$$s_{X,(P,\varpi)} : \Sigma(X) \bullet P \stackrel{.}{\longrightarrow} \Sigma(X \bullet P) \; . \qquad (2)$$

The extra structure on the parameter P above, in the form of a point $\varpi : V \longrightarrow P$, reflects the need of fresh variables in the definition of substitution for binding operators.

Free algebras with substitution structure in the model of Fiore, Plotkin and Turi [11] have already been considered by Hamana [13] as algebraic models for second-order syntax. The work presented here, however, goes a step further in this direction. Indeed, I give a general result describing free Σ-algebras with substitution structure

$$\begin{array}{ccc} X & \longrightarrow \; \mathcal{M}X \; \longleftarrow & \Sigma(\mathcal{M}X) \\ V & \longrightarrow \; \mathcal{M}X \; \longleftarrow & \mathcal{M}(X) \bullet \mathcal{M}(X) \end{array}$$

and develop a theory of strengths from which the monad substitution structure is seen to internalise as a family of maps

$$\mathcal{M}(X) \times (\mathcal{M}Y)^X \longrightarrow \mathcal{M}(Y) \; .$$

These structures arise by initial universal properties from which a syntactic theory for second-order syntax may be extracted. This I work out in some detail, exhibiting notions such as α-equivalence, variable binding, capture-avoiding simultaneous substitution, term metavariable, and meta-substitution. All these notions arise from the mathematical model, and are thus guaranteed to be correct by construction. Overall, the theory provides the foundational syntactic core for second-order multi-sorted equational theories, which are to be dealt with according to the mathematical theory of [10] in a subsequent paper in collaboration with Chung-Kil Hur.

With the model theory of first/second-order mono/multi-sorted syntax in place, I further initiate the development of a mathematical theory for dependently-sorted abstract syntax. The first obstacle in this respect has been that of finding an appropriate universe of discourse, embodying the notion of sort dependency and its associated substitution operation. In this respect, using ideas of Makkai [18] for modelling simple sort dependency, I first show how to extend the mono-sorted model of Fiore, Plotkin and Turi [11] (and its multi-sorted version) to encompass simple dependent sorts. Subsequently, I outline a further extension for the general case of sort dependency in the framework of Ehresmann's theory of sketches.

I. Second-order abstract syntax

This first part of the paper develops a mathematical theory for second-order syntax. The development is presented in three sections, respectively addressing categorical (Section I.1), model-theoretic (Section I.2), and syntactic (Section I.3) aspects of the theory.

I.1. Categorical theory

The purpose of this section is to provide some general abstract definitions (Sections I.1.1–I.1.3) and results (Sections I.1.4 and I.1.5) that are needed in the development of the model theory of second-order syntax.

I.1.1. Algebras

Parameterised algebras. For a functor $F : \mathscr{D} \times \mathscr{C} \longrightarrow \mathscr{C}$, an F-*algebra* consists of a *carrier* object $(D, C) \in \mathscr{D} \times \mathscr{C}$ together with a *structure map* $F(D, C) \longrightarrow C$ in $\mathscr{C}$. An F-algebra *homomorphism* $\big((D, C), \varphi\big) \longrightarrow \big((D', C'), \varphi'\big)$ is a map $(g, f) : (D, C) \longrightarrow (D', C')$ in $\mathscr{D} \times \mathscr{C}$ such that $f \circ \varphi = \varphi' \circ F(g, f)$. I will write F-$\mathbf{Alg}$ for the category of F-algebras and homomorphisms. Note that the traditional notions of algebra and homomorphism for an endofunctor are obtained as a special case (*viz.*, when $\mathscr{D} = \mathbf{1}$).

Parameterised initial-algebra functors. For a functor $F : \mathscr{D} \times \mathscr{C} \longrightarrow \mathscr{C}$, let $(\mu F(D), \mu_D)$ be an initial algebra of the endofunctor $F(D, -)$ on $\mathscr{C}$ for each $D \in \mathscr{D}$. Then, the mapping $D \longmapsto \mu F(D)$ extends to a functor $\mu F : \mathscr{D} \longrightarrow \mathscr{C}$ where, for all $g : D \longrightarrow D'$ in $\mathscr{D}$, the map $\mu F(g) : \mu F(D) \longrightarrow \mu F(D')$ in $\mathscr{C}$ is uniquely characterised by the fact that $\big(g, \mu F(g)\big)$ is an F-algebra homomorphism $\big((D, \mu F(D)), \mu_D\big) \longrightarrow \big((D', \mu F(D')), \mu_{D'}\big)$.

I.1.2. Actions and strengths

We need a broad generalisation of the notion of *strength* [15] (see (1) and (2) above) as a map of actions.

Actions. A $\mathscr{V}$-*action* for a monoidal category $(\mathscr{V}, I, \otimes)$ consists of a category $\mathscr{A}$ together with a functor $\odot : \mathscr{A} \times \mathscr{V} \longrightarrow \mathscr{A}$ and natural isomorphisms $A \odot I \xrightarrow{\cong} A$ and $(A \odot X) \odot Y \xrightarrow{\cong} A \odot (X \otimes Y)$ such that

$$\begin{array}{ccc} (A \odot I) \odot X & \xrightarrow{\cong} & A \odot (I \otimes X) \\ & \searrow^{\cong} \quad \swarrow^{\cong} & \\ & A \odot X & \end{array}$$

$$\begin{array}{ccc} & ((A \odot X) \odot Y) \odot Z & \\ \swarrow^{\cong} & & \searrow^{\cong} \\ (A \odot (X \otimes Y)) \odot Z & & (A \odot X) \odot (Y \otimes Z) \\ \downarrow{\cong} & & \downarrow{\cong} \\ A \odot ((X \otimes Y) \otimes Z) & \xrightarrow{\cong} & A \odot (X \otimes (Y \otimes Z)) \end{array}$$

for all $A \in \mathscr{A}$ and $X, Y, Z \in \mathscr{V}$. Such a $\mathscr{V}$-action is said to be *closed* if the endofunctor $(-) \odot X$ on $\mathscr{A}$ has a right adjoint for all $X \in \mathscr{V}$.

Every monoidal (closed) category $\mathscr{V}$ canonically induces a (closed) $\mathscr{V}$-action on $\mathscr{V}^n$, for $n \in \mathbb{N}$, by pointwise tensor product. More generally, every strong monoidal functor $U : \mathscr{V} \longrightarrow \mathscr{A}$ induces a $\mathscr{V}$-action on $\mathscr{A}$ given by $A, X \longmapsto A \otimes UX$. In particular, we will later consider the $(I/\mathscr{V})$-action on $\mathscr{V}$ induced by this construction for U the forgetful functor $I/\mathscr{V} \longrightarrow \mathscr{V}$, where $I/\mathscr{V}$ is equipped with the obvious monoidal structure for which U is strong monoidal.

Further, an important class of $\mathscr{V}$-actions arises from $\mathscr{V}$-enriched categories, for $\mathscr{V}$ symmetric monoidal closed, with tensors and powers (see [14]).

Strengths. Let $(\mathscr{A}, \odot)$ and $(\mathscr{A}', \odot')$ be $(\mathscr{V}, I, \otimes)$-actions. A $\mathscr{V}$*-strength* of type $(\mathscr{A}, \odot) \longrightarrow (\mathscr{A}', \odot')$ for a functor $F : \mathscr{A} \longrightarrow \mathscr{A}'$ is a natural transformation

$$\varphi_{A,X} : F(A) \odot' X \dot{\longrightarrow} F(A \odot X) : \mathscr{A} \times \mathscr{V} \longrightarrow \mathscr{A}'$$

such that

$$\begin{array}{ccc} F(A) \odot' I & \xrightarrow{\varphi_{A,I}} & F(A \odot I) \\ & {\scriptstyle\cong}\searrow \quad \swarrow{\scriptstyle\cong} & \\ & F(A) & \end{array}$$

$$\begin{array}{ccc} F(A) \odot' (X \otimes Y) & \xrightarrow{\varphi_{A, X\otimes Y}} & F(A \odot (X \otimes Y)) \\ \downarrow{\scriptstyle\cong} & & \uparrow{\scriptstyle\cong} \\ (F(A) \odot' X) \odot' Y & & F((A \odot X) \odot Y) \\ & {\scriptstyle\varphi_{A,X} \odot' Y}\searrow \quad \nearrow{\scriptstyle\varphi_{A\odot X, Y}} & \\ & F(A \odot X) \odot' Y & \end{array}$$

for all $A \in \mathscr{A}$ and $X, Y \in \mathscr{V}$.

Note that the usual notion of strength for an endofunctor on a monoidal category $(\mathscr{C}, I, \otimes)$ is recovered as that of $\mathscr{C}$-strength of type $(\mathscr{C}, \otimes) \longrightarrow (\mathscr{C}, \otimes)$. Natural examples of the above more general notion are part of the development of the paper.

I.1.3. Algebras with monoid structure

Following [11], I introduce a general notion of algebra with monoid structure. This definition will be used in the context of substitution monoidal structures (see Section I.2.2), and thus aims at formalising the notion of algebra with substitution structure.

Definition 1. Let $(\mathscr{C}, I, \otimes)$ be a monoidal category. For an endofunctor Σ on $\mathscr{C}$ with an $(I/\mathscr{C})$-strength s, let (Σ, s)-$\mathbf{Mon}$ be the category of (Σ, s)*-monoids* given by Σ-algebras $(a : \Sigma A \longrightarrow A)$ equipped with a monoid structure $(e : I \longrightarrow A \longleftarrow A \otimes A : m)$ subject to the following compatibility condition

$$\begin{array}{ccccc} \Sigma(A) \otimes A & \xrightarrow{s_{A,(A,e)}} & \Sigma(A \otimes A) & \xrightarrow{\Sigma m} & \Sigma A \\ \downarrow{\scriptstyle a \otimes \mathrm{id}} & & & & \downarrow{\scriptstyle a} \\ A \otimes A & & \xrightarrow[m]{\qquad\qquad} & & A \end{array}$$

Morphisms between (Σ, s)-monoids are maps between the underlying objects that are both Σ-algebra and monoid homomorphisms.

I.1.4. Free algebras with monoid structure

I proceed to give an analysis of free algebras with monoid structure suitable for extracting explicit syntactic descriptions in applications. The reader is advised to study this section and interpret its results in the context of Section I.2.2.

Theorem 2 ([6])**.** *Let $(\mathscr{C}, I, \otimes)$ be a monoidal closed category with binary coproducts $(+)$, and let Σ be an endofunctor on $\mathscr{C}$ with an $(I/\mathscr{C})$-strength s. For all $X \in \mathscr{C}$, an initial $(\Sigma + I + X\otimes)$-algebra carries the structure of a free (Σ, s)-monoid on X.*

Proof (outline). For an initial $(\Sigma + I + X\otimes)$-algebra $\mathcal{M}X$ with structure

$$[\tau_X, \varepsilon_X, \alpha_X] : \Sigma\mathcal{M}X + I + X \otimes \mathcal{M}X \xrightarrow{\cong} \mathcal{M}X$$

there is a unique map $\varsigma_X : \mathcal{M}X \otimes \mathcal{M}X \longrightarrow \mathcal{M}X$ such that

$$\begin{array}{ccccc} \Sigma(\mathcal{M}X) \otimes \mathcal{M}X & \xrightarrow{s_{\mathcal{M}X,(\mathcal{M}X,\varepsilon_X)}} & \Sigma(\mathcal{M}X \otimes \mathcal{M}X) & \xrightarrow{\Sigma\varsigma_X} & \Sigma\mathcal{M}X \\ \downarrow{\scriptstyle \tau_X \otimes \mathcal{M}X} & & & & \downarrow{\scriptstyle \tau_X} \\ \mathcal{M}X \otimes \mathcal{M}X & & \xrightarrow[\varsigma_X]{\qquad\qquad} & & \mathcal{M}X \end{array}$$

$$\begin{array}{ccc} I \otimes \mathcal{M}X & & \\ \downarrow{\scriptstyle \varepsilon_X \otimes \mathcal{M}X} & \searrow{\scriptstyle\cong} & \\ \mathcal{M}X \otimes \mathcal{M}X & \xrightarrow[\varsigma_X]{} & \mathcal{M}X \end{array}$$

$$\begin{array}{ccc} (X \otimes \mathcal{M}X) \otimes \mathcal{M}X \ \cong X \otimes (\mathcal{M}X \otimes \mathcal{M}X) & \xrightarrow{X \otimes \varsigma_X} & X \otimes \mathcal{M}X \\ \downarrow{\scriptstyle \alpha_X \otimes \mathcal{M}X} & & \downarrow{\scriptstyle \alpha_X} \\ \mathcal{M}X \otimes \mathcal{M}X & \xrightarrow[\varsigma_X]{\qquad\qquad} & \mathcal{M}X \end{array}$$

One then shows that the structure $(\tau_X, \varepsilon_X, \varsigma_X)$ on $\mathcal{M}X$ is a free (Σ, s)-monoid on X, with universal map given by $(X \cong X \otimes I \xrightarrow{X \otimes \varepsilon_X} X \otimes \mathcal{M}X \xrightarrow{\alpha_X} \mathcal{M}X)$. $\square$

Note that the three conditions in the above proof outline amount to a specification of the monoid multiplication ς_X by parameterised structural recursion on the initial $(\Sigma + I + X\otimes)$-algebra.

Lemma 3. *Let $(\mathscr{C}, I, \otimes)$ be a monoidal closed category with binary coproducts $(+)$, and let Σ be an endofunctor* *on $\mathscr{C}$ with an $(I/\mathscr{C})$-strength s. Assume further that an initial $(\Sigma + I + X\otimes)$-algebra exists for all $X \in \mathscr{C}$.*

*Then, the forgetful functor (Σ, s)-**Mon** $\longrightarrow \mathscr{C}$ has a left adjoint, and the induced monad on $\mathscr{C}$ has underlying functor $\mathcal{M} = \mu F$ for $F : \mathscr{C} \times \mathscr{C} \longrightarrow \mathscr{C}$ given by $(X, Y) \mapsto \Sigma Y + I + X \otimes Y$.*

The unit of the monad $\eta : \mathrm{Id} \dot{\longrightarrow} \mathcal{M}$ is given by the universal maps $X \longrightarrow \mathcal{M}X$ and its multiplication $\sigma : \mathcal{M}\mathcal{M} \dot{\longrightarrow} \mathcal{M}$ by the unique maps σ_X such that

$$\begin{array}{ccc} \Sigma\mathcal{M}\mathcal{M}X & \xrightarrow{\Sigma\sigma_X} & \Sigma\mathcal{M}X \\ {\scriptstyle \tau_{\mathcal{M}X}}\downarrow & & \downarrow{\scriptstyle \tau_X} \\ \mathcal{M}\mathcal{M}X & \xrightarrow[\sigma_X]{} & \mathcal{M}X \end{array}$$

$$\begin{array}{ccc} & I & \\ {\scriptstyle \varepsilon_{\mathcal{M}X}}\swarrow & & \searrow{\scriptstyle \varepsilon_X} \\ \mathcal{M}\mathcal{M}X & \xrightarrow[\sigma_X]{} & \mathcal{M}X \end{array}$$

$$\begin{array}{ccc} \mathcal{M}X \otimes \mathcal{M}\mathcal{M}X & \xrightarrow{\mathcal{M}X \otimes \sigma_X} & \mathcal{M}X \otimes \mathcal{M}X \\ {\scriptstyle \alpha_{\mathcal{M}X}}\downarrow & & \downarrow{\scriptstyle \varsigma_X} \\ \mathcal{M}\mathcal{M}X & \xrightarrow[\sigma_X]{} & \mathcal{M}X \end{array}$$

These three conditions amount to a specification of σ_X by structural recursion on the initial $(\Sigma + I + X\otimes)$-algebra in terms of the free (Σ, s)-monoid structure on X.

The following result provides a general form of initial-algebra semantics suitable for applications.

Corollary 4. *Let $(\mathscr{C}, I, \otimes)$ be a monoidal closed category with finite coproducts and colimits of ω-chains, and let Σ be an ω-cocontinuous endofunctor on $\mathscr{C}$ with an $(I/\mathscr{C})$-strength s. Then, the forgetful functor (Σ, s)-**Mon** $\longrightarrow \mathscr{C}$ has a left adjoint, and the initial $(\Sigma + I)$-algebra carries an initial (Σ, s)-monoid structure.*

I.1.5. Strengths for parameterised initial-algebra functors

A general result for inducing strengths on parameterised initial-algebra functors is given. The reader is advised to consider it in the context of Section I.2.3.

Theorem 5. *Let $\mathscr{C}$ be a closed $\mathscr{V}$-action and let $\mathscr{D}$ be a $\mathscr{V}$-action for a monoidal category $(\mathscr{V}, I, \otimes)$. Furthermore, for functors $F, G : \mathscr{D} \times \mathscr{C} \longrightarrow \mathscr{C}$, let $\phi_{((D,C),g),X} : F(D, C) \odot X \dot{\longrightarrow} F(D \odot X, C \odot X) : G\text{-}\mathbf{Alg} \times \mathscr{V} \longrightarrow \mathscr{C}$, $\gamma_{D,C,X} : G(D, C) \odot X \dot{\longrightarrow} G(D \odot X, C \odot X) : \mathscr{D} \times \mathscr{C} \times \mathscr{V} \longrightarrow \mathscr{C}$, and $\eta : G \dot{\longrightarrow} F : \mathscr{D} \times \mathscr{C} \longrightarrow \mathscr{C}$ be natural transformations such that*

$$\begin{array}{ccc} G(D,C) \odot X & \xrightarrow{\gamma_{D,C,X}} & G(D \odot X, C \odot X) \\ {\scriptstyle \eta_{D,C} \odot X}\downarrow & & \downarrow{\scriptstyle \eta_{D \odot X, C \odot X}} \\ F(D,C) \odot X & \xrightarrow[\phi_{((D,C),g),X}]{} & F(D \odot X, C \odot X) \end{array}$$

*for all $((D, C), g) \in G$-**Alg** and $X \in \mathscr{V}$.*

Then, for every $D \in \mathscr{D}$ and $X \in \mathscr{V}$, there exists a unique map $\sigma_{D,X} : \mu F(D) \odot X \longrightarrow \mu F(D \odot X)$ in $\mathscr{C}$ such that

$$\begin{array}{ccccc} & & F(D \odot X, \mu F(D) \odot X) & & \\ & {\scriptstyle \phi_{M_F(D),X}}\nearrow & & \searrow{\scriptstyle F(D \odot X, \sigma_{D,X})} & \\ F(D, \mu F(D)) \odot X & & & & F(D \odot X, \mu F(D \odot X)) \\ {\scriptstyle \mu_D \odot X}\downarrow & & & & \downarrow{\scriptstyle \mu_{D \odot X}} \\ \mu F(D) \odot X & & \xrightarrow[\sigma_{D,X}]{} & & \mu F(D \odot X) \end{array}$$

where $M_F(D)$ is the G-algebra with carrier $\big(D, \mu F(D)\big)$ and structure map

$$G\big(D, \mu F(D)\big) \xrightarrow{\eta_{D,\mu F(D)}} F\big(D, \mu F(D)\big) \xrightarrow{\mu_D} \mu F(D)\,.$$

Moreover, $\sigma = \{\, \sigma_{D,X} \,\}_{D \in \mathscr{D}, X \in \mathscr{V}}$ is a natural transformation $\mu F(D) \odot X \dot{\longrightarrow} \mu F(D \odot X) : \mathscr{D} \times \mathscr{V} \longrightarrow \mathscr{C}$, and if, for all $D \in \mathscr{D}$,

$$\begin{array}{ccc} F(D, \mu F(D)) \odot I & \xrightarrow{\phi_{M_F(D),I}} & F(D \odot I, \mu F(D) \odot I) \\ & {\scriptstyle \cong}\searrow \quad \swarrow{\scriptstyle \cong} & \\ & F(D, \mu F(D)) & \end{array}$$

and, for all $D \in \mathscr{D}$ and $X, Y \in \mathscr{V}$,

$$\begin{aligned} & F(D, \mu F(D)) \odot (X \otimes Y) \xrightarrow{\cong} \big(F(D, \mu F(D)) \odot X\big) \odot Y \xrightarrow{\phi_{M_F(D),X} \odot Y} F(D \odot X, \mu F(D) \odot X) \odot Y \\ & \quad \xrightarrow{F(D \odot X, \sigma_{D,X}) \odot Y} F\big(D \odot X, \mu F(D \odot X)\big) \odot Y \xrightarrow{\phi_{M_F(D \odot X),Y}} F\big((D \odot X) \odot Y, \mu F(D \odot X) \odot Y\big) \\ = \; & F(D, \mu F(D)) \odot (X \otimes Y) \xrightarrow{\phi_{M_F(D),X \otimes Y}} F\big(D \odot (X \otimes Y), \mu F(D) \odot (X \otimes Y)\big) \xrightarrow{\cong} F\big((D \odot X) \odot Y, (\mu F(D) \odot X) \odot Y\big) \\ & \quad \xrightarrow{F((D \odot X) \odot Y, \sigma_{D,X} \odot Y)} F\big((D \odot X) \odot Y, \mu F(D \odot X) \odot Y\big) \end{aligned}$$

then σ is a $\mathscr{V}$-strength for $\mu F : \mathscr{D} \longrightarrow \mathscr{C}$.

Corollary 6. *For a functor $F : \mathscr{D} \times \mathscr{C} \longrightarrow \mathscr{C}$, every closed action on $\mathscr{C}$, action on $\mathscr{D}$, and strength for F induce a strength for $\mu F : \mathscr{D} \longrightarrow \mathscr{C}$.*

I.2. Model theory

This section gives a model-theoretic treatment of second-order syntax as presented in the Introduction. The development is carried out in the context of mono-sorted syntax.

However, it directly generalises to the multi-sorted case (for which see [4, 5, 19, 22]).

After briefly reviewing the mathematical model of [11] supporting the algebraic treatment of variable binding and substitution (Sections I.2.1 and I.2.2), I consider operations of meta-renaming (Section I.2.3) and meta-substitution (Section I.2.4).

I.2.1. Variable binding

Recall that the model of [11] is given by the functor category $\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{S}et}^{\mathsf{F}}$ for F the category of finite sets (over a fixed countably infinite set of variables) and functions between them. For $P \in \boldsymbol{\mathcal{F}}$ and $\Gamma \in \mathsf{F}$, it is convenient to write $\Gamma \vdash p : P$ for $p \in P(\Gamma)$, and think of p as an element of type P in context Γ. For every such element, then, and context renaming $\rho : \Gamma \longrightarrow \Gamma'$ in F, it is furthermore convenient to write $p[\rho]$ for $P\rho(p)$; so that we have that $\Gamma' \vdash p[\rho] : P$ and can express the functoriality of P by the equations $p[\mathrm{id}_\Gamma] = p$ and $p[\rho][\rho'] = p[\rho;\rho']$ for all $\rho : \Gamma \longrightarrow \Gamma'$ and $\rho' : \Gamma' \longrightarrow \Gamma''$ in F.

The crucial ingredient in the model $\boldsymbol{\mathcal{F}}$ for interpreting variable binding is the presence of the object V of variables, given by the embedding $\mathsf{F} \hookrightarrow \boldsymbol{\mathcal{S}et}$, that provides an arity for variable binding. Indeed, for $P \in \boldsymbol{\mathcal{F}}$, one can describe the exponential P^V as $P^V(\Gamma) = \{\, [(v)p]_\approx \mid \Gamma, v \vdash p : P \,\}$ where the *α-equivalence relation* $\approx$ is defined by setting $(v)p \approx (v')p'$ iff $p[\mathrm{id}_\Gamma, v \mapsto v'] = p'$ (see [7]).

I.2.2. Substitution

The main structure for interpreting (simultaneous) substitution is given by a substitution tensor product (see [6, 8, 22]). In the model $\boldsymbol{\mathcal{F}}$, this is explicitly defined, for $P, Q \in \boldsymbol{\mathcal{F}}$, as

$$(P \bullet Q)(\Gamma) = \int^{\Delta \in \mathsf{F}} P(\Delta) \times (Q\Gamma)^\Delta$$

and consists of equivalence classes of triples

$$(\, \Delta \in \mathsf{F} \,,\; \Delta \vdash p : P \,,\; (\Gamma \vdash q_v : Q)_{v \in \Delta} \,)$$

under the equivalence relation generated by identifying $(\Delta, p, (q_{\rho v})_{v \in \Delta})$ and $(\Delta', p[\rho], (q_{v'})_{v' \in \Delta'})$ for all $\rho : \Delta' \longrightarrow \Delta$ in F. The substitution tensor product is closed and has the object V of variables as unit.

A monoid structure for the substitution tensor product on an object $P \in \boldsymbol{\mathcal{F}}$ amounts to giving functions

$$\Gamma \longrightarrow P(\Gamma) : v \longmapsto \varpi_v$$

and

$$P(\Delta) \times (P\Gamma)^\Delta \longrightarrow P(\Gamma) : p, \sigma \longmapsto p[\sigma]$$

for all $\Gamma, \Delta \in \mathsf{F}$, such that

$$v \in \Gamma, \rho \in \mathsf{F}(\Gamma, \Gamma') \;\vdash\; \varpi_v[\rho] = \varpi_{\rho v}$$

$$w \in \Delta, \sigma \in (P\Gamma)^\Delta \;\vdash\; \varpi_w[\sigma] = \sigma_w$$

$$p \in P\Delta \;\vdash\; p[\varpi] = p$$

$$p \in P\Delta, \rho \in \mathsf{F}(\Delta, \Delta'), \sigma \in (P\Gamma)^{\Delta'} \;\vdash\; p[\rho][\sigma] = p[\sigma_\rho]$$

$$p \in P\Delta, \sigma \in (P\Gamma)^\Delta, \rho \in \mathsf{F}(\Gamma, \Gamma') \vdash p[\sigma][\rho] = p[v \mapsto \sigma_v[\rho]]$$

$$p \in P\Delta, \sigma \in (P\Gamma)^\Delta, \varsigma \in (P\Gamma')^\Gamma \vdash p[\sigma][\varsigma] = p[v \mapsto \sigma_v[\varsigma]]$$

(See [7] for details.)

The specification of substitution for algebras of an endofunctor as axiomatised in Definition 1 requires that of a strength. In the model $\boldsymbol{\mathcal{F}}$, we have basic strengths for the substitution tensor product as follows:

1. $\boldsymbol{\mathcal{F}}$-strengths $\coprod_{i \in I}(P_i) \bullet Q \xrightarrow{\cong} \coprod_{i \in I}(P_i \bullet Q)$ for $\coprod$ and $(P \times P') \bullet Q \xrightarrow{\cong} (P \bullet Q) \times (P' \bullet Q)$ for $\times$ of type $\boldsymbol{\mathcal{F}}^2 \longrightarrow \boldsymbol{\mathcal{F}}$,
2. a $(V/\boldsymbol{\mathcal{F}})$-strength $s_{P,(Q,\varpi)} : (P)^V \bullet Q \overset{.}{\longrightarrow} (P \bullet Q)^V$ for $(-)^V$ of type $\boldsymbol{\mathcal{F}} \longrightarrow \boldsymbol{\mathcal{F}}$.

It is instructive to analyse the last strength. In elementary terms, it is induced by the mapping

$$(\nu)p \,;\, (q_v)_{v \in \Delta}$$
$$\mapsto (\nu')\big(p[\mathrm{id}_\Delta, \nu \mapsto \nu'] \,;\, (q_v[\Gamma \hookrightarrow (\Gamma, \nu')])_{v \in \Delta}, q'\big) \;,$$

for $\Delta, \nu \vdash p : P$ and $\Gamma \vdash q_v : Q$, where q' is the image of ν' under the pointed structure $\varpi : (\Gamma, \nu') \longrightarrow Q(\Gamma, \nu')$ of Q. Thus, one sees that the effect of the strength is intuitively to push the explicit substitution $(q_v)_{v \in \Delta}$ within the scope of the binder in the abstraction $(\nu)p$, by possibly renaming it to an abstraction with a fresh binder ν' to avoid capturing free variables, suitably renaming p and the q_v and extending the explicit substitution with a suitable assignment q' for ν'. A similar, though syntactic, mechanism is employed in the definition of substitution in the presence of binding constructs.

Corollary 7. *Every endofunctor Σ on $\boldsymbol{\mathcal{F}}$ built by composition from* Id, $\coprod$, $\times$, $(-)^V$ *comes equipped with a canonical $(V/\boldsymbol{\mathcal{F}})$-strength s for which the forgetful functor (Σ, s)-$\mathbf{Mon} \longrightarrow \boldsymbol{\mathcal{F}}$ has a left adjoint, with the functor underlying the induced monad being given by $\mathcal{M} = \mu F$ for $F : \boldsymbol{\mathcal{F}}^2 \longrightarrow \boldsymbol{\mathcal{F}} : (P, Q) \mapsto \Sigma Q + V + P \bullet Q$.*

In particular, this result applies to every endofunctor arising from an algebraic binding signature (see [11] and Section I.3). Moreover, one can use the theory of [9] to show that the forgetful functor (Σ, s)-$\mathbf{Mon} \longrightarrow \boldsymbol{\mathcal{F}}$ is monadic, but I will not dwell on this here.

I.2.3. Meta-renaming

Let Σ be an endofunctor on $\boldsymbol{\mathcal{F}}$, and let $\mathcal{M} = \mu F$ for $F : \boldsymbol{\mathcal{F}}^2 \longrightarrow \boldsymbol{\mathcal{F}} : (P, Q) \mapsto \Sigma Q + V + P \bullet Q$. The operation of meta-renaming for $\mathcal{M}$ internalises its functorial action and, as such (see [15]), it arises from a cartesian strength

$$\mathcal{M}(P)\times Q \stackrel{\cdot}{\longrightarrow} \mathcal{M}(P\times Q)\ .$$

Here, and in the light of Section I.1.4, I show how every cartesian strength on Σ induces one on $\mathcal{M}$. The crucial construction for achieving this is the law for distributing the categorical product over the substitution tensor product of the following proposition.

Proposition 8. *The family of maps* $d_{P,(P',\varpi),Q} : (P\bullet P')\times Q \longrightarrow (P\times Q)\bullet(P'\times Q) : \boldsymbol{\mathcal{F}}\times(V/\boldsymbol{\mathcal{F}})\times\boldsymbol{\mathcal{F}} \longrightarrow \boldsymbol{\mathcal{F}}$ *induced by the mapping*

$$\begin{array}{l}(\Delta\,;\,p\,;\,(p'_v)_{v\in\Delta})\,,\,q\\ \quad\mapsto\ \ \Gamma,\Delta\,;\\ \qquad\quad\big(p[\Delta\hookrightarrow(\Gamma,\Delta)],q[\Gamma\hookrightarrow(\Gamma,\Delta)]\big)\,;\\ \qquad\quad(\varpi_\Gamma(v),q)_{v\in\Gamma},(p'_v,q)_{v\in\Delta}\end{array}$$

where $p\in P(\Delta)$, $p'_v\in P'(\Gamma)$, $q\in Q(\Gamma)$, *defines a natural transformation such that*

$$\begin{array}{ccc}(P\bullet P')\times 1 & \xrightarrow{d_{P,(P',\varpi),1}} & (P\times 1)\bullet(P'\times 1)\\ & {\searrow}^{\cong}\quad P\bullet P'\quad {}^{\cong}{\swarrow} & \end{array}$$

and

$$\begin{array}{ccc}(P\bullet P')\times(Q\times Q') & & \\ \Big\downarrow{\cong} & \searrow^{d_{P,(P',\varpi),Q\times Q'}} & \\ & & (P\times(Q\times Q'))\bullet(P'\times(Q\times Q'))\\ ((P\bullet P')\times Q)\times Q' & & \Big\downarrow{\cong}\\ \Big\downarrow{d_{P,(P',\varpi),Q}\times Q'} & & ((P\times Q)\times Q')\bullet((P'\times Q)\times Q')\\ ((P\times Q)\bullet(P'\times Q))\times Q' & & \Big\downarrow{((P\times Q)\times Q')\bullet(h\times Q')}\\ \Big\downarrow{((P\times Q)\bullet h)\times Q'} & & ((P\times Q)\times Q')\bullet(P''\times Q')\\ & \swarrow_{d_{P\times Q,(P'',\varpi'),Q'}} & \\ ((P\times Q)\bullet P'')\times Q' & & \end{array}$$

for all $h : P'\times Q \longrightarrow P''$ *in* $\boldsymbol{\mathcal{F}}$ *for which*

$$\begin{array}{ccc}V\times Q & \xrightarrow{\pi_1} & V\\ \Big\downarrow{\varpi\times Q} & & \Big\downarrow{\varpi'}\\ P'\times Q & \xrightarrow[h]{} & P''\end{array}$$

The next two results provide evidence of the fundamental character of the above construction.

Proposition 9. *The canonical tensorial* $(V/\boldsymbol{\mathcal{F}})$*-strength* $s_{P,(Q,\varpi)} : P^V\bullet Q \stackrel{\cdot}{\longrightarrow} (P\bullet Q)^V$ *is the exponential transpose of the composite*

$$(P^V\bullet Q)\times V \xrightarrow{d_{P^V,(Q,\varpi),V}} (P^V\times V)\bullet(Q\times V) \xrightarrow{\epsilon_{P,V}\bullet\pi_1} P\bullet Q\ .$$

Theorem 10. *For* $R\in\boldsymbol{\mathcal{F}}$, *the exponential transpose of the family of natural transformations*

$$\begin{array}{ccc}(P^R\bullet Q)\times R & \xrightarrow{\qquad\qquad\cdot\qquad\qquad} & P\bullet Q\\ & {\searrow}_{d_{P^R,(Q,\varpi),R}}\quad (P^R\times R)\bullet(Q\times R)\quad {\nearrow}_{\epsilon_{P,R}\bullet\pi_1} & \end{array}$$

yields a tensorial $(V/\boldsymbol{\mathcal{F}})$*-strength for the endofunctor* $(-)^R$ *of type* $\boldsymbol{\mathcal{F}}\longrightarrow\boldsymbol{\mathcal{F}}$.

Proposition 8, used in the context of Theorem 5, yields the following.

Corollary 11. *For an endofunctor* Σ *on* $\boldsymbol{\mathcal{F}}$ *let* $\mathcal{M}=\mu F$ *for* $F:\boldsymbol{\mathcal{F}}^2\longrightarrow\boldsymbol{\mathcal{F}}:(P,Q)\longmapsto\Sigma Q+V+P\bullet Q$. *Then, every cartesian strength* c *for* Σ *induces a cartesian strength* $\widetilde{c}$ *for* $\mathcal{M}$ *given by the unique maps* $\widetilde{c}_{P,Q}:\mathcal{M}(P)\times Q\longrightarrow\mathcal{M}(P\times Q)$ *such that*

$$\begin{array}{ccccc}\Sigma(\mathcal{M}P)\times Q & \xrightarrow{c_{\mathcal{M}P,Q}} & \Sigma(\mathcal{M}(P)\times Q) & \xrightarrow{\Sigma(\widetilde{c}_{P,Q})} & \Sigma\mathcal{M}(P\times Q)\\ \Big\downarrow{\tau_P\times Q} & & & & \Big\downarrow{\tau_{P\times Q}}\\ \mathcal{M}(P)\times Q & & \xrightarrow[\widetilde{c}_{P,Q}]{\qquad\qquad\qquad} & & \mathcal{M}(P\times Q)\end{array}$$

$$\begin{array}{ccc}V\times Q & \xrightarrow{\pi_1} & V\\ \Big\downarrow{\varepsilon_P\times Q} & & \Big\downarrow{\varepsilon_{P\times Q}}\\ \mathcal{M}(P)\times Q & \xrightarrow[\widetilde{c}_{P,Q}]{} & \mathcal{M}(P\times Q)\end{array}$$

$$\begin{array}{ccccc} & & (P\times Q)\bullet(\mathcal{M}(P)\times Q) & & \\ & \nearrow^{d_{P,(\mathcal{M}P,\varepsilon_P),Q}} & & \searrow^{(P\times Q)\bullet\widetilde{c}_{P,Q}} & \\ (P\bullet\mathcal{M}P)\times Q & & & & (P\times Q)\bullet\mathcal{M}(P\times Q)\\ \Big\downarrow{\alpha_P\times Q} & & & & \Big\downarrow{\alpha_{P\times Q}}\\ \mathcal{M}(P)\times Q & & \xrightarrow[\widetilde{c}_{P,Q}]{\qquad\qquad\qquad} & & \mathcal{M}(P\times Q)\end{array}$$

Thus, whenever Σ has a cartesian strength c, it follows that $\mathcal{M}$ acquires an internal *meta-renaming structure* (see [7]):

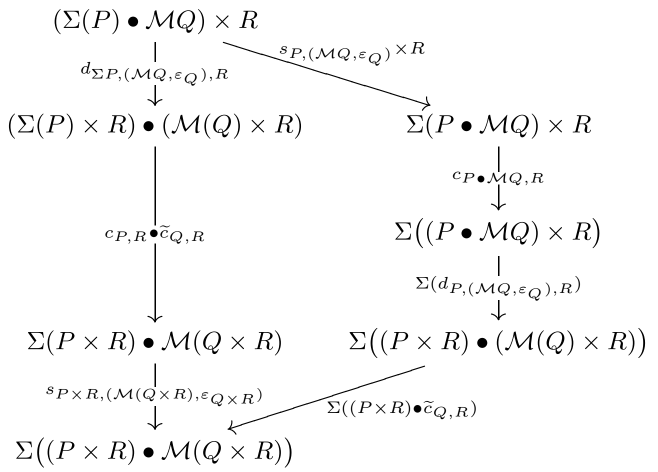
$$\begin{array}{ccccc}Q & \xrightarrow{\eta_Q} & \mathcal{M}(Q) & \xleftarrow{\qquad\rho_{P,Q}\qquad} & \mathcal{M}(P)\times Q^P\\ & & {\nwarrow}_{\mathcal{M}(\epsilon_{P,Q})}\quad \mathcal{M}(P\times Q^P)\quad {\swarrow}_{\widetilde{c}_{P,Q^P}} & & \end{array}$$

for all $P,Q\in\boldsymbol{\mathcal{F}}$.

I.2.4. Meta-substitution

Let Σ be an endofunctor on $\boldsymbol{\mathcal{F}}$, and let $\mathcal{M}=\mu F$ for $F:\boldsymbol{\mathcal{F}}^2\longrightarrow\boldsymbol{\mathcal{F}}:(P,Q)\longmapsto\Sigma Q+V+P\bullet Q$. We have just seen that a cartesian strength for Σ induces a meta-renaming structure for $\mathcal{M}$. We have also seen in Section I.1.4 that a tensorial $(V/\boldsymbol{\mathcal{F}})$-strength equips $\mathcal{M}$ with a monad structure arising from substitution. I show next that when the cartesian and tensorial strengths are compatible, then so are the meta-renaming and monad structures; in which case, one further obtains a meta-substitution structure.

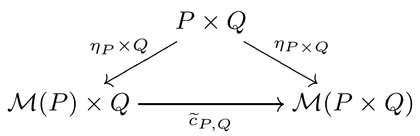
Definition 12. A cartesian strength c and a tensorial $(V/\boldsymbol{\mathcal{F}})$-strength s for Σ are said to be *compatible* whenever

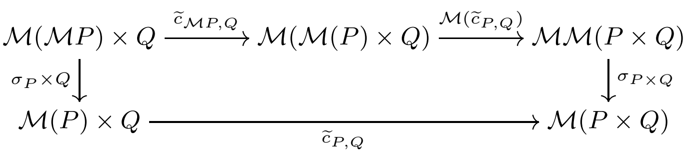


for all $P, Q, R \in \boldsymbol{\mathcal{F}}$.

This definition is justified by the following result.

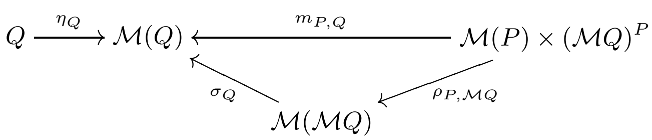
Theorem 13. *For every compatible cartesian strength c and tensorial $(V/\boldsymbol{\mathcal{F}})$-strength s for Σ, the induced cartesian strength $\widetilde{c}$ for $\mathcal{M}$ is compatible with the free (Σ, s)-monoid monad structure. That is,*





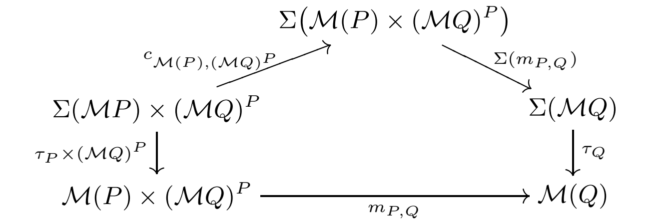
for all $P, Q \in \boldsymbol{\mathcal{F}}$.

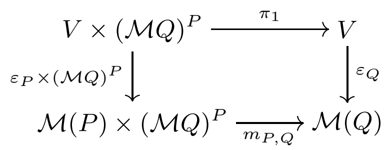
In the situation of the theorem, thus, $\mathcal{M}$ comes equipped with a *meta-substitution structure* (see [7]):

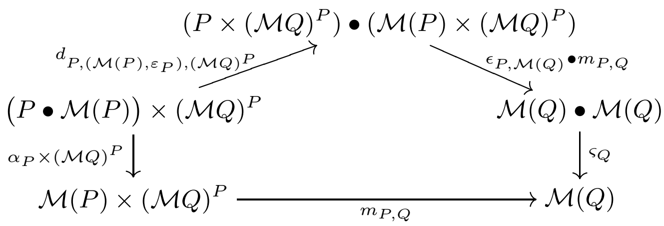


for all $P, Q \in \boldsymbol{\mathcal{F}}$.

The meta-substitution operation $m_{P,Q}$ is universally characterised as the unique map such that





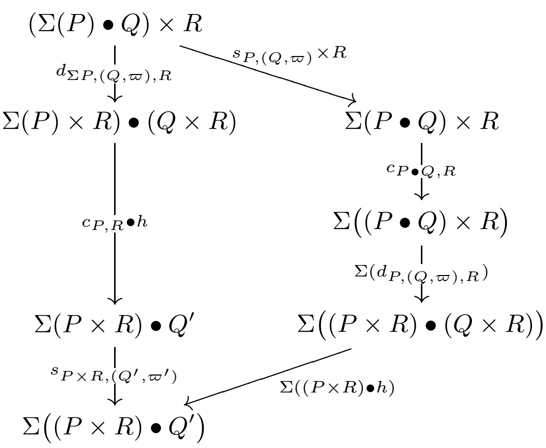


These three conditions amount to a specification of $m_{P,Q}$ by parameterised structural recursion on the initial $(\Sigma + V + P\bullet)$-algebra in terms of the free (Σ, s)-monoid structure on Q.

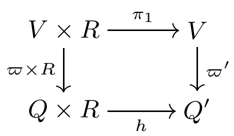
Main examples of meta-substitution structure arise from the following result.

Theorem 14. *Every endofunctor Σ on $\boldsymbol{\mathcal{F}}$ built by composition from* Id, $\coprod$, $\times$, $(-)^V$ *comes equipped with a canonical cartesian strength c that is compatible with the canonical tensorial $(V/\boldsymbol{\mathcal{F}})$-strength s.*

Proof (outline). One shows that, for all $P, R \in \boldsymbol{\mathcal{F}}$ and $(Q, \varpi), (Q', \varpi') \in V/\boldsymbol{\mathcal{F}}$,



for all $h : Q \times R \longrightarrow Q'$ in $\boldsymbol{\mathcal{F}}$ for which



$\square$

I.3. Syntactic theory

I will now proceed to synthesise syntactic structure from the preceding model theory. To this end, I consider a class of syntactic signatures that induce signature endofunctors for which the monad of free algebras with substitution embodies second-order abstract syntax (see also [13]). Indeed, we will see that: (i) syntactic terms with variable binding (subject to α-equivalence) and built from term metavariables arise as free algebras with substitution; (ii) the model-theoretic substitution structure amounts to the syntactic operation of simultaneous capture-avoiding substitution; (iii) the model-theoretic meta-substitution structure provides a syntactic operation of substitution for term metavariables.

Binding signatures. A *binding signature* (see, *e.g.*, [1]) Σ is given by a family of sets $\{\, \Sigma(n) \,\}_{n \in \mathbb{N}^*}$. Every such induces the signature endofunctor

$$\Sigma(P) = \coprod_{n \in \mathbb{N}^*} \Sigma(n) \times \prod_{i \in |n|} P^{V^{n_i}}$$

on $\boldsymbol{\mathcal{F}}$.

By Corollary 7 and Theorems 14 and 13, signature endofunctors admit free algebras with both substitution and meta-substitution structures.

Syntax. For a signature endofunctor Σ, the carrier $\mathcal{M}(X) \in \boldsymbol{\mathcal{F}}$ of the free Σ-algebra with substitution structure on $X \in \boldsymbol{\mathcal{F}}$ is constructed as the colimit of the ω-chain $\langle {F_X}^n(0) \rangle_{n \in \omega}$ for $F_X(Y) = \Sigma Y + V + X \bullet Y$.

We wish to consider terms in *term-metavariable contexts*. Such contexts are defined as families $\mathfrak{X} \in \boldsymbol{\mathcal{S}et}^{\mathbb{N}}$, where one interprets $\mathfrak{X}(n)$ as the set of term metavariables of *valence* n. Every term-metavariable context $\mathfrak{X}$ freely induces a term-metavariable object

$$\overline{\mathfrak{X}} = \coprod_{n \in \mathbb{N}} \mathfrak{X}(n) \times V^n$$

in $\boldsymbol{\mathcal{F}}$. It follows that $\mathcal{M}(\overline{\mathfrak{X}})$ can be syntactically presented by the following rules:

$$\frac{}{\Gamma \vdash \lfloor x \rfloor : \mathcal{M}(\overline{\mathfrak{X}})} \; (x \in \Gamma)$$

$$\frac{\Gamma, x_1^{(i)}, \ldots, x_{n_i}^{(i)} \vdash t_i : \mathcal{M}(\overline{\mathfrak{X}}) \; (i = 1, \ldots, |n|)}{\Gamma \vdash \mathsf{f}\big(\ldots, (x_1^{(i)}, \ldots, x_{n_i}^{(i)}) t_i, \ldots\big) : \mathcal{M}(\overline{\mathfrak{X}})} \begin{pmatrix} n \in \mathbb{N}^* \\ \mathsf{f} \in \Sigma(n) \end{pmatrix}$$

$$\frac{\Gamma \vdash t_i : \mathcal{M}(\overline{\mathfrak{X}}) \;\; (i = 1, \ldots, n)}{\Gamma \vdash \mathrm{M}[t_1, \ldots, t_n] : \mathcal{M}(\overline{\mathfrak{X}})} \; \big(n \in \mathbb{N}, \mathrm{M} \in \mathfrak{X}(n)\big)$$

where terms are identified by α-equivalence according to the convention that in $\mathsf{f}\big(\ldots, (x_1^{(i)}, \ldots, x_{n_i}^{(i)}) t_i, \ldots\big)$ the $x_j^{(i)}$ are bound in t_i.

Substitution. The operation of substitution

$$\mathcal{M}(\overline{\mathfrak{X}}) \bullet \mathcal{M}(\overline{\mathfrak{X}}) \longrightarrow \mathcal{M}(\overline{\mathfrak{X}})$$

provides functions

$$\mathcal{M}(\overline{\mathfrak{X}})_\Delta \times (\mathcal{M}(\overline{\mathfrak{X}})_\Gamma)^\Delta \longrightarrow \mathcal{M}(\overline{\mathfrak{X}})_\Gamma \quad (\Gamma, \Delta \in \mathsf{F})$$

mapping

$$\Delta \vdash t : \mathcal{M}(\overline{\mathfrak{X}}) \quad \text{and} \quad \{\, \Gamma \vdash u_z : \mathcal{M}(\overline{\mathfrak{X}}) \,\}_{z \in \Delta}$$

to

$$\Gamma \vdash t\{\, u_z \,\}_{z \in \Delta} : \mathcal{M}(\overline{\mathfrak{X}})$$

given by:

- $\lfloor x \rfloor \{\, u_z \,\}_z = u_x$.
- $\mathsf{f}(\ldots, (x_1^{(i)}, \ldots, x_{n_i}^{(i)}) t_i, \ldots) \{\, u_z \,\}_{z \in \Delta}$
  $= \mathsf{f}\big(\ldots, (y_1^{(i)}, \ldots, y_{n_i}^{(i)})\, t_i\{\, u'_z \,\}_{z \in (\Delta, x_1^{(i)}, \ldots, x_{n_i}^{(i)})}, \ldots\big)$
  with $y_j^{(i)} \notin \Gamma$ and where u'_z is $\lfloor y_j^{(i)} \rfloor$ if $z = x_j^{(i)}$ and u_z otherwise.
- $\mathrm{M}[\ldots, t_i, \ldots] \{\, u_z \,\}_z = \mathrm{M}[\ldots, t_i\{\, u_z \,\}_z, \ldots]$.

Meta-substitution. The operation of meta-substitution

$$\mathcal{M}(\overline{\mathfrak{X}}) \times \big(\mathcal{M}(\overline{\mathfrak{X}'})\big)^{\overline{\mathfrak{X}}} \longrightarrow \mathcal{M}(\overline{\mathfrak{X}'})$$

yields functions

$$\mathcal{M}(\overline{\mathfrak{X}})_\Gamma \times \prod_{n \in \mathbb{N}, \mathrm{M} \in \mathfrak{X}(n)} \big(\mathcal{M}(\overline{\mathfrak{X}'})^{V^n}\big)_\Gamma \longrightarrow \mathcal{M}(\overline{\mathfrak{X}'})_\Gamma$$

mapping

$$\Gamma \vdash t : \mathcal{M}(\overline{\mathfrak{X}})$$

and

$$\Big\{\, (x_1^{(\mathrm{M})}, \ldots, x_n^{(\mathrm{M})}) t_{\mathrm{M}} \mid \Gamma, x_1^{(\mathrm{M})}, \ldots, x_n^{(\mathrm{M})} \vdash t_{\mathrm{M}} : \mathcal{M}(\overline{\mathfrak{X}'}) \,\Big\}_{\substack{n \in \mathbb{N} \\ \mathrm{M} \in \mathfrak{X}(n)}}$$

to

$$\Gamma \vdash t \Big\{\, (x_1^{(\mathrm{M})}, \ldots, x_n^{(\mathrm{M})}) t_{\mathrm{M}} \,\Big\}_{n \in \mathbb{N}, \mathrm{M} \in \mathfrak{X}(n)} : \mathcal{M}(\overline{\mathfrak{X}'})$$

given by:

- $\lfloor x \rfloor \,\big\{\, (\vec{x}^{(\mathrm{M})}) t_{\mathrm{M}} \,\big\}_{\mathrm{M}} = \lfloor x \rfloor$.
- $\mathsf{f}\big(\ldots, (\vec{x}) t, \ldots\big) \,\big\{\, (\vec{x}^{(\mathrm{M})}) t_{\mathrm{M}} \,\big\}_{\mathrm{M}}$
  $= \mathsf{f}\,\big(\ldots, (\vec{x})\, t\big\{\, (\vec{x}^{(\mathrm{M})}) t_{\mathrm{M}} \,\big\}_{\mathrm{M}}, \ldots\big)$.
- $\mathrm{N}[t_1, \ldots, t_n] \,\big\{\, (\vec{x}^{(\mathrm{M})}) t_{\mathrm{M}} \,\big\}_{\mathrm{M}} = t_{\mathrm{N}}\{\, u_z \,\}_{z \in (\Gamma, \vec{x}^{(\mathrm{N})})}$
  where u_z is $t_i\big\{\, (\vec{x}^{(\mathrm{M})}) t_{\mathrm{M}} \,\big\}_{\mathrm{M}}$ if $z = x_i^{(\mathrm{N})}$ and $\lfloor z \rfloor$ otherwise.

Of course, the facts that substitution and meta-substitution are well-defined (in that they respect their corresponding typing) and satisfy their specifications (in that they satisfy their respective monoid laws) is a direct consequence of the mathematical theory.

As a final remark, I note that the characterisation of free algebras with substitution as initial algebras leads to an induction proof principle [17] for reasoning about second-order syntax (see [7], and also [21]).

II. Dependently-sorted abstract syntax

This second part of the paper initiates the development of algebraic models for dependently-sorted syntax (though see also [6, 8]).

As a matter of motivation and illustration, in this extended abstract I mainly focus on algebraic models with substitution in the context of simple dependent sorts (Sections II.1–II.3), and only sketch the general case (Section II.4). In the same vein, I also restrict attention to the case of first-order syntax. However, the models (which generalise that of [11]) embody enough structure (in the form of suitable arity objects) to accommodate binding operators. Details of the overall development will appear elsewhere.

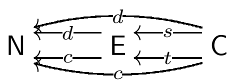
II.1. Simple dependent sorts

The first ingredient needed to provide a treatment of dependently-sorted syntax is a mathematical formulation of system of dependent sorts.

Simple sort dependency. The approach to dependent sorts of this section stems from the work of Makkai [18]. I motivate it here by considering the example of the system of dependent sorts needed for the specification of 2-dimensional graphs, where there is a sort N of nodes, a sort E of edges depending on the sort N of nodes by means of domain/codomain dependencies, and a sort C of 2-cells depending on the sorts N of nodes and E of edges by means of suitably compatible domain/codomain and source/target dependencies. Syntactically, this may be expressed by sort judgements along the following lines (see, *e.g.*, [3]):

$$\begin{cases} & \vdash \ \mathsf{N} \text{ sort} \\ d, c : \mathsf{N} & \vdash \ \mathsf{E}(d,c) \text{ sort} \\ d, c : \mathsf{N}, s, t : \mathsf{E}(d,c) & \vdash \ \mathsf{C}(d,c,s,t) \text{ sort} \end{cases}$$

Such syntactic representations do not directly reflect the mathematical structure of dependent sorts and, to this end, it is better to consider graphical representations. These turn out to be certain *simple* categories [18, §1]; *viz.*, one-way [16], skeletal, with finite fan-out [20]. For instance, the graphical representation of the above system of dependent sorts is the simple category

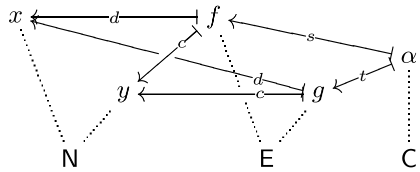
S = 

$$d \circ s = d \circ t = d \qquad c \circ s = c \circ t = c$$

Contexts. The graphical view of systems of dependent sorts as simple categories $\mathbb{S}$ leads to a straightforward notion of context for them; *viz.*, *finite functors* $\mathbb{S} \longrightarrow \mathcal{S}\!et$, see [18, §4]. For example, the syntactic context

$$x, y : \mathsf{N}, f, g : \mathsf{E}(x,y), \alpha : \mathsf{C}(x,y,f,g) \qquad (3)$$

amounts to the finite functor $\mathsf{S} \longrightarrow \mathcal{S}\!et$ with elements x, y, f, g, α depicted by the following graphic



Note that the variations of the context (3) obtained by permuting x and y and/or f and g have the same graphical representation.

The full subcategory of $\mathcal{S}\!et^{\mathbb{S}}$ consisting of the finite functors is denoted $\underline{\mathrm{Fin}}[\mathbb{S}, \mathcal{S}\!et]$. Here I take the category of elements $\boldsymbol{\mathcal{E}}(\Gamma)$ of a functor $\Gamma : \mathbb{S} \longrightarrow \mathcal{S}\!et$ to have set of objects $\mathrm{E}(\Gamma) = \{ (x : S) \mid S \in \mathbb{S}, x \in \Gamma(S) \}$ and morphisms $s : (x : S) \longrightarrow (\Gamma(s)(x) : S')$ in $\boldsymbol{\mathcal{E}}(\Gamma)$ for all $s : S \longrightarrow S'$ in $\mathbb{S}$, and say that the functor Γ is *finite* whenever its set of elements $\mathrm{E}(\Gamma)$ is.

Simple dependent sorts. A *simple system of dependent sorts* is defined to be a countable sequence $\big(\Gamma_i \vdash \mathsf{S}_i\big)_{i \geq 1}$ such that (i) $\mathsf{S}_i \neq \mathsf{S}_j$ for all $i \neq j$ and (ii) $\Gamma_i \in \underline{\mathrm{Fin}}[\mathbb{S}_{i-1}, \mathcal{S}\!et]$ for all $i \geq 1$, where the sequence of simple categories $(\mathbb{S}_i)_{i \geq 0}$ is inductively defined by setting $\mathbb{S}_0$ to be the empty category and $\mathbb{S}_i$, for $i \geq 1$, to be the category obtained from $\mathbb{S}_{i-1}$ by adding the object S_i together with morphisms $x : \mathsf{S}_i \longrightarrow \mathsf{S}$ for all $(x : \mathsf{S}) \in \boldsymbol{\mathcal{E}}(\Gamma_i)$ subject to the following dependency compatibility condition:

$$s \circ x = x' : \mathsf{S}_i \longrightarrow \mathsf{S}' \text{ in } \mathbb{S}_i$$

for all $s : (x : \mathsf{S}) \longrightarrow (x' : \mathsf{S}')$ in $\boldsymbol{\mathcal{E}}(\Gamma_i)$. Of course, the simple category associated to a simple system of dependent sorts $\big(\Gamma_i \vdash \mathsf{S}_i\big)_{i \geq 1}$ is given by $\bigcup_{i \geq 0} \mathbb{S}_i$.

Simple systems of dependent sorts are simple in two respects: (i) they correspond to countable simple categories and (ii) coincide up to isomorphism with the syntactic sort structures of Cartmell [3] without operators.

II.2. Algebraic models

I now show how signatures are to be interpreted algebraically. I will do this in the context of the dependently-sorted algebraic theories of Cartmell [3], familiarity with which is assumed.

A *simple dependently-sorted signature* is given by: (i) a countable sequence of introductory sort judgements $(\Gamma_i \vdash S_i)_{i \geq 1}$ such that every $(\Gamma_{n+1} \vdash S_{n+1})$ is derivable from $(\Gamma_1 \vdash S_1, \ldots, \Gamma_n \vdash S_n)$; and (ii) a countable sequence of introductory operator judgements $(\Delta_i \vdash F_i)_{i \geq 1}$ such that every $(\Delta_{n+1} \vdash F_{n+1})$ is derivable from $(\Gamma_i \vdash S_i)_{i \geq 1}$ and $(\Delta_1 \vdash F_1, \ldots, \Delta_n \vdash F_n)$.

Example 15. An illustrative fragment of a simple signature for lists follows.

$$\{ \ \vdash \mathsf{A}\ \mathrm{sort}\,, \quad \vdash \mathsf{N}\ \mathrm{sort}\,, \quad x : \mathsf{N} \vdash \mathsf{L}(x)\ \mathrm{sort}$$

$$\left\{ \begin{array}{rcl} n : \mathsf{N} & \vdash & \mathsf{succ}(n) : \mathsf{N} \\ x : \mathsf{A}, n : \mathsf{N}, \ell : \mathsf{L}(n) & \vdash & \mathsf{cons}(x, n, \ell) : \mathsf{L}(\mathsf{succ}(n)) \\ n : \mathsf{N}, \ell : \mathsf{L}(\mathsf{succ}(n)) & \vdash & \mathsf{tail}(n, \ell) : \mathsf{L}(n) \end{array} \right.$$

Note that I use the formal, rather than informal, syntax of [3].

The interpretation of a simple dependently-sorted signature takes place in a category with finite limits and an initial object, say $\mathscr{C}$, and is given in stages as follows. First, one obtains a simple category $\mathbb{S}$ from the system of dependent sorts as explained in the previous section, and considers $\mathscr{C}^{\mathbb{S}}$ as universe of discourse. Then, the operator judgement $(\Delta_1 \vdash F_1)$ provides a signature endofunctor Σ_1 on $\mathscr{C}^{\mathbb{S}}$ together with a category of algebraic models $\Sigma_1\text{-}\mathbf{Mod} \hookrightarrow \Sigma_1\text{-}\mathbf{Alg}$. More generally, each $(\Delta_{n+1} \vdash F_{n+1})$ provides a signature functor $\Sigma_{n+1} : \Sigma_n\text{-}\mathbf{Mod} \longrightarrow \mathscr{C}^{\mathbb{S}}$ together with a category of algebraic models $\Sigma_{n+1}\text{-}\mathbf{Mod} \hookrightarrow \Sigma_{n+1}\text{-}\mathbf{Alg}$ equipped with a forgetful functor $\Sigma_{n+1}\text{-}\mathbf{Mod} \longrightarrow \Sigma_n\text{-}\mathbf{Mod}$. Finally, the model of the signature is the limit of

$$\mathscr{C}^{\mathbb{S}} \longleftarrow \Sigma_1\text{-}\mathbf{Mod} \longleftarrow \cdots \longleftarrow \Sigma_n\text{-}\mathbf{Mod} \longleftarrow \cdots$$

As a notational convention, let $\Sigma_0\text{-}\mathbf{Mod} = \mathscr{C}^{\mathbb{S}}$ and let X be the object of $\mathscr{C}^{\mathbb{S}}$ underlying an algebraic model $\underline{X} \in \Sigma_{n-1}\text{-}\mathbf{Mod}$. The signature functor Σ_n induced by an operator judgement $\big(\Delta \vdash \mathsf{f}(\ldots) : \mathsf{S}(t_1, \ldots, t_k)\big)$ has action given by setting:

- $(\Sigma_n \underline{X})_s = [\![t_i]\!]_{\underline{X}} : [\![\Delta]\!]_{\underline{X}} \longrightarrow X_{\mathsf{S}_i}$ for all non-identity maps $s : \mathsf{S} \to \mathsf{S}_i$;
- $(\Sigma_n \underline{X})_s = X_s$ for all non-identity maps s in the image of the forgetful functor $\mathsf{S}/\mathbb{S} \to \mathbb{S}$; and
- $(\Sigma_n \underline{X})_s = \mathrm{id}_0$ for all other non-identity maps.

Here the empty context is interpreted as the terminal object and a context $\big(\Delta', x : \mathsf{S}'(\ldots, t_i', \ldots)\big)$ as the limit of the diagram

$$\begin{array}{ccc} [\![\Delta']\!]_{\underline{X}} & & X_{\mathsf{S}'} \\ & {\scriptstyle [\![t_i']\!]_{\underline{X}}} \searrow \quad \swarrow {\scriptstyle X_{s_i}} & \\ & X_{\mathsf{S}_i'} & \end{array}$$

ranging over the non-identity maps $s_i : \mathsf{S}' \longrightarrow \mathsf{S}_i'$ in $\mathbb{S}$. Terms are interpreted as expected.

A Σ_n-*algebra* is an object $\underline{X} \in \Sigma_{n-1}\text{-}\mathbf{Mod}$ together with a map $\Sigma_n \underline{X} \longrightarrow X$ in $\mathscr{C}^{\mathbb{S}}$. A Σ_n-*model* is a Σ_n-algebra $(\underline{X}, \xi)$ such that $\xi_{\mathsf{S}'} = \mathrm{id}_{X_{\mathsf{S}'}}$ for all $\mathsf{S}' \neq \mathsf{S}$ in the image of the forgetful functor $\mathsf{S}/\mathbb{S} \to \mathbb{S}$. Free models may be constructed according to the theory of free constructions for equational systems of [9].

Example 16. The universe of discourse associated to the signature of Example 15 is given by the category $\mathscr{C} \times \mathscr{C}^{\rightarrow}$, and the signature functors and associated models are as follows:

- $\Sigma_1(A, L \to N) = (0, 0 \to N)$ and thus Σ_1-models are structures $\big((A, L \to N), N \to N\big)$.
- $\Sigma_2\big((A, L \xrightarrow{\ell} N), N \xrightarrow{s} N\big)$
  $= \big(0, A \times L \xrightarrow{\pi_2} L \xrightarrow{\ell} N \xrightarrow{s} N\big)$
  and thus Σ_2-models are structures
  $\big((A, L \xrightarrow{\ell} N), N \xrightarrow{s} N, A \times L \xrightarrow{c} L\big)$
  such that $\ell \circ c = s \circ \ell \circ \pi_2 : A \times L \longrightarrow N$.
- $\Sigma_3\big((A, L \xrightarrow{\ell} N), N \xrightarrow{s} N, A \times L \xrightarrow{c} L\big)$
  $= (0, s^* L \xrightarrow{s^*\ell} N)$,
  where $s^*\ell : s^* L \longrightarrow N$ is the pullback of $\ell : L \longrightarrow N$ along $s : N \longrightarrow N$, and thus Σ_3-models are structures
  $\big((A, L \xrightarrow{\ell} N), N \xrightarrow{s} N, A \times L \xrightarrow{c} L, s^* L \xrightarrow{t} L\big)$
  such that $\ell \circ c = s \circ \ell \circ \pi_2 : A \times L \longrightarrow N$ and $\ell \circ t = s^*\ell : s^* L \longrightarrow N$.

II.3. Substitution

In view of the previous two sections, one is lead to consider the universe of discourse for dependently-sorted abstract syntax given by

$$\big(\boldsymbol{\mathcal{S}et}^{\underline{\mathrm{Fin}}[\mathbb{S}, \boldsymbol{\mathcal{S}et}]}\big)^{\mathbb{S}}\ , \text{ for simple categories } \mathbb{S}\,. \qquad (4)$$

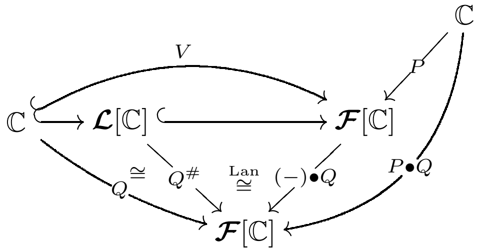
The intuitive idea behind this construction being that, for a variable set $X \in \big(\boldsymbol{\mathcal{S}et}^{\underline{\mathrm{Fin}}[\mathbb{S}, \boldsymbol{\mathcal{S}et}]}\big)^{\mathbb{S}}$, an $\mathbb{S}$-sort S, and an $\mathbb{S}$-context Γ, the set $X_{\mathsf{S}}(\Gamma)$ consists of the elements in X of sort S in context Γ. (Note that in the absence of dependency between sorts the simple category under consideration is discrete and one recovers the model for multi-sorted abstract syntax with variable binding, see [4, 5, 19, 22].)

As already emphasised in the paper, the crucial notion for treating substitution is that of substitution tensor product. To be able to introduce it in a conceptual manner I need recall the following universal characterisation of categories of contexts due to Makkai [18, §4]: for $\mathbb{S}$ a simple category, the category $\underline{\mathrm{Fin}}[\mathbb{S}, \boldsymbol{\mathcal{S}et}]^{\mathrm{op}}$ is the free finite-limit completion of $\mathbb{S}$.

Writing $\boldsymbol{\mathcal{L}}[\mathbb{C}]$ for the free finite-limit completion of a small category $\mathbb{C}$ (*viz.*, the opposite of the full subcategory of finitely presentable objects of $\boldsymbol{\mathcal{S}et}^{\mathbb{C}}$), I will more generally introduce a canonical substitution monoidal structure $(V, \bullet)$ on models

$$\boldsymbol{\mathcal{F}}[\mathbb{C}]^{\mathbb{C}}\ , \text{ where } \boldsymbol{\mathcal{F}}[\mathbb{C}] = \boldsymbol{\mathcal{S}et}^{\boldsymbol{\mathcal{L}}[\mathbb{C}]^{\mathrm{op}}}\ ,$$

for $\mathbb{C}$ an arbitrary small category. This monoidal structure is given by the following construction



so that

$$V_C(\Gamma) = \mathcal{L}[\mathbb{C}](\Gamma, C)$$

and

$$\begin{aligned} &(P \bullet Q)_C(\Gamma) \\ &= \big((P_C) \bullet Q\big)(\Gamma) \\ &= \int^{\Delta \in \mathcal{L}[\mathbb{C}]} P_C(\Delta) \times \lim_{(x\,:\,D) \in \mathcal{E}(\Delta)} Q_D(\Gamma) \end{aligned}$$

where, for $\Delta \in \mathcal{L}[\mathbb{C}]$, the category of elements $\mathcal{E}(\Delta)$ has objects $(x : D)$ with $D \in \mathbb{C}$ and $x : \Delta \longrightarrow D$ in $\mathcal{L}[\mathbb{C}]$, and morphisms $\delta : (x : D) \longrightarrow (\delta \circ x : D')$ for all $\delta : D \longrightarrow D'$ in $\mathbb{C}$.

(I note in passing that this monoidal structure can be generalised to a Kleisli composition operation (see [8]). In fact, it can also be recast in the categorical setting of Power and Tanaka (see, *e.g.*, [22]). However, the instantiation of their abstract notion of typed binding signature is not relevant to dependently-sorted syntax.)

It is instructive to see how the above substitution monoidal structure accounts for the heavy dependency present in the operation of substitution in the context of dependent sorts. To this end, for $P \in (\mathcal{S}et^{\underline{\mathrm{Fin}}[\mathbb{S},\mathcal{S}et]})^{\mathbb{S}}$ with $\mathbb{S}$ a simple category, visualise each $p \in P_{\mathsf{S}}(\Gamma)$, for S an $\mathbb{S}$-sort and Γ an $\mathbb{S}$-context, as a dependent judgement

$$\Gamma \vdash p : \mathsf{S}(\ldots, p_i, \ldots)$$

where $p_i = P_{s_i}(\Gamma)(p)$ for $s_i : \mathsf{S} \longrightarrow \mathsf{S}_i$ an $\mathbb{S}$-dependency. Then, a natural transformation $P \bullet P \longrightarrow P$ provides compatible mappings of the form

$$\Delta \vdash p : \mathsf{S}(\ldots, p_i, \ldots)\,;\Big(\Gamma \vdash q_j : \mathsf{S}_j(\ldots)\Big)_{\left(x_j : \mathsf{S}_j(\ldots)\right) \in \Delta}$$
$$\longmapsto \Gamma \vdash p[\ldots, q_j, \ldots] : \mathsf{S}(\ldots, p_i[\ldots, q_j, \ldots], \ldots)$$

where $\Gamma \vdash q_j : \mathsf{S}_j(\ldots, q_k, \ldots)$ if $x_j : \mathsf{S}_j(\ldots, x_k, \ldots)$. It follows that the notion of monoid with respect to the substitution monoidal structure abstractly specifies the substitution operation in the context of dependent sorts.

I now show how the algebraic models for simple dependently-sorted signatures of the previous section are to be extended to incorporate substitution. The inductive step of the construction to follow is based on the fact that, since the endofunctor $(-) \bullet Q$ on $\mathcal{F}[\mathbb{C}]$ preserves finite limits and the substitution tensor product $(P \bullet Q)_C$ is given pointwise as $(P_C) \bullet Q$, there is a canonical action on Σ_n-models and tensorial strength as follows

$$(\Sigma_{n+1}\, M) \bullet Q \cong \Sigma_{n+1}(M \bullet Q)\,.$$

- A Σ_0-*model with substitution* is a monoid (P, ϖ, ς) in $\big(\mathcal{S}et^{\underline{\mathrm{Fin}}[\mathbb{S},\mathcal{S}et]}\big)^{\mathbb{S}}$ with respect to the substitution monoidal structure.
- A Σ_{n+1}-*model with substitution* $(\underline{P}, \varpi, \varsigma)$ is given by a Σ_{n+1}-model $\underline{P} = (\,|\underline{P}|, \Sigma_{n+1}|\underline{P}| \longrightarrow P\,)$ and a Σ_n-model with substitution $(\,|\underline{P}|, \varpi, \varsigma\,)$ such that $\varsigma : P \bullet P \longrightarrow P$ is a Σ_{n+1}-homomorphism $\underline{P} \bullet P \longrightarrow \underline{P}$; that is, such that

$$\begin{array}{ccc} (\,\Sigma_{n+1}\,|\underline{P}|\,) \bullet P \cong \Sigma_{n+1}(\,|\underline{P}| \bullet P\,) & \xrightarrow{\Sigma_{n+1}(\varsigma)} & \Sigma_{n+1}|\underline{P}| \\ \downarrow & & \downarrow \\ P \bullet P & \xrightarrow[\varsigma]{} & P \end{array}$$

Example 17. As a follow up of Examples 15 and 16, note that a model with substitution for the signature of Example 15 is given by an underlying object $P \in \big(\mathcal{S}et^{\mathsf{F}^{\mathsf{S}}}\big)^{\mathsf{S}}$, for $\mathsf{S} = \boxed{\mathsf{A} \quad \mathsf{L} \longrightarrow \mathsf{N}}$, equipped with a Σ_3-model structure $(s : P_{\mathsf{N}} \to P_{\mathsf{N}}, c : P_{\mathsf{A}} \times P_{\mathsf{L}} \to P_{\mathsf{L}}, t : s^* P_{\mathsf{L}} \longrightarrow P_{\mathsf{L}})$ on $(P_{\mathsf{A}}, P_{\mathsf{L}} \longrightarrow P_{\mathsf{N}})$ as in Example 16 together with a monoid structure (ϖ, ς) on P subject to the following compatibility conditions

$$\begin{array}{ccc} P_{\mathsf{N}} \bullet P & \xrightarrow{\varsigma_{\mathsf{N}}} & P_{\mathsf{N}} \\ {\scriptstyle s \bullet P}\downarrow & & \downarrow{\scriptstyle s} \\ P_{\mathsf{N}} \bullet P & \xrightarrow[\varsigma_{\mathsf{N}}]{} & P_{\mathsf{N}} \end{array}$$

$$\begin{array}{ccc} (P_{\mathsf{A}} \times P_{\mathsf{L}}) \bullet P \cong (P_{\mathsf{A}} \bullet P) \times (P_{\mathsf{L}} \bullet P) & \xrightarrow{\varsigma_{\mathsf{A}} \times \varsigma_{\mathsf{L}}} & P_{\mathsf{A}} \times P_{\mathsf{L}} \\ {\scriptstyle c \bullet P}\downarrow & & \downarrow{\scriptstyle c} \\ P_{\mathsf{L}} \bullet P & \xrightarrow[\varsigma_{\mathsf{L}}]{} & P_{\mathsf{L}} \end{array}$$

$$\begin{array}{ccc} (s^* P_{\mathsf{L}}) \bullet P \cong s^*(P_{\mathsf{L}} \bullet P) & \xrightarrow{s^* \varsigma_{\mathsf{L}}} & s^* P_{\mathsf{L}} \\ {\scriptstyle t \bullet P}\downarrow & & \downarrow{\scriptstyle t} \\ P_{\mathsf{L}} \bullet P & \xrightarrow[\varsigma_{\mathsf{L}}]{} & P_{\mathsf{L}} \end{array}$$

II.4. Sketches

The main virtue of the universes of discourse of (4) is their simplicity; and indeed this is what is needed in certain applications (see, *e.g.*, [18]). However, these models carry an inherent limitation: the restriction to simple dependency (in contexts and signatures).

I will now sketch how the approach is to be extended to include more general notions of context and signature. The main idea here is to consider graphical representations of dependently-sorted signatures, with sorts together with their dependencies and operators. This is naturally provided by Ehresmann's concept of sketch; more specifically, by that of (certain kind of) finite limit sketch. (For the general theory of sketches, the reader may consult [2] and, for their specific application to dependently-sorted algebra, the work of Taylor [23, Chapter VIII].)

Every sketch gives rise to a theory (or classifying category) containing a universal model. For the sketch of a dependently-sorted signature, the theory provides the category of contexts, with the class of models under consideration determining the kind of contexts that one is interested in. For instance, the category of contexts *with equality types* arises as the theory of the universal model amongst those in categories *with finite limits* (see, *e.g.*, [23, Section 8.3]).

For a dependently-sorted signature (sketch) $\mathbb{S}$, let $\mathcal{C}[\mathbb{S}]$ be its associated category of contexts (theory). Generalising the construction of (5), the category

$$\mathbf{Mod}_{\mathbb{S}}\big(\widehat{\mathcal{C}[\mathbb{S}]}\big)$$

of models of $\mathbb{S}$ in $\widehat{\mathcal{C}[\mathbb{S}]} = \mathcal{S}et^{\mathcal{C}[\mathbb{S}]^{\mathrm{op}}}$ acquires a substitution monoidal structure. One is thus led to consider the universe of discourse

$$\mathbf{Mon}\big(\mathbf{Mod}_{\mathbb{S}}\big(\widehat{\mathcal{C}[\mathbb{S}]}\big)\big)$$

of models with monoid structure, and indeed its free objects embody dependently-sorted abstract syntax with substitution structure.

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