The Algebra of Directed Acyclic Graphs

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Abstract. We give an algebraic presentation of directed acyclic graph structure, introducing a symmetric monoidal equational theory whose free PROP we characterise as that of finite abstract dags with input/output interfaces. Our development provides an initial-algebra semantics for dag structure.

Keywords: dag, PROP, symmetric monoidal equational theory, bialgebra, Hopf algebra, topological sorting, initial-algebra and categorical semantics

Dedicated to Samson Abramsky on the occasion of his 60^{th} birthday

1 Introduction

This work originated in a question of Robin Milner in connection to explorations he was pursuing on possible extensions to his theory of bigraphs [7]. The particular direction that concerns us here is the generalisation of the spatial dimension of bigraphs from a tree hierarchy to a directed acyclic graph (dag) structure.

In [6], Milner provided axioms for bigraphical structure, axiomatising tree-branching structure by means of the equational theory of commutative monoids. As for the axiomatisation of dag structure, he foresaw that it would also involve the dual theory of commutative comonoids and, in conversation with the first author, raised the question on how these two structures should interact. In considering the problem, it soon became clear that the axioms in question were those of commutative bialgebras (where the monoid structure is a comonoid homomorphism and, equivalently, the comonoid structure is a monoid homomorphism) that are degenerate in that the composition of the comultiplication followed by the multiplication collapses to the identity. This gives the axiomatics of wiring for dag structure.

The natural setting for presenting our work is the categorical language of PROPs; specifically relying on the concept of free PROP, which roughly corresponds to the symmetric strict monoidal category freely generated by a symmetric monoidal equational theory. Indeed, our main result characterises the free PROP on the theory D of degenerate commutative bialgebras with a node (endomap) as that of finite abstract dags with input/output interfaces, see Section 5. Let us give an idea of why this is so.

It is important to note that the theory D is the sum of two sub-theories: the theory R of degenerate commutative bialgebras and the theory N_1 of a node (endomap). Each of these theories captures a different aspect of dag structure. The free PROP on R provides relational edge structure; while the free PROP on N_1 introduces node structure. Thus, the free PROP on their sum, which is essentially obtained by interleaving both structures, results in dag structure. A main aim of the paper is to give a simple technical development that formalises these intuitions.

This work falls within a central theme of Samson Abramsky's research: the mathematical study of syntactic structure, an example of which in the context of PROs is his characterisation of Temperley-Lieb structure [1].

2 Directed acyclic graphs

2.1 Dags. A *directed acyclic graph (dag)* is a graph with directed edges in which there are no cycles. Formally, a directed graph is a pair $(N, R \subseteq N \times N)$ consisting of a set of nodes N and a binary relation R on it that specifies a directed edge from a node n to another one m whenever $(n, m) \in R$. The acyclicity condition of a dag (N, R) is ensured by requiring that the transitive closure R^+ of the relation R is irreflexive; *i.e.* $(n, n) \notin R^+$ for all $n \in N$.

2.2 Idags. We will deal here with a slight generalisation of the notion of dag. An *interfaced dag (idag)* is a tuple of sets I, O, N and a binary relation $R \subseteq (I + N) \times (O + N)$, for $+$ the sum of sets, subject to the acyclicity condition $(n, n) \notin (p \circ R \circ \imath)^+$ for all $n \in N$, where the relations $i \subseteq N \times (I + N)$ and $p \subseteq (O + N) \times N$ respectively denote the injection of N into $I + N$ and the projection of $O + N$ onto N.

Informally, idags are dags extended with interfaces. An idag (I, O, N, R), also referred to as an (I, O)-dag, is said to have input interface I and output interface O; N is its set of internal nodes. Fig. 1 depicts two examples with input and output sets of ordinals, where for $n \in \mathbb{N}$ we adopt the notation $\underline{n}$ for the ordinal $\{0, \ldots, n-1\}$.

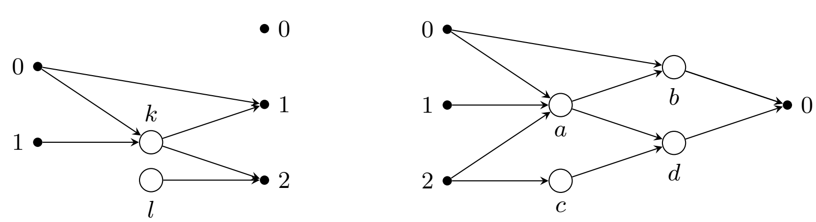


Fig. 1. A $(\underline{2}, \underline{3})$-dag and a $(\underline{3}, \underline{1})$-dag.

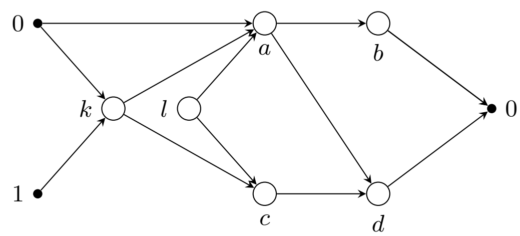


Fig. 2. The concatenation of the idags of Fig. 1.

The notion of dag is recovered as that of idag with empty sets of input and output nodes. Idags also generalise binary relations, as these are in bijective correspondence with idags without internal nodes.

2.3 Operations on idags. The extension of dags with interfaces allows for two basic operations on them.

The *concatenation* operation $D' \circ D$ of an (I, M)-dag $D = (N, R)$ and an (M, O)-dag $D' = (N', R')$ is the (I, O)-dag that retains the hierarchy information of both idags except that edges in R from input and internal nodes in D to intermediate nodes in M become redirected to target internal and output nodes in D' as specified by R'. Formally, $D' \circ D = (N + N', R'')$ where R'' is the composite

$$I{+}N{+}N' \xrightarrow{R+\mathrm{id}} M{+}N{+}N' \cong M{+}N'{+}N \xrightarrow{R'+\mathrm{id}} O{+}N'{+}N \cong O{+}N{+}N' \ .$$

The *juxtaposition* operation $D \oplus D'$ of an (I, O)-dag $D = (N, R)$ and an (I', O')-dag $D' = (N', R')$ is the $(I + I', O + O')$-dag $D \oplus D' = (N + N', R'')$ where R'' is the composite

$$I + I' + N + N' \cong I + N + I' + N' \xrightarrow{R+R'} O + N + O' + N' \cong O + O' + N + N' \ .$$

Thus, juxtaposition puts two idags side by side, without modifying their hierarchies.

2.4 The category of finite abstract idags. As we are to look at idags abstractly, we need a notion that identifies those that are essentially the same. Accordingly, we set two (I, O)-dags (N, R) and (N', R') to be isomorphic whenever there exists a bijection $\sigma : N \cong N'$ such that $(\mathrm{id} + \sigma) \circ R = R' \circ (\mathrm{id} + \sigma)$.

Abstract (I, O)*-dags* are then defined to be equivalence classes of isomorphic (I, O)-dags. The operations of concatenation and juxtaposition respect isomorphism and one can use them to endow abstract idags with the structure of a symmetric monoidal category. We will restrict attention to the finite case: The category **Dag** has objects given by finite sets and homs $\mathbf{Dag}(I, O)$ given by abstract (I, O)-dags with a finite set of internal nodes. These are equipped with composition operation given by concatenation and identities given by identity relations. Furthermore, the juxtaposition operation provides a symmetric tensor product with unit the empty set.

The aim of the paper is to give an algebraic presentation characterising **Dag**. The appropriate setting for establishing our result is that of PROPs, to which we now turn.

3 Product and permutation categories

3.1 PROPs. A *PROduct and Permutation category (PROP)*, see [5], is a symmetric strict monoidal category with objects the natural numbers and tensor product given by addition. This definition is often relaxed in practice, allowing symmetric strict monoidal categories with underlying commutative monoid structure on objects isomorphic to the commutative monoid of natural numbers. A typical example is the additive monoid of finite ordinals $\big(\{\underline{n} \mid n \in \mathbb{N}\}, \underline{0}, \oplus\big)$, for which $\underline{n} \oplus \underline{m} = \underline{n+m}$.

The main example of PROP to be studied in the paper follows.

Example. The category **Dag** is equivalent to the PROP **D** consisting of its full subcategory determined by the finite ordinals.

PROPs describe algebraic structure, with the category $\mathsf{Mod}_{\mathbf{P}}(\mathscr{C})$ of *functorial models* of a PROP **P** in a symmetric monoidal category $\mathscr{C}$ given by symmetric monoidal functors $\mathbf{P} \to \mathscr{C}$ and symmetric monoidal natural transformations between them.

3.2 Free PROPs. As remarked by Mac Lane in [5], "[a] useful construction yields the free PROP [...] with given generators and relations"; the usefulness residing in it being "adapted to the study of universal algebra". We briefly recall the construction and its universal characterisation.

A *signature* consists of a set of operators O together with an assignment $O \to \mathbb{N} \times \mathbb{N}$ of arity/coarity pairs to operators. In this context, it is usual to use the notation $o : n \to m$ to indicate that the operator o is assigned the arity/coarity pair (n, m). For a signature Σ, we let $\mathrm{E}(\Sigma)$ consist of the expressions with arity/coarity pairs generated by the language of symmetric strict monoidal categories with underlying commutative monoid the additive natural numbers together with the operators in Σ (*cf.* [3]). A *symmetric monoidal presentation* on a signature Σ is then a set of pairs of expressions in $\mathrm{E}(\Sigma)$ with the same arity/coarity pair. A *symmetric monoidal equational theory* consists of a signature together with a symmetric monoidal presentation on it. An *algebra* for a symmetric monoidal equational theory $(\Sigma, \mathcal{T})$ in a symmetric monoidal category is an object A equipped with morphisms $A^{\otimes n} \to A^{\otimes m}$ for every operator of arity/coarity pair (n, m) in Σ such that the interpretation of every equation in $\mathcal{T}$ is satisfied. We write $\mathsf{Alg}_{(\Sigma,\mathcal{T})}(\mathscr{C})$ for the category of $(\Sigma, \mathcal{T})$-algebras and homomorphisms in $\mathscr{C}$.

The free PROP $\mathbf{P}[\Sigma, \mathcal{T}]$ on a symmetric monoidal theory $(\Sigma, \mathcal{T})$ has homs $\mathbf{P}[\Sigma, \mathcal{T}](n, m)$ given by the quotient of the set of expressions $\mathrm{E}(\Sigma)$, with arity n

and coarity m, under the laws of symmetric strict monoidal categories and the presentation $\mathcal{T}$. It is universally characterised by a natural equivalence

$$\mathsf{Mod}_{\mathbf{P}[\Sigma,\mathcal{T}]}(\mathscr{C}) \simeq \mathsf{Alg}_{(\Sigma,\mathcal{T})}(\mathscr{C})$$

for $\mathscr{C}$ ranging over symmetric monoidal categories.

4 Examples of free PROPs

We give examples of symmetric monoidal equational theories together with abstract characterisations of their induced free PROPs.

4.1 Empty theory. The free PROP $\mathbf{P}$ on the *empty symmetric monoidal theory* (with no operators and no equations) is the initial symmetric strict monoidal category, *i.e.* the groupoid of finite ordinals and bijections.

4.2 Nodes. For a set L, the free PROP $\mathbf{N}_L$ on the symmetric monoidal theory of *nodes* $\mathsf{N}_L = \big(\{\lambda : 1 \to 1\}_{\lambda \in L}, \emptyset\big)$ is the free symmetric strict monoidal category on the free monoid $(L^\star, \varepsilon, \cdot)$. Explicitly, $\mathbf{N}_L$ has finite ordinals as objects and homs $\mathbf{N}_L(\underline{n}, \underline{m}) = \mathbf{P}(\underline{n}, \underline{m}) \times (L^\star)^n$ with identities $(\mathrm{id}_n, \varepsilon)$ and composition $(\tau, w) \circ (\sigma, v) = \big(\tau \circ \sigma, (w_{\sigma(i)} \cdot v_i)_{0 \le i < n}\big)$.

4.3 Idempotent objects. The symmetric monoidal theory of an *idempotent object* has signature with operators $\Delta : 1 \to 2$ and $\nabla : 2 \to 1$ subject to the presentation

$$\nabla \circ \Delta = \mathrm{id}_1 \ : 1 \to 1 \ , \quad \Delta \circ \nabla = \mathrm{id}_2 \ : 2 \to 2 \ .$$

The free PROP $\mathbf{V}$ on it is a groupoid, with hom $\mathbf{V}(1,1)$ given by Thompson's group V, see [2].

4.4 Commutative monoids and commutative comonoids. The symmetric monoidal theory of *commutative monoids* has signature with operators $\eta : 0 \to 1$ and $\nabla : 2 \to 1$ subject to the presentation

$$\begin{gathered} \nabla \circ (\eta \otimes \mathrm{id}_1) = \mathrm{id}_1 = \nabla \circ (\mathrm{id}_1 \otimes \eta) \ : 1 \to 1 \ , \\ \nabla \circ (\nabla \otimes \mathrm{id}_1) = \nabla \circ (\mathrm{id}_1 \otimes \nabla) \ : 3 \to 1 \ , \\ \nabla \circ \gamma_{1,1} = \nabla \ : 2 \to 2 \end{gathered}$$

where γ denotes the symmetry. The free PROP on it is the category of finite ordinals and functions.

The dual symmetric monoidal theory is that of *commutative comonoids*. It has signature with operators $\epsilon : 1 \to 0$ and $\Delta : 1 \to 2$ subject to the presentation

$$\begin{gathered} (\epsilon \otimes \mathrm{id}_1) \circ \Delta = \mathrm{id}_1 = (\mathrm{id}_1 \otimes \epsilon) \circ \Delta \ : 1 \to 1 \ , \\ (\Delta \otimes \mathrm{id}_1) \circ \Delta = (\mathrm{id}_1 \otimes \Delta) \circ \Delta \ : 1 \to 3 \ , \\ \gamma_{1,1} \circ \Delta = \Delta \ : 2 \to 2 \ . \end{gathered}$$

The free PROP on it is of course the opposite of the category of finite ordinals and functions.

4.5 Commutative bialgebras. The symmetric monoidal theory B of *commutative bialgebras* has signature with operators $\eta : 0 \to 1$, $\nabla : 2 \to 1$, $\epsilon : 1 \to 0$, and $\Delta : 1 \to 2$ subject to the presentation consisting of that of commutative monoids, commutative comonoids, and the following

$$\begin{gathered}\epsilon \circ \eta = \mathrm{id}_0 \ \ : 0 \to 0 \ \ , \\ \epsilon \circ \nabla = \epsilon \otimes \epsilon \ \ : 2 \to 0 \ \ , \quad \Delta \circ \eta = \eta \otimes \eta \ \ : 0 \to 2 \ \ , \\ \Delta \circ \nabla = (\nabla \otimes \nabla) \circ (\mathrm{id}_1 \otimes \gamma_{1,1} \otimes \mathrm{id}_1) \circ (\Delta \otimes \Delta) \ \ : 2 \to 2 \ \ .\end{gathered}$$

The symmetric monoidal theory of *degenerate commutative bialgebras* extends the above with the equation

$$\nabla \circ \Delta = \mathrm{id}_1 \ \ : 1 \to 1 \ \ .$$

The free PROP $\mathbf{B}$ on the symmetric monoidal theory of commutative bialgebras has homs $\mathbf{B}(n,m) = \mathbb{N}^{n \times m}$ under matrix composition. Accordingly, the free PROP $\mathbf{R}$ on the symmetric monoidal theory of degenerate commutative bialgebras is the category of finite ordinals and relations. See *e.g.* [5, §10], [8], and [4].

4.6 Commutative Hopf algebras. The symmetric monoidal theory of *commutative Hopf algebras* extends that of commutative bialgebras with an *antipode* operator $s : 1 \to 1$ subject to the laws:

$$\begin{gathered}s \circ \eta = \eta \ \ : 0 \to 1 \ \ , \nabla \circ (s \oplus s) = s \circ \nabla \ \ : 2 \to 1 \ \ , \\ \epsilon \circ s = \epsilon \ \ : 1 \to 0 \ \ , \ (s \oplus s) \circ \Delta = \Delta \circ s \ \ : 1 \to 2 \ \ , \\ \nabla \circ (s \oplus \mathrm{id}_1) \circ \Delta = \eta \circ \epsilon = \nabla \circ (\mathrm{id}_1 \oplus s) \circ \Delta \ \ : 1 \to 1 \ \ .\end{gathered}$$

Its free PROP $\mathbf{H}$ has homs $\mathbf{H}(n,m) = \mathbb{Z}^{n \times m}$ under matrix composition.[1]

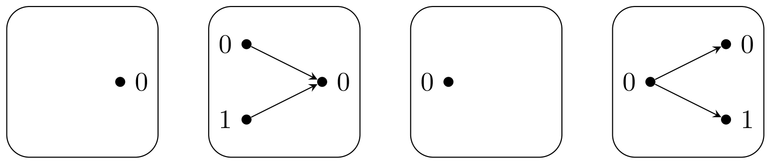
4.7 Commutative monoids with a node. The symmetric monoidal theory of *commutative monoids with a node* is the sum of the theory of commutative monoids and the theory of a single node. Its free PROP $\mathbf{F}$ has homs consisting of interfaced forests. Precisely, $\mathbf{F}$ is the sub-PROP of $\mathbf{D}$ determined by the interfaced dags (N, R) with R a total function, see [6].

5 The algebra of idags

5.1 Algebraic structure. The generator $\underline{1}$ of the PROP $\mathbf{D}$ carries two important algebraic structures:

[1] We are grateful to Ross Duncan and Aleks Kissinger for bringing this example to our attention.

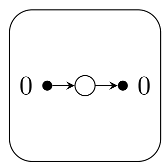
1. the degenerate commutative bialgebra



$\eta : \underline{0} \to \underline{1}$ $\quad\nabla : \underline{2} \to \underline{1}$ $\quad\epsilon : \underline{1} \to \underline{0}$ $\quad\varDelta : \underline{1} \to \underline{2}$

and

2. the node



$\lambda : \underline{1} \to \underline{1}$

These respectively induce universal injections of PROPs as follows

$$\begin{tikzcd} \mathbf{R} \arrow[dr] & & \mathbf{N}_1 \arrow[dl] \\ & \mathbf{D} & \end{tikzcd} \tag{1}$$

The main result of the paper is that together the PROPs $\mathbf{R}$ and $\mathbf{N}_1$ characterise the PROP $\mathbf{D}$.

Theorem 1. *The PROP* $\mathbf{D}$ *is free on the symmetric monoidal theory* D *of* degenerate commutative bialgebras with a node *(i.e. the sum of the theory* R *of degenerate commutative bialgebras and the theory* N_1 *of a node).*

The theorem is proved by establishing the universal property of the free PROP by means of the following lemma, whose proof occupies the rest of the section.

Lemma 1. *The cospan (1) is a pushout of symmetric monoidal categories for the following span of universal PROP injections*

$$\begin{tikzcd} & \mathbf{P} \arrow[dl] \arrow[dr] & \\ \mathbf{R} & & \mathbf{N}_1 \end{tikzcd}$$

5.2 Categorical interpretation. We start by giving an interpretation of finite idags on degenerate commutative bialgebras with a node in arbitrary symmetric monoidal categories. Specifically, for every D-algebra

$$(A, \eta_A : I \to A, \nabla_A : A \otimes A \to A, \epsilon_A : A \to I, \varDelta_A : A \to A \otimes A, \lambda_A : A \to A)$$

in a symmetric monoidal category $\mathscr{C}$ we will define mappings

$$\mathcal{D}[\![-]\!]_A : \mathbf{D}(\underline{n}, \underline{m}) \to \mathscr{C}(A^{\otimes n}, A^{\otimes m})$$

extending the interpretations for $\mathbf{R}$ and $\mathbf{N}_1$, respectively induced by the R-algebra $(A, \eta_A, \nabla_A, \epsilon_A, \Delta_A)$ and the N_1-algebra (A, λ_A), as follows

$$\begin{tikzcd}
\mathbf{R}(\underline{n}, \underline{m}) \arrow[r] \arrow[dr, "{\mathcal{R}[\![-]\!]_A}"'] & \mathbf{D}(\underline{n}, \underline{m}) \arrow[d] & \mathbf{N}_1(\underline{n}, \underline{m}) \arrow[l] \arrow[dl, "{\mathcal{N}_1[\![-]\!]_A}"] \\
& \mathscr{C}(A^{\otimes n}, A^{\otimes m}) &
\end{tikzcd}$$

For dag structure, in stark contrast with tree structure, there is no direct definition of the interpretation function by structural induction, and a more involved approach to defining it is necessary. This proceeds in two steps as follows.

1. We give an interpretation $\mathcal{D}_\sigma[\![D]\!]_A$ parameterised by topological sortings σ of D.
2. We show that the interpretation is independent of the topological sorting, in that $\mathcal{D}_\sigma[\![D]\!]_A = \mathcal{D}_{\sigma'}[\![D]\!]_A$ for all topological sortings σ and σ' of D.

A *topological sorting* of a finite $(\underline{n}, \underline{m})$-dag $D = (N, R)$ is a bijection

$$\sigma : [N] \to N \quad , \quad \text{for } [N] = \big\{\, 0, \ldots, |N|-1 \,\big\}$$

such that

$$\forall\, 0 \leq i, j < |N|.\, \big(\imath_2(\sigma_i), \imath_2(\sigma_j)\big) \in R \implies i < j$$

where $\imath_2$ denotes the second sum injection. Every such topological sorting induces a canonical decomposition in $\mathbf{D}$ as follows:

$$D = D^\sigma_{|N|} \circ (\mathrm{id}_{n+|N|-1} \oplus \lambda) \circ D^\sigma_{|N|-1} \circ \cdots \circ (\mathrm{id}_n \oplus \lambda) \circ D^\sigma_0 \tag{2}$$

where, for $0 \leq k < |N|$, each $D^\sigma_k \in \mathbf{D}\big(\underline{n} \oplus \underline{k}, \underline{n} \oplus \underline{k} \oplus \underline{1}\big)$ corresponds to the relation $R^\sigma_k = (\imath_{n+k} \cup \overline{R}^\sigma_k) \in \mathbf{R}\big(\underline{n} \oplus \underline{k}, \underline{n} \oplus \underline{k} \oplus \underline{1}\big)$ with $\imath_{n+k}$ the inclusion relation and $\overline{R}^\sigma_k$ encoding the edges from the input nodes $0 \leq i < n$ and the internal nodes σ_ℓ for $0 \leq \ell < k$ to the internal node σ_k; while $D^\sigma_{|N|} \in \mathbf{D}\big(\underline{n} \oplus [N], \underline{m}\big)$ corresponds to the relation $R^\sigma_{|N|} \in \mathbf{R}\big(\underline{n} \oplus [N], \underline{m}\big)$ encoding the edges from the input and the internal nodes to the output nodes. Explicitly, for $0 \leq k < |N|$ and $0 \leq j < m$,

- $\forall\, 0 \leq i < n.\, (i, n+k) \in \overline{R}^\sigma_k \iff \big(\imath_1(i), \imath_2(\sigma_k)\big) \in R$
- $\forall\, 0 \leq \ell < k.\, (n+\ell, n+k) \in \overline{R}^\sigma_k \iff \big(\imath_2(\sigma_\ell), \imath_2(\sigma_k)\big) \in R$
- $\forall\, 0 \leq i < n.\, (i, j) \in R^\sigma_{|N|} \iff \big(\imath_1(i), \imath_1(j)\big) \in R$
- $\forall\, 0 \leq \ell < |N|.\, (n+\ell, j) \in R^\sigma_{|N|} \iff \big(\imath_2(\sigma_\ell), \imath_1(j)\big) \in R$

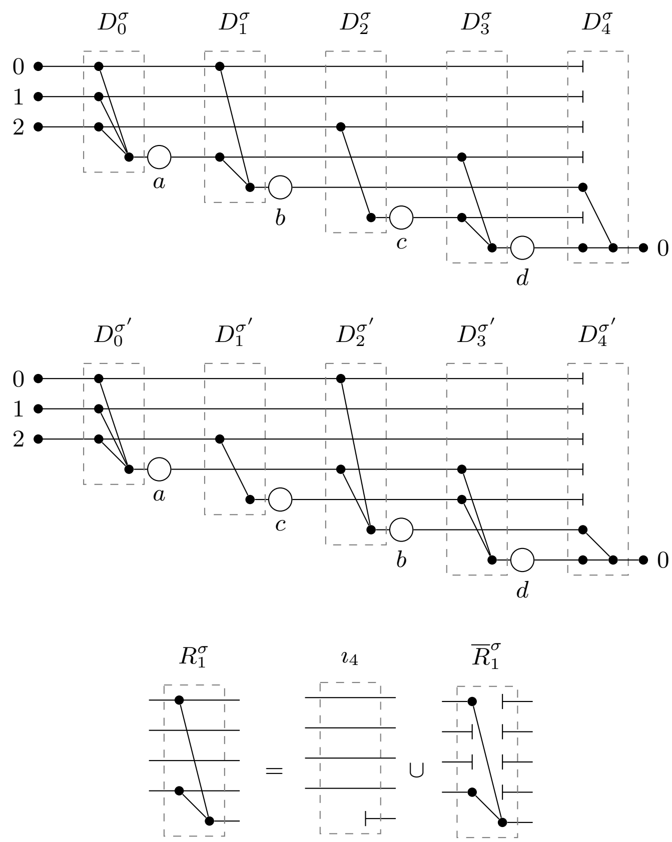


Fig. 3. Decompositions of the $(\underline{3},\underline{1})$-dag of Fig. 1 for the topological sortings $\sigma = (a, b, c, d)$ and $\sigma' = (a, c, b, d)$; and the decomposition of an auxiliary relation.

where $\imath_1$ and $\imath_2$ respectively denote the first and second sum injections. See Fig. 3 for two sample decompositions.

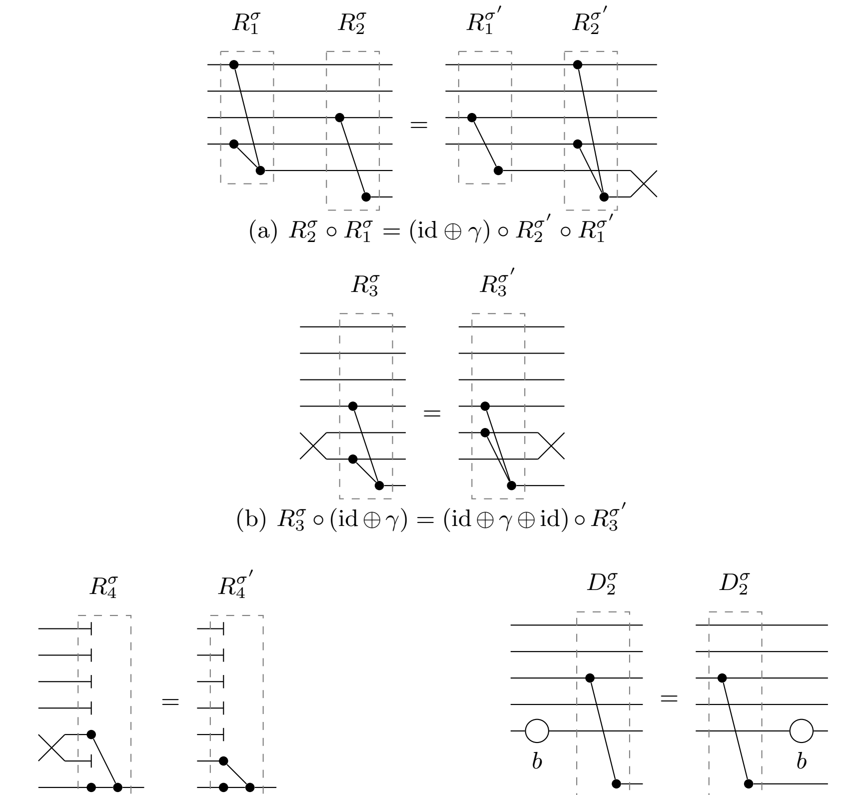
For a finite $(\underline{n}, \underline{m})$-dag D, we are led to define $\mathcal{D}_\sigma[\![D]\!]_A : A^{\otimes n} \to A^{\otimes m}$ as the composite

$$\mathcal{R}[\![R^\sigma_{|N|}]\!]_A \circ (\mathrm{id}_{n+|N|-1} \otimes \lambda_A) \circ \mathcal{R}[\![R^\sigma_{|N|-1}]\!]_A \circ \cdots \circ (\mathrm{id}_n \otimes \lambda_A) \circ \mathcal{R}[\![R^\sigma_0]\!]_A \ .$$

The above definitions have been specifically chosen so that the properties to follow are readily established.

A first remark is that the interpretation is invariant under isomorphism.

Proposition 1. *Let $D = (N, R)$ and $D' = (N', R')$ be two finite $(\underline{n}, \underline{m})$-dags isomorphic by means of a bijection $\beta : N \cong N'$. If σ is a topological sorting of D, then $\sigma' = \beta \circ \sigma$ is a topological sorting of D' and $\mathcal{D}_\sigma[\![D]\!]_A = \mathcal{D}_{\sigma'}[\![D']\!]_A$.*



(a) $R_2^\sigma \circ R_1^\sigma = (\mathrm{id} \oplus \gamma) \circ R_2^{\sigma'} \circ R_1^{\sigma'}$

(b) $R_3^\sigma \circ (\mathrm{id} \oplus \gamma) = (\mathrm{id} \oplus \gamma \oplus \mathrm{id}) \circ R_3^{\sigma'}$

(c) $R_4^\sigma \circ (\mathrm{id} \oplus \gamma \oplus \mathrm{id}) = R_4^{\sigma'}$

(d) $D_2^\sigma \circ (\mathrm{id} \oplus \lambda) = (\mathrm{id} \oplus \lambda \oplus \mathrm{id}_1) \circ D_2^\sigma$

Fig. 4. Graphical demonstration of the identities in Lemma 2 for the $(\underline{3},\underline{1})$-dag of Fig. 1.

Proof. Because one has by construction that $R_i^\sigma = R'^{\sigma'}_i$ for all $0 \leq i \leq |N|$.

More fundamental is the independence of the interpretation under topological sorting.

Lemma 2. *Let σ and σ' be two topological sortings of a finite $(\underline{n}, \underline{m})$-dag $D = (N, R)$ with $|N| \geq 2$. If σ and σ' differ only by the transposition of two adjacent indices, say $\sigma'_i = \sigma_{i+1}$ and $\sigma'_{i+1} = \sigma_i$ for $0 \leq i < |N| - 1$, then the following identities hold:*

1. $R_j^\sigma = R_j^{\sigma'}$ *for all* $0 \leq j < i$,
2. $R_{i+1}^\sigma \circ R_i^\sigma = (\mathrm{id}_{n+i} \oplus \gamma) \circ R_{i+1}^{\sigma'} \circ R_i^{\sigma'}$,
3. $R_j^\sigma \circ (\mathrm{id}_{n+i} \oplus \gamma \oplus \mathrm{id}_{j-i-2}) = (\mathrm{id}_{n+i} \oplus \gamma \oplus \mathrm{id}_{j-i-1}) \circ R_j^{\sigma'}$ *for all* $i+1 < j < |N|$,
4. $R_{|N|}^\sigma \circ (\mathrm{id}_{n+i} \oplus \gamma \oplus \mathrm{id}_{|N|-i-2}) = R_{|N|}^{\sigma'}$,
5. $\mathcal{R}[\![R_{i+1}^\tau]\!] \circ (\mathrm{id}_{A^{\otimes n+i}} \otimes \lambda_A) = (\mathrm{id}_{A^{\otimes n+i}} \otimes \lambda_A \otimes \mathrm{id}_A) \circ \mathcal{R}[\![R_{i+1}^\tau]\!]$ *for* $\tau = \sigma, \sigma'$.

Proof. The identities (1–4) follow by construction. Identity (5) is a consequence of the following general fact: for every $R \in \mathbf{R}(\underline{k+1}, \underline{k+2})$ such that, for all $j \in \underline{k+2}$, $(k, j) \in R$ iff $j = k$ one has $R = (\mathrm{id}_k \oplus \gamma) \circ (R' \oplus \mathrm{id}_1)$ for $R' \in \mathbf{R}(\underline{k}, \underline{k+1})$; so that, for all $f : A \to A$,

$$\begin{aligned}
\mathcal{R}[\![R]\!]_A \circ (\mathrm{id}_{A^{\otimes k}} \otimes f) &= (\mathrm{id}_{A^{\otimes k}} \otimes \gamma) \circ (\mathcal{R}[\![R']\!]_A \otimes \mathrm{id}_A) \circ (\mathrm{id}_{A^{\otimes k}} \otimes f) \\
&= (\mathrm{id}_{A^{\otimes k}} \otimes \gamma) \circ (\mathrm{id}_{A^{\otimes k+1}} \otimes f) \circ (\mathcal{R}[\![R']\!]_A \otimes \mathrm{id}_A) \\
&= (\mathrm{id}_{A^{\otimes k}} \otimes f \otimes \mathrm{id}_A) \circ (\mathrm{id}_{A^{\otimes k}} \otimes \gamma) \circ (\mathcal{R}[\![R']\!]_A \otimes \mathrm{id}_A) \\
&= (\mathrm{id}_{A^{\otimes k}} \otimes f \otimes \mathrm{id}_A) \circ \mathcal{R}[\![R]\!]_A
\end{aligned}$$

Proposition 2. *For any two topological sortings σ, σ' of a finite $(\underline{n}, \underline{m})$-dag D,*

$$\mathcal{D}_\sigma[\![D]\!]_A = \mathcal{D}_{\sigma'}[\![D]\!]_A : A^{\otimes n} \to A^{\otimes m} \quad .$$

Proof. It is enough to establish the equality for σ and σ' as in the hypothesis of Lemma 2. Let us then assume this situation.

By Lemma 2 (1), we have

$$\begin{aligned}
&(\mathrm{id} \otimes \lambda_A) \circ \mathcal{R}[\![R^\sigma_{i-1}]\!]_A \circ \cdots \circ (\mathrm{id} \otimes \lambda_A) \circ \mathcal{R}[\![R^\sigma_0]\!]_A \\
&\quad = \ (\mathrm{id} \otimes \lambda_A) \circ \mathcal{R}[\![R^{\sigma'}_{i-1}]\!]_A \circ \cdots \circ (\mathrm{id} \otimes \lambda_A) \circ \mathcal{R}[\![R^{\sigma'}_0]\!]_A
\end{aligned}$$

so that we need only show

$$\begin{aligned}
&\mathcal{R}[\![R^\sigma_{|N|}]\!]_A \circ (\mathrm{id} \otimes \lambda_A) \circ \mathcal{R}[\![R^\sigma_{|N|-1}]\!]_A \circ \cdots \circ (\mathrm{id} \otimes \lambda_A) \circ \mathcal{R}[\![R^\sigma_i]\!]_A \\
&\quad = \ \mathcal{R}[\![R^{\sigma'}_{|N|}]\!]_A \circ (\mathrm{id} \otimes \lambda_A) \circ \mathcal{R}[\![R^{\sigma'}_{|N|-1}]\!]_A \circ \cdots \circ (\mathrm{id} \otimes \lambda_A) \circ \mathcal{R}[\![R^{\sigma'}_i]\!]_A
\end{aligned}$$

For this we calculate in three steps as follows:

1. $(\mathrm{id} \otimes \lambda_A) \circ \mathcal{R}[\![R^\sigma_{i+1}]\!]_A \circ (\mathrm{id} \otimes \lambda_A) \circ \mathcal{R}[\![R^\sigma_i]\!]_A$
   $= \ (\mathrm{id} \otimes \lambda_A) \circ (\mathrm{id} \otimes \lambda_A \otimes \mathrm{id}_A) \circ \mathcal{R}[\![R^\sigma_{i+1}]\!]_A \circ \mathcal{R}[\![R^\sigma_i]\!]_A$
   , by Lemma 2 (5)
   $= \ (\mathrm{id} \otimes \lambda_A) \circ (\mathrm{id} \otimes \lambda_A \otimes \mathrm{id}_A) \circ (\mathrm{id} \otimes \gamma) \circ \mathcal{R}[\![R^{\sigma'}_{i+1}]\!]_A \circ \mathcal{R}[\![R^{\sigma'}_i]\!]_A$
   , by Lemma 2 (2)
   $= \ (\mathrm{id} \otimes \gamma) \circ (\mathrm{id} \otimes \lambda_A) \circ (\mathrm{id} \otimes \lambda_A \otimes \mathrm{id}_A) \circ \mathcal{R}[\![R^{\sigma'}_{i+1}]\!]_A \circ \mathcal{R}[\![R^{\sigma'}_i]\!]_A$
   $= \ (\mathrm{id} \otimes \gamma) \circ (\mathrm{id} \otimes \lambda_A) \circ \mathcal{R}[\![R^{\sigma'}_{i+1}]\!]_A \circ (\mathrm{id} \otimes \lambda_A) \circ \mathcal{R}[\![R^{\sigma'}_i]\!]_A$
   , by Lemma 2 (5)

2. $(\mathrm{id}_{A^{\otimes n+|N|-1}} \otimes \lambda_A) \circ \mathcal{R}[\![R^\sigma_{|N|-1}]\!]_A \circ \cdots$

$$\begin{aligned}
&\cdots \circ (\mathrm{id}_{A^{\otimes n+i+2}} \otimes \lambda_A) \circ \mathcal{R}[\![R^\sigma_{i+2}]\!]_A \circ (\mathrm{id}_{A^{\otimes n+i}} \otimes \gamma) \\
&= (\mathrm{id}_{A^{\otimes n+|N|-1}} \otimes \lambda_A) \circ \mathcal{R}[\![R^\sigma_{|N|-1}]\!]_A \circ \cdots \\
&\qquad \cdots \circ (\mathrm{id}_{A^{\otimes n+i+2}} \otimes \lambda_A) \circ (\mathrm{id}_{A^{\otimes n+i}} \otimes \gamma \otimes \mathrm{id}_A) \circ \mathcal{R}[\![R^{\sigma'}_{i+2}]\!]_A \\
&\qquad \text{, by Lemma 2 (3)} \\
&= (\mathrm{id}_{A^{\otimes n+|N|-1}} \otimes \lambda_A) \circ \mathcal{R}[\![R^\sigma_{|N|-1}]\!]_A \circ \cdots \\
&\qquad \cdots \circ (\mathrm{id}_{A^{\otimes n+i}} \otimes \gamma \otimes \mathrm{id}_A) \circ (\mathrm{id}_{A^{\otimes n+i+2}} \otimes \lambda_A) \circ \mathcal{R}[\![R^{\sigma'}_{i+2}]\!]_A \\
&\ \vdots \\
&= (\mathrm{id}_{A^{\otimes n+i}} \otimes \gamma \otimes \mathrm{id}_{A^{\otimes |N|-i-2}}) \circ (\mathrm{id}_{A^{\otimes n+|N|-1}} \otimes \lambda_A) \circ \mathcal{R}[\![R^{\sigma'}_{|N|-1}]\!]_A \circ \cdots \\
&\qquad \cdots \circ (\mathrm{id}_{A^{\otimes n+i+2}} \otimes \lambda_A) \circ \mathcal{R}[\![R^{\sigma'}_{i+2}]\!]_A
\end{aligned}$$

3. $\mathcal{R}[\![R^\sigma_{|N|}]\!]_A \circ (\mathrm{id}_{A^{\otimes n+i}} \otimes \gamma \otimes \mathrm{id}_{A^{\otimes |N|-i-2}}) = \mathcal{R}[\![R^{\sigma'}_{|N|}]\!]_A$, by Lemma 2 (4)

5.3 Compositionality. We show that the interpretation of finite idags is compositional for the operations of concatenation and juxtaposition.

Proposition 3. *Let $D = (N, R)$ be a finite $(\underline{n}, \underline{m})$-dag topologically sorted by σ and $D' = (N', R')$ a finite $(\underline{m}, \underline{\ell})$-dag topologically sorted by σ'. Write σ'/σ for the topological sorting of the concatenation $(\underline{n}, \underline{\ell})$-dag $D' \circ D = (N + N', R'')$ according to σ and then σ' (that is, with $(\sigma'/\sigma)_i = \sigma_i$ for $0 \le i < |N|$ and $(\sigma'/\sigma)_{|N|+j} = \sigma'_j$ for $0 \le j < |N'|$). Then,*

$$\mathcal{D}_{\sigma'}[\![D']\!]_A \circ \mathcal{D}_\sigma[\![D]\!]_A = \mathcal{D}_{\sigma'/\sigma}[\![D' \circ D]\!]_A \ \ .$$

Proof. The result follows from the definition of the interpretation function and the following identities:

1. $R^\sigma_i = R''^{(\sigma'/\sigma)}_i$ for all $0 \le i < |N|$,
2. $R'^{\sigma'}_j \circ \big(R^\sigma_{|N|} \oplus \mathrm{id}_j\big) = \big(R^\sigma_{|N|} \oplus \mathrm{id}_{j+1}\big) \circ R''^{(\sigma'/\sigma)}_{|N|+j}$ for all $0 \le j < |N'|$,
3. $R'^{\sigma'}_{|N'|} \circ \big(R^\sigma_{|N|} \oplus \mathrm{id}_{|N'|}\big) = R''^{(\sigma'/\sigma)}_{|N+N'|}$.

Proposition 4. *Let $D = (N, R)$ be a finite $(\underline{n}, \underline{m})$-dag topologically sorted by σ and let $D' = (N', R')$ be a finite $(\underline{n'}, \underline{m'})$-dag topologically sorted by σ'. The $(\underline{n+n'}, \underline{m+m'})$-dag $D \oplus D' = (N + N', R'')$ obtained by juxtaposition is topologically sorted by σ'/σ and*

$$\mathcal{D}_\sigma[\![D]\!]_A \otimes \mathcal{D}_{\sigma'}[\![D']\!]_A = \mathcal{D}_{\sigma'/\sigma}[\![D \oplus D']\!]_A \ \ .$$

Proof. The result follows from the definition of the interpretation function and the following identities:

1. $R''^{(\sigma'/\sigma)}_i = R^\sigma_i$ for all $0 \le i < |N|$,

2. $R''^{(\sigma'/\sigma)}_{|N|+j} = \mathrm{id}_{n+|N|} \oplus R'^{\sigma'}_{j}$ for all $0 \leq j < |N'|$,
3. $R''^{(\sigma'/\sigma)}_{|N+N'|} = R^{\sigma}_{|N|} \oplus R'^{\sigma'}_{|N'|}$.

Proof (of Lemma 1). For a cone

$$\begin{tikzcd}
& \mathbf{P} \arrow[dl] \arrow[dr] \arrow[dd, "G"] & \\
\mathbf{R} \arrow[dr, "F"'] & & \mathbf{N}_1 \arrow[dl, "H"] \\
& \mathscr{C} &
\end{tikzcd}$$

of symmetric monoidal categories, consider the D-algebra

$$A = G1$$
$$\eta_A = (I \cong F0 \xrightarrow{F\eta} A)\ , \quad \nabla_A = (A \otimes A \cong F(2) \xrightarrow{F\nabla} A)$$
$$\epsilon_A = (A \xrightarrow{F\epsilon} F0 \cong I)\ , \quad \varDelta_A = (A \xrightarrow{F\varDelta} F(2) \cong A \otimes A)$$
$$\lambda_A = (A \xrightarrow{H\lambda} A)$$

and define the unique mediating functor $\mathbf{D} \to \mathscr{C}$ to map $D \in \mathbf{D}(\underline{n}, \underline{m})$ to the composite

$$\mathcal{D}[\![D]\!]_A = \big(G(n) \cong A^{\otimes n} \xrightarrow{\mathcal{D}_\sigma[\![(N,R)]\!]_A} A^{\otimes m} \cong G(m) \big)$$

for a topological sorting σ of a representation (N, R) of the abstract idag D. (The symmetric monoidal structure of this functor is inherited from that of G.)

6 Conclusion

We have given an algebraic presentation of dag structure in the categorical language of PROPs, establishing that the PROP of finite abstract interfaced dags is universally characterised as being free on the symmetric monoidal equational theory of degenerate commutative bialgebras with a node. A main contribution in this respect has been a simple proof that provides an initial-algebra semantics for dag structure.

The technique introduced in the paper is robust and can be adapted to a variety of similar results. Firstly, one may drop the degeneracy condition on bialgebras. In this case, the free PROP on the sum of the symmetric monoidal equational theories B and N_1 consists of idags with edges weighted by positive natural numbers. These can be formalised as structures $\big(I, O, N, R \in \mathbb{N}^{(I+N)\times(O+N)}\big)$ such that $\big(I, O, N, \{(x, y) \mid R(x, y) \neq 0\}\big)$ is an idag. Secondly, one may introduce an antipode operator. In this case, the free PROP on the sum of the symmetric monoidal equational theories H and N_1 consists of idags with edges weighted by non-zero integers. Analogously, these can be formalised as structures $\big(I, O, N, R \in \mathbb{Z}^{(I+N)\times(O+N)}\big)$ such that $\big(I, O, N, \{(x, y) \mid R(x, y) \neq 0\}\big)$ is an idag. Of course, these two weightings respectively come from the structure of **B** and **H**, see §§ **4.5** and **4.6**. Finally, one may generalise from N_1 to N_L for a set of labels L. The resulting free PROPs consist of the appropriate versions of L-labelled idags.

In another direction, one may consider extending the symmetric monoidal theory D with equations involving the node. As suggested to us by Samuel Mimram, an interesting possibility is to introduce the equation

$$\lambda = \nabla \circ (\lambda \oplus \mathrm{id}_1) \circ \varDelta \ : 1 \to 1 \ .$$

According to the canonical decomposition (2), the effect of this equation on the free PROP **D** is to force on idags D the identification

$$\begin{aligned} D &= D^\sigma_{|N|} \circ (\mathrm{id}_{n+|N|-1} \oplus \lambda) \circ D^\sigma_{|N|-1} \circ \cdots \circ (\mathrm{id}_n \oplus \lambda) \circ D^\sigma_0 \\ &= D^\sigma_{|N|} \circ \big(\mathrm{id}_{n+|N|-1} \oplus (\nabla \circ (\lambda \oplus \mathrm{id}_1) \circ \varDelta)\big) \circ D^\sigma_{|N|-1} \circ \cdots \\ &\qquad \cdots \circ \big(\mathrm{id}_n \oplus (\nabla \circ (\lambda \oplus \mathrm{id}_1) \circ \varDelta)\big) \circ D^\sigma_0 \\ &= D^+ \end{aligned}$$

for D^+ the transitive closure of D. The free PROP consists then of transitive idags. For another example, one may consider introducing the equations

$$\lambda \circ \eta = \eta \ : 0 \to 1 \ , \quad \epsilon \circ \lambda = \epsilon \ : 1 \to 0 \ .$$

The resulting free PROP is that of idags with no dangling internal nodes.

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