Rough Notes on Presheaves

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Topics: Categories, functors, natural transformations, (co)limits, stable colimits, adjunctions, exponentials, presheaves, category of elements, discrete fibrations, Yoneda embedding, representable presheaves, density, free cocompletions, essential geometric morphisms, splitting idempotents / Cauchy completion, (co)limits and exponentials in presheaf categories, subobject classifier, topoi, subpresheaves and sieves, the topos of presheaves, preservation properties of the Yoneda embedding, orthogonality, Grothendieck topologies, sheaves, (sub)canonical topology, close and dense subobjects, double negation topology, separated presheaves, associated sheaf functor, Grothendieck relations, glueing.

Categories : objects & morphisms, equipped with a (partial) law of composition.

Small categories → have a (small) set of morphisms (and hence also of objects).

Locally small categories → have (small) sets of homs.

Examples:

1. **Set** = sets and functions.

2. **Poset** = partially ordered sets and monotone functions.

3. $\mathbf{Poset}_\wedge$ = partially ordered sets with bounded binary infs and stable (i.e. monotone and pullback preserving) functions.

4. **Graph**: Labelled directed graphs $N \underset{\mathrm{cod}}{\overset{\mathrm{dom}}{\leftleftarrows}} E$ and graph homomorphisms.

5. **Monoids** are one object categories.

6. **Preorders** are categories with at most one morphism in each hom

7. **Cpo** = ω-complete partial orders and ω-continuous functions.

8. **Monoid actions**: let M be a monoid; an M-action is a set X equipped with an action $X \times M \xrightarrow{\cdot} X$ such that $x \cdot e = x$ and $(x \cdot m) \cdot n = x \cdot (m \cdot n)$; a morphism $(X, \cdot_X) \xrightarrow[h]{} (Y, \cdot_Y)$ is a function $h: X \to Y$ such that

$$h(x \cdot_X m) = h(x) \cdot_Y m .$$

[Cf. dynamical systems]

9. $\mathbb{F}, \mathbb{I}, \mathbb{S}, \mathbb{B}$: finite sets and all, injective, surjective, bijective functions.

Functors

A functor is a morphism of categories. A functor $F: \mathcal{X} \to \mathcal{Y}$ is given by assignments

$$F: |\mathcal{X}| \to |\mathcal{Y}|$$

$$F_{X,X'}: \mathcal{X}(X, X') \to \mathcal{Y}(F(X), F(X'))$$

such that

$$F(\mathrm{id}_X) = \mathrm{id}_{FX}, \quad F(fg) = F(f)\,F(g)$$

Cat is the category of small categories and functors.

Natural transformations

A natural transformation is a morphism of functors. Given functors $F, G : \mathcal{X} \to \mathcal{Y}$ a natural transformation $\varphi : F \to G$ is given by a family

$$\{ \varphi_X : FX \to GX \text{ in } \mathcal{Y} \}_{X \in |\mathcal{X}|}$$

such that, for every $f : X \to X'$ in $\mathcal{X}$,

$$\begin{tikzcd} FX \arrow[r, "\varphi_X"] \arrow[d, "Ff"'] & GX \arrow[d, "Gf"] \\ FX' \arrow[r, "\varphi_{X'}"'] & GX' \end{tikzcd}$$

Example Let $\underline{\underline{\omega}} = (0 \to 1 \to \cdots \to n \to \cdots)$

A functor $F : \underline{\omega} \to \mathcal{X}$ amounts to a sequence

$$F0 \to F1 \to \cdots \to Fn \to \cdots \quad \text{in } \mathcal{X}$$

and a natural transformation $\varphi : F \to G : \underline{\omega} \to \mathcal{X}$ amounts to

$$\begin{tikzcd} F \arrow[d, "\varphi"'] & F_0 \arrow[r] \arrow[d, "\varphi_0"'] & F_1 \arrow[r] \arrow[d, "\varphi_1"] & \cdots \arrow[r] & F_n \arrow[r] \arrow[d, "\varphi_n"] & \cdots & \text{in } \mathcal{Y} \\ G & G_0 \arrow[r] & G_1 \arrow[r] & \cdots \arrow[r] & G_n \arrow[r] & \cdots & \end{tikzcd}$$

[We will go back to this kind of example later on.]

Constructions in categories

Initial object

$$0 \overset{\exists!}{\dashrightarrow} X \ \forall$$

$$\mathcal{X}(0,X) \cong 1$$

Terminal object

$$\forall X \overset{\exists!}{\dashrightarrow} 1$$

$$1 \cong \mathcal{X}(X,1)$$

(Binary) Coproducts

$$\begin{tikzcd}
 & A+B \arrow[dd, dashed, "\exists!"] & \\
A \arrow[ur] \arrow[dr] & & B \arrow[ul] \arrow[dl] \\
 & X & \\
 & \forall &
\end{tikzcd}$$

$$\mathcal{X}(A+B,X) \cong \mathcal{X}(A,X) \times \mathcal{X}(B,X)$$

(Binary) Products

$$\begin{tikzcd}
 & \forall X \arrow[dl] \arrow[dr] \arrow[dd, dashed, "\exists!"] & \\
A & & B \\
 & A \times B \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

$$\mathcal{X}(X,A) \times \mathcal{X}(X,B) \cong \mathcal{X}(X,A\times B)$$

Colimits $\left(\mathcal{J} \xrightarrow{\Delta} \mathcal{X}\right)$ Limits

$$\begin{tikzcd}
 & \Delta \arrow[dl] \arrow[dr, "\forall"] & \\
C \arrow[rr, dashed, "\exists!"'] & & X
\end{tikzcd}$$

$$\mathcal{X}(C,X) \cong \underline{\mathrm{Cone}}(\Delta,X)$$

(That is,

$$\underbrace{\mathcal{X}(\mathrm{colim}\,\Delta,X) \cong [\Delta, K_X]}_{\text{adjoint situation}}$$
)

$$\begin{tikzcd}
 & \Delta & \\
X \arrow[ur, "\forall"] \arrow[rr, dashed, "\exists!"'] & & L \arrow[ul]
\end{tikzcd}$$

$$\mathrm{Cone}(X,\Delta) \cong \mathcal{X}(X,L)$$

Example $\mathcal{J} = \underline{\omega}$

$$\begin{tikzcd}
\Delta_0 \arrow[r] \arrow[d] \arrow[drr] & \Delta_1 \arrow[r] \arrow[dl] \arrow[dr] & \cdots \arrow[r] & \Delta_n \arrow[r] \arrow[dlll] \arrow[dl] & \cdots \\
C \arrow[rr, dashed, "\exists!"'] & & X & &
\end{tikzcd}$$

Colimits in Set

sums:
$$\coprod_{i\in I} A_i = \biguplus_{i\in I} A_i$$
$$\begin{tikzcd} \biguplus_{i\in I} A_i \\ A_i \arrow[u, "\underline{in}_i"'] \end{tikzcd}$$

coequalisers:
$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r] & C \end{tikzcd}$$

$C = A+B/\approx$ where $f(a) \sim g(a)$ $\forall a \in A$

$\approx$ is the equivalence relation generated by

push outs:
$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[d] \\ A \arrow[r] & P \end{tikzcd}
\qquad
\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[d, "\underline{in}_2"] \arrow[dd, bend left] \\ A \arrow[r, "\underline{in}_1"'] \arrow[dr, bend right] & A+B \arrow[d, "coeq(\underline{in}_1 f, \underline{in}_2 g)"] \\ & P \end{tikzcd}$$

$P = A+B/\approx$ where $\underline{in}_1(fc) \sim \underline{in}_2(gc)$ $\forall c \in C$.

$\approx$ is the equivalence relation generated by

filtered colimits: $\Delta : \mathcal{G} \to \underline{Set}$

$\mathcal{G}$:
(1) non-empty
(2) $\forall_{n}^{m} \exists\, l$ ($m \to l$, $n \to l$)
(3) $\begin{tikzcd} n \arrow[r, shift left, "\forall"] \arrow[r, shift right] & m \arrow[r, "\exists"] & l \end{tikzcd}$

$$\begin{tikzcd} \Delta n \arrow[d] \\ \operatorname{colim}\Delta = \biguplus_{n\in|\mathcal{G}|} \Delta n \Big/ \approx \end{tikzcd}$$

where $\forall x \in \Delta n, y \in \Delta m$,
$x \approx y$ iff $\exists\ n \xrightarrow{f} l,\ m \xrightarrow{g} l$ s.t. $\Delta(f)(x) = \Delta(g)(y)$.

Limits in Set

products: $\prod_{i\in I} A_i = \{(a_i)_{i\in I} \mid a_i \in A_i\}$

$$\begin{tikzcd} \prod_{i\in I} A_i \arrow[d, "\pi_i"] \\ A_i \end{tikzcd}$$

equalisers:

$$\begin{tikzcd} E \arrow[r, tail, "eq"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

$$E = \{a \in A \mid f(a) = g(a)\}$$

pullbacks:

$$\begin{tikzcd} P \arrow[r, "g"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}
\qquad
\begin{tikzcd} P \arrow[d, "eq"] \arrow[dr, bend left, "g"] \arrow[dd, bend right, "p"'] & \\ A\times B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "!"] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

$$P = \{(a,b) \in A\times B \mid f(a) = g(b)\}$$

Remark:

$$\begin{tikzcd} S \arrow[r, "id"] \arrow[d, "id"'] & S \arrow[d, "f"] \\ S \arrow[r, "f"'] & A \end{tikzcd}$$ is a pullback

iff

f is a mono (in the sense that $\forall x, y : Z \to S$, $fx = fy \Rightarrow x = y$).

(In Set, a map is a mono iff it is injective.)

Remark: If $\begin{tikzcd} P \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & D \arrow[d, tail, "m"] \\ B \arrow[r, "f"'] & A \end{tikzcd}$ is a pullback where m is a mono, then p is a mono.

In Set, we have

$$\begin{tikzcd} P = \{(b,d) \in B \times D \mid f(b) = m(d)\} \arrow[r] \arrow[d] & D \arrow[d, tail, "m"] \\ B \arrow[r, "f"'] & A \end{tikzcd}$$

and

$$P \cong f^{-1}(\text{image}(m)).$$

I.e. pulling back a mono along a map is like taking inverse image.

Remark: Given a mono $m : P \rightarrowtail A$ thought of as a "predicate" on A, on a "generalised" element $x : X \to A$ we set

$$x \in P \text{ iff}_{def} \quad x \text{ factors through } m$$

(iff there exists (a necessarily unique) $x' : X \to P$ such that $x = m x'$)

(Check this definition in Set for a subset $P \hookrightarrow A$ and a "global" element $x : 1 \to A$.)

Remark: In Set, colimits are stable under pullbacks.

For finite coproducts this means that:

(1) if $\begin{matrix} X \\ \downarrow \\ 0 \end{matrix}$ then $X \cong 0$. (One also says that the initial object is strict.)

(2) In the situation

$$\begin{tikzcd} C_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "f"] & C_2 \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \\ A \arrow[r, "\amalg_1"'] & A+B & B \arrow[l, "\amalg_2"] \end{tikzcd}$$

The diagram $C_1 \to C \leftarrow C_2$ is a coproduct.

(Intuitively, $C \cong C_1 + C_2 \cong f^{-1}(A) + f^{-1}(B)$.)

__Adjunctions__

There are many equivalent definitions of adjoint situations, see Mac Lane's book for instance.

$F: \mathcal{X} \to \mathcal{Y}$ is left adjoint to $G: \mathcal{Y} \to \mathcal{X}$ iff there is a bijective correspondence

$$\frac{FX \to Y}{X \to GY} \qquad \text{i.e. } \mathcal{Y}(FX, Y) \cong \mathcal{X}(X, GY)$$

natural in X and Y.

__Remark__: Left adjoints preserve colimits.

An argument for the preservation of initial object is:

$$\mathcal{Y}(F0, Y) \cong \mathcal{X}(0, GY) \cong 1 \Rightarrow F0 \cong 0.$$

For arbitrary colimits (though below I will only do it for binary sums) one can proceed as follows:

First note the following fact:

$$X \cong X' \text{ iff } \Big\{ \underbrace{\mathcal{X}(A,X) \overset{\varphi_A}{\cong} \mathcal{X}(A,X')}_{\text{natural in } A} \Big\}_{A \in |\mathcal{X}|}$$

def: $X \cong X'$ iff $\exists f : X \to X'$ and $g : X' \to X$ s.t. $gf = id$ & $fg = id$.

i.e. for all $f : A \to B$ in $\mathcal{X}$ and $h : B \to X$ in $\mathcal{X}$

$$\varphi_A(h \circ f) = \varphi_B(h) \circ f : A \to X'$$

(Write down the proof.)

Using this, the preservation of binary sums for a left adjoint F comes from the following natural isomorphism:

$$\mathcal{Y}(F(A+B), Y) \cong \mathcal{X}(A+B, GY) \cong \mathcal{X}(A, GY) \times \mathcal{X}(B, GY)$$

$$\cong \mathcal{Y}(FA, Y) \times \mathcal{Y}(FB, Y) \cong \mathcal{Y}(F(A) + F(B), Y)$$

(Generalise the above to arbitrary colimits.)

<u>Remark</u>: Note that in <u>Set</u> we have

$$\underset{\text{in } \underline{Set}}{S \cong S'} \text{ iff } \underline{Set}(1, S) \cong \underline{Set}(1, S')$$

However this is not the case in all categories; for instance, it is not the case that $P \cong P'$ iff $\underline{\underline{Poset}}(1,P) \cong \underline{\underline{Poset}}(1,P')$ (Why?)

but it is the case that

$$\underset{\text{in } \underline{Poset}}{P \cong P'} \text{ iff } \underbrace{\underline{Poset}(\Sigma, P) \cong \underline{Poset}(\Sigma, P')}_{\text{natural in } \Sigma} \qquad \left(\Sigma \overset{\text{def}}{=} \begin{smallmatrix} \top \\ | \\ \bot \end{smallmatrix} \right)$$

Exponentiation is defined as right adjoint to product with an object (= "context extension").

$$\frac{X \times A \longrightarrow Y}{X \longrightarrow A \Rightarrow Y} \qquad \text{i.e. } \mathcal{C}(X \times A, Y) \cong \mathcal{C}(X, A \Rightarrow Y)$$

$$- \times A \dashv A \Rightarrow (-) : \mathcal{C} \to \mathcal{C} \quad , \ \mathcal{C} \text{ with products}$$

Def A category is said to be cartesian if it has finite products. A cartesian category is said to be closed if it has exponentials.

Def A category is said to be bicartesian if it has finite coproducts.

Remark. In a bicartesian closed category we have:

$$0 \xrightarrow{\cong} 0 \times A$$

$$(X \times A) + (Y \times A) \xrightarrow{\cong} (X + Y) \times A$$

because (the left adjoint $- \times A$) preserves colimits.

Exercise: Show that in a cartesian closed category, an initial object (whenever it exists) is strict (i.e. $X \to 0 \Rightarrow X \cong 0$).

Presheaves

Def The dual $\mathcal{C}^{op}$ of a category $\mathcal{C}$ is the category with

$$|\mathcal{C}^{op}| \overset{def}{=} |\mathcal{C}| \quad , \quad \mathcal{C}^{op}(A, B) \overset{def}{=} \mathcal{C}(B, A)$$

$$id^{\mathbb{C}^{op}}_A =_{df} id^{\mathbb{C}}_A \quad , \quad f \circ_{\mathbb{C}^{op}} g =_{df} g \circ_{\mathbb{C}} f .$$

Def The category of presheaves $\hat{\mathbb{C}}$ over a small category $\mathbb{C}$ is the category $\underline{Set}^{\mathbb{C}^{op}}$ with objects given by functors

$$F : \mathbb{C}^{op} \to \underline{Set}$$

and morphisms given by natural transformations

$$\varphi : F \xrightarrow{\cdot} G : \mathbb{C}^{op} \to \underline{Set} .$$

Thus a presheaf is given by a mapping

$$\begin{array}{c} A \\ f \uparrow \\ B \end{array} \text{ in } \mathbb{C} \quad \longmapsto \quad \begin{array}{c} F(A) \\ \downarrow Ff \\ F(B) \end{array} \text{ in } \underline{Set}$$

(note the reverse of direction)

such that

$$F(id_A) = id_{FA} \ ,$$

$$F(f \circ_{\mathbb{C}} g) = F(g) \circ_{\underline{Set}} F(f) \qquad C \xrightarrow{g} B \xrightarrow{f} A \text{ in } \mathbb{C}$$

There are many ways of visualising presheaves, we explore some below.

- <u>Presheaves as actions</u>: to give a presheaf is equivalent to give families

$$\{ F(A) \}_{A \in |\mathbb{C}|} \ , \ \{ \mathbb{C}(B,A) \times F(A) \xrightarrow[\bullet_{BA}]{} F(B) \}_{B,A \in |\mathbb{C}|}$$

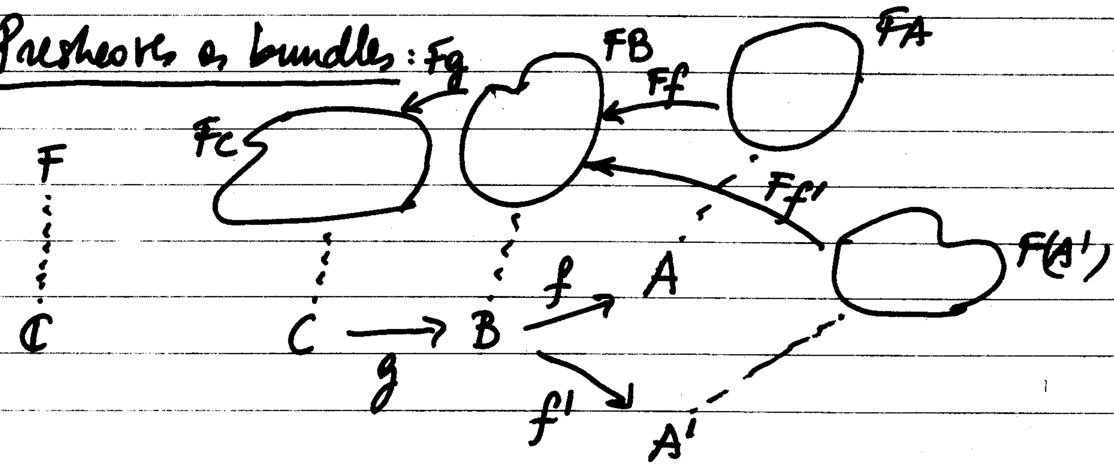
such that

$$\forall x \in FA,\quad id_A \cdot x = x$$

$$\forall x \in FC,\ \forall\ C \xrightarrow{g} B \xrightarrow{f} A \text{ in } \mathbb{C},$$

$$f\cdot(g\cdot x) = (f\circ g)\cdot x$$

- Presheaves as bundles:

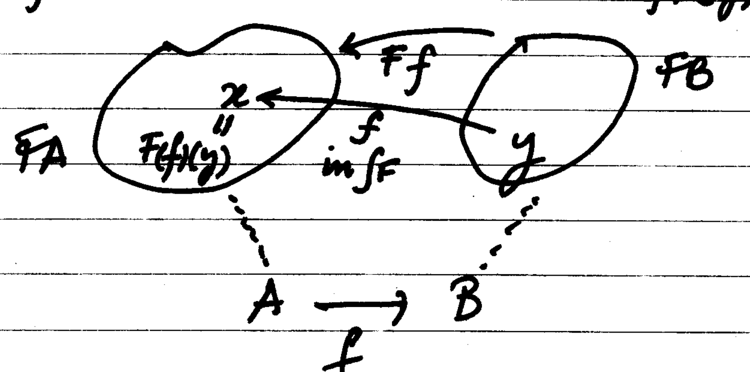


We can turn the "picture" above $\mathbb{C}$ into a category, called the category of elements of the presheaf F and sometimes denoted as $\int F$, as follows:

objects $\quad |\int F| = \coprod_{A\in|\mathbb{C}|} F(A) = \{(A,x) \mid A\in|\mathbb{C}|,\ x\in F(A)\}$

morphisms $\quad$ A morphism $(A,x) \to (B,y)$ in $\int F$ is a morphism

$$f : A \to B \text{ in } \mathbb{C} \text{ such that } F(f)(y) = x\,.$$



identities and composition are given as in $\mathbb{C}$.

__Remark__: Note that the category of elements $\int F$ comes equipped with a "projection" functor

$$\begin{array}{ccc} \int F & & (A,x) \\ {\scriptstyle \pi_F}\downarrow & & \downarrow \\ \mathbb{C} & & A \end{array}$$

with the following property: for every

$$\begin{array}{c} (B,y) \\ \vdots \\ A \xrightarrow[f]{} B \end{array}$$

there exists a unique $(A,x) \longrightarrow (B,y)$, namely $f : (A, F(f)(y)) \to (B,y)$, such that

$$\begin{array}{c} (A,x) \longrightarrow (B,y) \\ \vdots \\ A \xrightarrow[f]{} B \end{array}.$$

This kind of functors are called __discrete fibrations__. More formally, a functor

$$\begin{array}{c} \mathbb{E} \\ {\scriptstyle p}\downarrow \\ \mathbb{C} \end{array}$$

is called a discrete fibration if, for every

$$\begin{array}{ccc} T & & \text{in } \mathbb{E} \\ \vdots & & \\ A \xrightarrow[f]{} B = p(T) & & \text{in } \mathbb{C} \end{array}$$

typically denoted $f^*(B) \longrightarrow T$

There exists a unique $S \xrightarrow{\varphi} T$ in $\mathbb{E}$, such that

$$\begin{array}{c} S \xrightarrow{\varphi} T \\ \Big\downarrow \\ p(\varphi) = f \\ p(S) = A \longrightarrow B = p(T) \end{array}$$

Out of a discrete fibration $\begin{array}{c}\mathbb{E}\\ \downarrow p \\ \mathbb{C}\end{array}$ we can manufacture a presheaf F over $\mathbb{C}$ as follows:

- for $A \in |\mathbb{C}|$, $F(A) =_{df} \{ S \in |\mathbb{E}| : p(S) = A \} = p^{-1}(A)$
- for $f : A \to B$ in $\mathbb{C}$,

$$F(f) : F(B) \longrightarrow F(A) : T \longmapsto f^*(B) .$$

In fact the category of presheaves $\hat{\mathbb{C}}$ over $\mathbb{C}$ and the category of discrete fibrations $\underline{DFib}/\mathbb{C}$ over $\mathbb{C}$ are equivalent (prove it!).

It's time for examples:

1. $\hat{\mathbb{1}} \simeq \underline{Set}$
2. $\hat{\mathbb{2}} \simeq \underline{Set} \times \underline{Set}$.
3. For a monoid M viewed as a one-object category,

$$\hat{M} \simeq (\text{left})\ M\text{-actions} .$$

4. For $\mathbb{C} = \boxed{n \underset{\text{cod}}{\overset{\text{dom}}{\leftleftarrows}} e}$, $\hat{\mathbb{C}}$ has objects $F_n \underset{F(\text{cod})}{\overset{F(\text{dom})}{\leftleftarrows}} F_e$ which are nothing but labelled directed graphs with set of nodes F_n, set of edges F_e, and domain and codomain mappings respectively given by $F(\text{dom})$ and $F(\text{cod})$.

A natural transformation $\varphi : F \to G$ is given by mappings $\varphi_n : F_n \to G_n$ and $\varphi_e : F_e \to G_e$ between nodes and edges such that

$$\begin{array}{ccc} F_n & \xleftarrow{F\text{dom}} & F_e \\ \varphi_n \downarrow & & \downarrow \varphi_e \\ G_n & \xleftarrow[G\text{dom}]{} & G_e \end{array} \qquad \begin{array}{ccc} F_n & \xleftarrow{F\text{cod}} & F_e \\ \varphi_n \downarrow & & \downarrow \varphi_e \\ G_n & \xleftarrow[G\text{cod}]{} & G_e \end{array}$$

That is, natural transformations are nothing but graph homomorphisms.

5. For $\mathbb{C} = \underline{\underline{\omega}}^{op} = \boxed{0 \leftarrow 1 \leftarrow \cdots \leftarrow n \leftarrow \cdots}$, $\hat{\mathbb{C}}$ has objects

$$F_0 \to F_1 \to \cdots \to F_n \to \cdots \qquad \text{in } \underline{\underline{Set}}$$

Thought of as sets through time (Lawvere) with morphisms

$$\begin{array}{ccccccc} F & & F_0 \to F_1 \to \cdots \to F_n \to \cdots \\ \varphi \downarrow & & \varphi_0 \downarrow \quad \downarrow \varphi_1 \quad \cdots \quad \varphi_n \downarrow \quad \cdots \\ G & & G_0 \to G_1 \to \cdots \to G_n \to \cdots \end{array}$$

6. For $\mathbb{C} = \underline{\underline{\omega}} = \boxed{0 \to 1 \to \cdots \to n \to \cdots}$, $\hat{\mathbb{C}}$ has objects

$$F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_n \leftarrow \cdots \qquad \text{in } \underline{\underline{Set}}$$

which are better pictured as "forests" (Joyal) with F_n the set of nodes at level n and with the mappings $F_{n+1} \to F_n$ corresponding to the assignment of a parent. (Note that the category of elements $\int F$ provides exactly this representation!)

A natural transformation corresponds to a root-preserving forest homomorphism.

(*Exercise* Let A be a set (of labels) and consider the poset $\underline{P}(A)$ of strings in A ordered by prefix. Interpret $\widehat{\underline{P}(A)}$ along the above lines. [Notice that $\underline{P}(1) \cong \underline{\omega}$.])

Exercise
7. Consider

$$\mathbb{C} = \begin{tikzcd} 1 \arrow[loop left, "id"] \arrow[r, shift left=2, "\top"] \arrow[r, shift right=2, "\perp"'] & \Sigma \arrow[l, "!" description] \arrow[loop above, "\top"] \arrow[loop right, "id"] \arrow[loop below, "\perp"] \end{tikzcd}$$

The full subcategory of Poset determined by 1 and $\Sigma = (\perp \le \top)$

and interpret $\hat{\mathbb{C}}$ along the lines of example 4.

Consider

$$\mathbb{D} = \begin{tikzcd} \Sigma \arrow[loop above, "\top"] \arrow[loop right, "id"] \arrow[loop below, "\perp"] \end{tikzcd}$$

the full subcategory of Poset determined by Σ.

and compare $\hat{\mathbb{C}}$ with $\hat{\mathbb{D}}$.

8. It is not always easy to picture/visualise/etc. presheaves; for instance, consider presheaves over the monoid of ω-continuous endomorphisms on $\bar{\omega} = \boxed{0 \le 1 \le \dots \le n \le \dots \le \infty}$. (Fiore & Rosolini)

What have we done by moving from $\mathbb{C}$ to $\hat{\mathbb{C}}$? Many things...
(... but not as many as we would like, that's one of the reasons for looking at sheaves later on.)

Yoneda embedding

Def A functor $F : \mathcal{X} \to \mathcal{Y}$ is an embedding (i.e. full and faithful) if, for every $A, B \in |\mathcal{X}|$, the mapping

$$F_{A,B} : \mathcal{X}(A,B) \to \mathcal{Y}(FA, FB)$$

is a bijection.

The category $\hat{\mathbb{C}}$ of presheaves (over $\mathbb{C}$) contains a "copy" of $\mathbb{C}$. Indeed, we have an embedding

$$y : \mathbb{C} \hookrightarrow \hat{\mathbb{C}} \qquad \text{(Yoneda)}$$

$$C \mapsto y(C) =_{\mathrm{def}} \mathbb{C}(-,C) = \mathbb{C}^{op}(C,-)$$

I.e.

$$y(C)(X) = \mathbb{C}(X,C) \quad \forall X \in |\mathbb{C}|$$

$$\begin{aligned} y(C)(f) : \mathbb{C}(Y,C) &\to \mathbb{C}(X,C) \qquad \forall X \xrightarrow{f} Y \text{ in } \mathbb{C} \\ h &\mapsto h \circ f \end{aligned}$$

$$f \Big\downarrow \begin{matrix} C \\ \\ D \end{matrix} \quad \mapsto \quad \begin{matrix} y(C) \\ \big\downarrow y(f) \\ y(D) \end{matrix} \quad \text{where} \quad \begin{aligned} \mathbb{C}(X,C) &\xrightarrow{y(f)_X} \mathbb{C}(X,D) \\ h &\longmapsto f \circ h \end{aligned}$$

(Cayley)

Exercise: Consider the Yoneda embedding for $\mathbb{C}$ a monoid (as usual thought of as a one object category) and for $\mathbb{C}$ a partially ordered set (Dedekind).

That, y is an embedding can be seen from the famous Yoneda lemma:

we have a natural bijective correspondence

$$\hat{\mathbb{C}}(y(C),P) \cong P(C) \qquad (*)$$

for all $C\in|\mathbb{C}|$ and $P\in|\hat{\mathbb{C}}|$. (That is, to give a natural transformation $y(C) \dashrightarrow P$ is to give an element of $P(C)$.)

A neat example in which to see the correspondence (*) is the following: take $\mathbb{C} = \underline{\omega}^{op} = \boxed{0\to 1\to \cdots \to n \to \cdots}$, then we have

$$\begin{tikzcd}
y(k) \arrow[d, "\varphi"] & \overset{\overset{0}{\shortparallel}}{[k,0]} \arrow[r] \arrow[d, "\varphi_0"] & \overset{\overset{0}{\shortparallel}}{[k,1]} \arrow[r] \arrow[d, "\varphi_1"] & \cdots \arrow[r] & \overset{\overset{0}{\shortparallel}}{[k,k-1]} \arrow[r] \arrow[d, "\varphi_{k-1}"] & \overset{\overset{1}{\shortparallel}}{[k,k]} \arrow[r] \arrow[d, "\varphi_k"] & \overset{\overset{1}{\shortparallel}}{[k,k+1]} \arrow[r] \arrow[d, "\varphi_{k+1}"] & \cdots \\
P & P_0 \arrow[r] & P_1 \arrow[r] & \cdots \arrow[r] & P_{k-1} \arrow[r] & P_k \arrow[r] & P_{k+1} \arrow[r] & \cdots
\end{tikzcd}$$

where, for $0\le n\le k-1$, $[k,n]=0$ and φ_n is the unique map $0\to P_n$; for $n\geq k+1$, $[k,n]=\underset{\{k\to n\}}{\underset{\shortparallel}{1}}$ and (by naturality)

$$\varphi_n(k\to n) = P(k\to n)\left(\varphi_k(k\xrightarrow{id}k)\right)$$

Hence, φ is completely determined by the element $\varphi_k(k\xrightarrow{id}k)$ in P_k!

In general, the bijective correspondence (*) is given by

$$\hat{\mathbb{C}}(y(C),P) \xrightarrow{\cong} P(C)$$

$$\varphi \longmapsto \varphi_C(id_C)$$

Corollary (of the Yoneda lemma)
$P\cong Q$ in $\hat{\mathbb{C}}$ iff $\hat{\mathbb{C}}(y(A),P) \cong \hat{\mathbb{C}}(y(A),Q)$ nat. in A

(Cf. pp. 8)

Exercise Find the inverse mapping.

The Yoneda lemma justifies the following extremely useful conventions (Lawvere):

- for $A \xrightarrow{f} B$ in $\mathbb{C}$, we write

$$A \xrightarrow{f} B \text{ in } \widehat{\mathbb{C}} \quad \text{for} \quad y(A) \xrightarrow{y(f)} y(B) \text{ in } \widehat{\mathbb{C}}$$

- for $A \in |\mathbb{C}|$, $P \in |\widehat{\mathbb{C}}|$, and $x \in P(A)$, we write

$$A \xrightarrow{x} P \text{ in } \widehat{\mathbb{C}} \quad \text{for} \quad y(A) \xrightarrow{\hat{x}} P \text{ in } \widehat{\mathbb{C}}$$

(where $\hat{x}_A(\mathrm{id}_A) = x$)

With these notations, we have:

- for $f : B \to A$ in $\mathbb{C}$, $P \in |\widehat{\mathbb{C}}|$, and $x \in P(A)$, the composite

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[rr, bend right, "x \circ f"'] & A \arrow[r, "x"] & P \end{tikzcd}$$

sometimes denoted $x \cdot_P f$ corresponds to $P(f)(x) \in P(B)$.

- for $\varphi : P \dot{\to} Q$ in $\widehat{\mathbb{C}}$, $A \in |\mathbb{C}|$, and $x \in P(A)$, the composite

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[rr, bend right, "\varphi \circ x"'] & P \arrow[r, "\varphi"] & Q \end{tikzcd}$$

corresponds to $\varphi_A(x) \in Q(A)$.

- for $\varphi : P \to Q$ in $\widehat{\mathbb{C}}$, $f : B \to A$ in $\mathbb{C}$, and $x \in P(A)$, the commutativity of

$$\begin{tikzcd} B \arrow[d, "f"'] \arrow[dr, "x\circ f"] & \\ A \arrow[r, "x"] \arrow[dr, "\varphi\circ x"'] & P \arrow[d, "\varphi"] \\ & Q \end{tikzcd}$$

amounts to the equality

$$\varphi_B\big(P(f)(x)\big) = Q(f)\big(\varphi_A(x)\big)$$

which is exactly the naturality of φ:

$$\begin{tikzcd} P(A) \arrow[r, "\varphi_A"] \arrow[d, "P(f)"'] & Q(A) \arrow[d, "Q(f)"] \\ P(B) \arrow[r, "\varphi_B"'] & Q(B) \end{tikzcd}$$

In fact a natural transformation $P \dot{\to} Q$ in $\widehat{\mathbb{C}}$ is a mapping associating each $x : A \to P$ in $\widehat{\mathbb{C}}$ ($A \in |\mathbb{C}|$) with an $\tilde{x} : A \to Q$ in $\widehat{\mathbb{C}}$ such that, for all $f : B \to A$ in $\mathbb{C}$, $\tilde{x}\circ f = \widetilde{x\circ f}$.

- for $\varphi, \phi : P \to Q$ in $\widehat{\mathbb{C}}$,

$$\varphi = \phi \text{ iff } \forall A \in |\mathbb{C}|, \forall x : A \to P \text{ in } \widehat{\mathbb{C}}, \ \varphi x = \phi x .$$

(This is the generalisation of <u>extensionality</u> from $\widehat{\mathbb{1}} \cong \underline{Set}$ to an arbitrary presheaf category.)

- For $P \in |\widehat{\mathbb{C}}|$, we have the isomorphism of categories

$$\int P \cong y_{/P} \qquad \text{(Why?)}$$

where for a functor $F : \mathcal{X} \to \mathcal{Y}$ and an object $Y \in |\mathcal{Y}|$, the (slice) category $F/_Y$ has objects given by pairs $(X \in |\mathcal{X}|,\ f : FX \to Y \text{ in } \mathcal{Y})$ and morphisms $(X, f) \to (X', f')$ given by morphisms $h : X \to X'$ in $\mathcal{X}$ such that

$$\begin{tikzcd}
FX \arrow[rr, "Fh"] \arrow[dr, "f"'] & & FX' \arrow[dl, "f'"] \\
& Y &
\end{tikzcd} \quad \text{in } \mathcal{Y}$$

commutes.

(Question: What is the discrete fibration associated to $y(c)$?)

Density

We have a canonical ("domain") diagram

$$\Delta_P : y/P \longrightarrow \hat{\mathbb{C}} : \begin{array}{c} A \\ x \downarrow \\ P \end{array} \longmapsto A$$

with respect to which there is a canonical cone

$$p : \Delta \dot{\to} P \ , \quad p\left(\begin{array}{c} A \\ \downarrow x \\ P \end{array}\right) =_{df} x : A \to P$$

which is colimiting. Indeed, for every cone $q : \Delta \dot{\to} Q$ in $\hat{\mathbb{C}}$ (That is, we have the following situation

$$\begin{tikzcd}
\Delta : & A \arrow[r, "f"] \arrow[rr, bend right, "q_x"'] \arrow[d, dotted, dash] & B \arrow[r, "q_y"] \arrow[d, dotted, dash] & Q & \text{in } \hat{\mathbb{C}} \\
y/_P : & A \arrow[r, "f"] \arrow[dr, "x"'] & B \arrow[d, "y"] & & \\
& & P & &
\end{tikzcd}$$

)

There exists a unique mediating morphism

$$\mu : P \dot{\longrightarrow} Q \text{ in } \hat{\mathbb{C}} \;, \; \mu(x) =_{\mathrm{def}} q_x .$$

(Fill-in and check the details.)

Hence, we have shown that

every presheaf is a canonical colimit of representables

or, in other words, that

The representables are dense (or a dense generator) in $\hat{\mathbb{C}}$

where the missing definitions are as follows:

<u>Def</u> A presheaf is said to be <u>representable</u> if it is isomorphic to one of the form $y(c)$.

<u>Def</u> A set of objects $D \subseteq |\mathcal{E}|$ is said to be <u>a dense generator</u> in $\mathcal{E}$ if, writing $\mathbb{D}$ for the full subcategory of $\mathcal{E}$ determined by D and $J : \mathbb{D} \hookrightarrow \mathcal{E}$ for the corresponding embedding, the canonical diagram

$$J/_C \xrightarrow{\Delta_C} \mathcal{E} : \begin{array}{c} d \\ \downarrow \\ C \end{array} \mapsto d \qquad (C \in |\mathcal{E}|)$$

has the canonical cone

$$\gamma : \Delta_C \to C \;, \; \gamma\left(\begin{array}{c} d \\ \downarrow x \\ C \end{array} \right) =_{\mathrm{def}} x : d \to C$$

as colimit.

Diagramatically,

$$\begin{tikzcd} d \arrow[r, "\forall"] \arrow[rr, "\exists x"] \arrow[d, "x"'] & d' \arrow[r, "\exists x'"] \arrow[dl, "x'"] & z \\ c \arrow[urr, dashed, "\exists !"'] & & \end{tikzcd}$$

<u>Prop</u>: If $\mathbb{D}$ is a dense generator in $\mathcal{C}$ then we have the following factorisation

$$\begin{tikzcd} \mathbb{D} \arrow[r, hook, "y"] \arrow[dr, "J"'] & \hat{\mathbb{D}} \\ & \mathcal{C} \arrow[u, dashed, "y'"'] \end{tikzcd} \qquad y'(c) \overset{\text{def}}{=} \mathcal{C}(J(-), c) \quad \text{(} c \mapsto y'(c) \text{)}$$

and y' is an embedding (i.e full and faithful). Indeed, to see that

$$\mathcal{C}(c, c') \longrightarrow \hat{\mathbb{D}}(\mathcal{C}(J-, c), \mathcal{C}(J-, c'))$$
$$f \longmapsto y'(f) : x \mapsto fx$$

is a bijection, observe that for every natural transformation $\varphi : \mathcal{C}(J-, c) \to \mathcal{C}(J-, c')$ we have

$$\begin{tikzcd} d \arrow[r, "\forall"] \arrow[rr, "\varphi x"] \arrow[d, "x"'] & d' \arrow[r, "\varphi x'"] \arrow[dl, "x'"] & c' \\ c \arrow[urr, dashed, "\exists !"'] & & \end{tikzcd}$$

(Make the argument precise.)

Exercise: Conversely, for $D \subseteq |\mathbb{G}|$, show that if $y': \mathbb{G} \to \hat{\mathbb{D}}$ is an embedding then D is a dense generator.

The importance of the above proposition is that it allows us to embed locally small categories in presheaf categories. Here are some examples:

1. Σ is a dense generator in Poset, hence

$$\underline{\underline{\text{Poset}}} \hookrightarrow \hat{\mathbb{C}}$$

for $\mathbb{C}$ the monoid of monotone endomorphisms on Σ. (Cf. (7) on pp. 15)

2. $\Sigma \times \Sigma$ is a dense generator in $\underline{\underline{\text{Poset}}}_\wedge$ (defined on pp. 0), hence

$$\underline{\underline{\text{Poset}}}_\wedge \hookrightarrow \hat{\mathbb{C}}$$

for $\mathbb{C}$ the monoid of stable endomorphisms on $\Sigma \times \Sigma$.

3. $\bar{\omega} = \boxed{0 \leq 1 \leq \cdots \leq n \leq \cdots \leq \infty}$ is a dense generator in $\underline{\underline{\text{Cpo}}}$, hence

$$\underline{\underline{\text{Cpo}}} \hookrightarrow \hat{\mathbb{C}}$$

for $\mathbb{C}$ the monoid of continuous endomorphisms on $\bar{\omega}$ (Cf. (8) on pp. 15)

Free cocompletion. We exhibit a universal property of the presheaf construction as a free cocompletion. (Exercise: Show that $\hat{\mathbb{C}}$ is cocomplete; construct colimits pointwise.)

We start generalising the construction in the proposition on page 22: a functor $\mathbb{C} \to \mathbb{D}$ can be "extended" to a functor $\mathbb{D} \to \hat{\mathbb{C}}$ as follows:

$$\begin{tikzcd} \mathbb{C} \arrow[rr, hook, "y"] \arrow[dr, "F"'] & & \hat{\mathbb{C}} \\ & \mathbb{D} \arrow[ur, bend right, "F_*"'] & \end{tikzcd} \qquad \begin{array}{c} F_*(D) =_{df} \mathbb{D}(F(-), D) \\ \uparrow \\ D \end{array}$$

<u>Claim:</u>

$$\begin{tikzcd} \mathbb{C} \arrow[rr, hook, "y"] \arrow[dr, "F"'] & & \hat{\mathbb{C}} \arrow[dl, dashed, "F^{\#}"'] \arrow[dl, phantom, "\dashv", shift left=3] \\ & \mathbb{D} \arrow[ur, bend right, "F_*"'] & \end{tikzcd}$$

$\mathbb{D}$ cocomplete

Note that if such an $F^{\#}$ exists then it has to preserve colimits (because it is a left adjoint), then we should have

$$F^{\#}(P) \cong F^{\#}\big(\operatorname*{colim}_{i \in \int P} y(\pi_i)\big)$$

$$\cong \operatorname*{colim}_{i \in \int P} F^{\#}(y\,\pi_i)$$

$$\cong \operatorname*{colim}_{i \in \int P} F(\pi_i)$$

Hence we define

$$F^{\#}(P) =_{df} \operatorname{colim}\Big(\int P \xrightarrow{\pi} \mathbb{C} \xrightarrow{F} \mathbb{D}\Big).$$

Before checking the adjoint situation

$$F^{\#} \dashv F_*$$

we observe the free cocompletion property: consider

$$\begin{tikzcd} \mathbb{C} \arrow[r, hook, "y"] \arrow[dr, "Hy"'] & \hat{\mathbb{C}} \arrow[d, "H"] \\ & \mathbb{D} \end{tikzcd}$$

H cocontinuous

$\mathbb{D}$ cocomplete

Then

$$(Hy)^{\#}(P) =_{\mathrm{df}} \mathrm{colim}\left(\textstyle\int P \xrightarrow{\pi} \mathbb{C} \xhookrightarrow{y} \hat{\mathbb{C}} \xrightarrow{H} \mathbb{D}\right)$$

$$\cong H\left(\mathrm{colim} \textstyle\int P \xrightarrow{\pi} \mathbb{C} \xhookrightarrow{y} \hat{\mathbb{C}}\right)$$

$$\cong H(P)$$

Hence

$$\begin{tikzcd} \mathbb{C} \arrow[r, hook, "y"] \arrow[dr, "\forall"'] & \hat{\mathbb{C}} \arrow[d, dashed, "\exists!"] \\ & \mathbb{D} \end{tikzcd}$$

(up to iso)

cocontinuous

$\mathbb{D}$ cocomplete

We check that $F^{\#} \dashv F_*$:

$$\hat{\mathbb{C}}(P, F_*(D))$$

$$\cong \hat{\mathbb{C}}\left(\mathrm{colim}_{i \in \int P}\, y(\pi_i),\ \mathbb{D}(F(-), D)\right)$$

$$\cong \lim_{i \in (\int P)^{op}} \hat{\mathbb{C}}\left(y(\pi_i),\ \mathbb{D}(F(-), D)\right)$$

$$\cong \lim_{i \in (\int P)^{op}} \mathcal{D}(F\pi i, D)$$

$$\cong \mathcal{D}(\operatorname{colim}_{i \in \int P} F\pi i, D)$$

$$\cong \mathcal{D}(F^{\#}(P), D)$$

(*Remark*: $\mathcal{X}(\operatorname{colim}(\Delta \xrightarrow{H} \mathcal{X}), X)$
$$\cong \lim(\Delta^{op} \xrightarrow{\mathcal{X}(H-,X)} \underline{\underline{Set}}) \,.)$$

We consider now an important special case of the above construction: For a functor $f : \mathbb{A} \to \mathbb{B}$, we have the following situation

$$\begin{tikzcd}
\mathbb{A} \arrow[r, hook, "y_{\mathbb{A}}"] \arrow[d, "f"'] & \widehat{\mathbb{A}} \arrow[d, shift right=6, "f_!"'] \arrow[d, shift left=6, "f_*"] \\
\mathbb{B} \arrow[r, hook, "y_{\mathbb{B}}"'] & \widehat{\mathbb{B}} \arrow[u, "f^*"']
\end{tikzcd}
\qquad f_! \dashv f^* \dashv f_*$$

(an essential geometric morphism)

where we have already encountered the adjunction $f_! \dashv f^*$ as:

$$f_! =_{def} (y_{\mathbb{B}} f)^{\#}$$

$$f^* =_{def} (y_{\mathbb{B}} f)_* \,, \quad f^*(Q) =_{def} \widehat{\mathbb{B}}(y_{\mathbb{B}} f-, Q) \cong Q(f-)$$

and where, since in particular we want

$$\begin{array}{cc}
y(b) \longrightarrow f_*(P) & \text{in } \hat{\mathbb{B}} \\
\hline\hline
\mathbb{B}(f-,b) \cong f^*(y\,b) \longrightarrow P & \text{in } \hat{\mathbb{A}}
\end{array},$$

we set

$$f_*(P)(b) = \hat{\mathbb{A}}\big(\mathbb{B}(f-,b),\, P\big)$$

to obtain $f^* \dashv f_*$. (Fill in the details.)

In the above situation we have the following proposition:

> f is an embedding
>
> iff
>
> $f!$ is an embedding
>
> iff
>
> The unit $P \to f^* f_! P$ $(P \in |\hat{\mathbb{A}}|)$ of $f! \dashv f^*$ is an iso.

The last "iff" is a general result (see Mac Lane, for instance). As for the first "iff" it is left as an exercise, though below I'll check that $P \cong f^* f_! P$ $(P \in |\hat{\mathbb{A}}|)$. Indeed, we have

$$\begin{array}{c}
(f^*(f_! P))(a) \\
\hline\hline
y_{\mathbb{A}}(a) \longrightarrow f^*(f!\,P) \\
\hline\hline
y_{\mathbb{B}}(fa) \cong f!(y_{\mathbb{A}}\,a) \longrightarrow f!(P) \\
\hline\hline
(f!\,P)(f\,a)
\end{array}$$

and

$$(f_!P)(fa) \cong \left(\operatorname{colim}_{i\in\int P} y_{\mathbb{B}}(f\pi_i)\right)(fa)$$

$$\cong \operatorname{colim}_{i\in\int P} \left(y_{\mathbb{B}}(f\pi_i)(fa)\right)$$

$$\cong \operatorname{colim}_{i\in\int P} \mathbb{B}(fa, f\pi_i)$$

$$\cong \operatorname{colim}_{i\in\int P} \mathbb{A}(a, \pi_i)$$

$$\cong \operatorname{colim}_{i\in\int P} \left(y_{\mathbb{A}}(\pi_i)(a)\right)$$

$$\cong \left(\operatorname{colim}_{i\in\int P} y_{\mathbb{A}}(\pi_i)\right)(a)$$

$$\cong P(a).$$

(Make sure you understand all the steps.)

<u>Exercise</u>: For an embedding f show that f_* is also an embedding, or equivalently that the counit $f^*f_*P \to P$ (for $P \in |\hat{\mathbb{A}}|$) of the adjunction $f^* \dashv f_*$ is an iso. [Hint: There is a simple abstract proof.]

<u>Example/Exercise</u> Consider the essential geometric morphism induced by the "hidding" functor

$$\underline{P}(A \uplus \{z\}) \xrightarrow{h} \underline{P}(A)$$

$$z^{n_1}a_1 \ldots z^{n_\ell}a_\ell z^{n_{\ell+1}} \longmapsto a_1 \ldots a_\ell$$

$(a_i \in A)$

where $\underline{P}(X)$ is the poset of strings on X ordered by prefix. (See (6) on pp. 15.)

Splitting idempotents (Freyd) / Cauchy Completion (Lawvere)

Performing the presheaf construction over different categories may yield the same result (though it usually provides a different perspective). We start by looking at the simplest such example:

$$\begin{tikzcd} \mathbb{A} = & a \arrow[loop above, "e"] \arrow[d, "j"'] & (e^2 = e) \\ \mathbb{B} = & b \arrow[loop left, "id"] \arrow[r, shift left, tail, "i"] & a \arrow[l, shift left, two heads, "p"] \arrow[loop above, "e = ip"] \end{tikzcd} \qquad \text{induces} \qquad \begin{tikzcd} \mathbb{A} \arrow[r, hook] \arrow[d, "j"'] & \hat{\mathbb{A}} \arrow[d, shift right=3, "j!"'] \arrow[d, shift left=3, "j_*"] \\ \mathbb{B} \arrow[r, hook] & \hat{\mathbb{B}} \arrow[u, "j^*"' description] \end{tikzcd} \qquad j! \dashv j^* \dashv j_*$$

and we have

$$\begin{tikzcd} P(a) \arrow[loop above, "Pe"] \arrow[d, "j!"'] \\ P(b) \arrow[r, shift left, tail] & P(a) \arrow[l, shift left, two heads] \arrow[loop above, "Pe"] \end{tikzcd} \quad \dashv \quad \begin{tikzcd} Q(a) \arrow[loop above, "Qe"] \\ Q(b) \arrow[r, shift left, tail] & Q(a) \arrow[l, shift left, two heads] \arrow[loop above, "Qe"] \arrow[ul, "j^*"] \end{tikzcd} \quad \dashv \quad \begin{tikzcd} P(a) \arrow[loop above, "Pe"] \arrow[d, "j_*"] \\ P(b) \arrow[r, shift left, hook] & P(a) \arrow[l, shift left, two heads, "Pe"] \arrow[loop above, "Pe"] \end{tikzcd}$$

$P(b) \overset{def}{=} \mathrm{Coeq}(Pe, id)$

where
$$\begin{tikzcd} P(a) \arrow[r, shift left, "Pe"] \arrow[r, shift right, "id"'] & P(a) \arrow[r, two heads, "coeq"] \arrow[d, "Pe"'] & P(b) \arrow[dl, dotted, "\exists!"] \\ & P(a) & \end{tikzcd}$$

$P(b) \overset{def}{=} \underline{Fix}(Pe) = \{x \in P(a) \mid (Pe)(x) = x\}$

Exercise: Obtain the above explicit descriptions of $j!$,

j^*, and j_* from the general ones given on pp. 26 and 27.

<u>Exercise</u> We know that $id \cong j^* j_!$ and that $j^* j_* \cong id$. Show that $j_! j^* \cong id$ and $id \cong j_* j^*$, and that $j_! \cong j_*$.

Hence, the presheaf categories $\hat{\mathbb{A}}$ and $\hat{\mathbb{B}}$ are equivalent.

In general, categories that only differ for a splitting of idempotents yield the same category of presheaves. More precisely:

<u>Thm</u> Let $j : \mathbb{A} \hookrightarrow \mathbb{B}$ be an embedding such that, for every $b \in |\mathbb{B}|$, there exists an idempotent $a \circlearrowleft e$ in $\mathbb{A}$ such that b comes from splitting $j(e)$; that is,

$$id \circlearrowright b \underset{p}{\overset{i}{\rightleftarrows}} j(a) \circlearrowleft j(e) = ip \quad \text{in } \mathbb{B}.$$

Then, we have the equivalence of categories

$$\hat{\mathbb{A}} \simeq \hat{\mathbb{B}}$$

(In fact, the adjunctions $j_! \dashv j^* \dashv j_*$ yield the desired equivalence.)

We proceed to show that in the above situation,

$$(*) \qquad j_! j^*(Q) \cong Q \qquad \text{for all } Q \in |\hat{\mathbb{B}}|$$

We start with the following:

Exercise

(1) If $\mathrm{id} \circlearrowright B \underset{p}{\overset{i}{\rightleftarrows}} A \circlearrowleft e = ip$ then $A \underset{\mathrm{id}}{\overset{e}{\rightrightarrows}} A \xrightarrow{p} B$ is an absolute coequaliser. (Recall that a colimit is said to be absolute (Paré) if it is preserved by all functors.)

(2) For $e^2 = e$, if $A \underset{\mathrm{id}}{\overset{e}{\rightrightarrows}} A \overset{p}{\twoheadrightarrow}$ is a coequaliser then for $i : B \to A$ given by

$$\begin{tikzcd} A \arrow[r, shift left, "e"] \arrow[r, shift right, "\mathrm{id}"'] & A \arrow[r, "p"] \arrow[d, "e"'] & B \arrow[dl, dotted, "\exists!\, i"] \\ & A & \end{tikzcd}$$

we have that $\mathrm{id} \circlearrowright B \underset{p}{\overset{i}{\rightleftarrows}} A \circlearrowleft e$.

Going back to $(**)$ on pp. 30, let us write $\overline{Q}$ for $j_! j^* Q$. We have that

$$j^* \overline{Q} = j^* j_! j^* Q \cong j^* Q$$

or in other words that

$$\overline{Q} j \cong Q j$$

Thus for every splitting

$$\mathrm{id} \circlearrowright b \underset{p}{\overset{i}{\rightleftarrows}} ja \circlearrowleft je = ip \qquad \text{in } \mathbb{B}$$

we have

$$\begin{array}{ccc} \mathrm{id} \circlearrowright Q(b) & \underset{Qp}{\overset{Qi}{\rightleftarrows}} & Q(ja) \circlearrowleft Qje \\ & & \wr\wr \\ \mathrm{id} \circlearrowright \overline{Q}(b) & \underset{\overline{Q}p}{\overset{\overline{Q}i}{\rightleftarrows}} & \overline{Q}(ja) \end{array}$$

and hence (e.g. using the above exercise) we have that

$$\begin{array}{c} Q(b) \\ \wr\wr \\ \overline{Q}(b) \end{array}$$

The fact that this isomorphism is natural in $b \in |\mathbb{B}|$ is an instructive check that it's left as an <u>exercise</u>.

There is a universal way in which to split the idempotents of a category (in a formal way). The construction (due to Freyd) is as follows.

<u>Def</u> $\underline{\mathrm{Split}}(\mathbb{A})$ has objects given by the idempotents of $\mathbb{A}$ with morphisms ${}^{e}\!\!\circlearrowright a \to a' \circlearrowleft^{e'}$ given by maps $f : a \to a'$ in $\mathbb{A}$ such that $fe = f = fe'$. The identities are given by $id_e = e$ and composition is given as in $\mathbb{A}$. The universal embedding $\mathbb{A} \hookrightarrow \underline{\mathrm{Split}}(\mathbb{A})$ maps $a \in |\mathbb{A}|$ to $id_a \in |\underline{\mathrm{Split}}(\mathbb{A})|$.

<u>Exercises</u>

(1) Describe the universal property of $\mathbb{A} \hookrightarrow \underline{\mathrm{Split}}(\mathbb{A})$

(2) Show that $\{ id_a \mid a \in |\mathbb{A}| \} \subseteq |\underline{\mathrm{Split}}(\mathbb{A})|$ is a dense generator in $\underline{\mathrm{Split}}(\mathbb{A})$.

(3) Show that $\underline{\mathrm{Split}}(\mathbb{A}) \simeq \underline{\mathrm{Split}}(\underline{\mathrm{Split}}(\mathbb{A}))$.

Clearly we have that $\widehat{\mathbb{A}} \simeq \widehat{\underline{\mathrm{Split}}(\mathbb{A})}$.
(Why?)

Exercises

(1) For $\mathbb{C}$ the monoid of monotone maps on Σ, show that $\underline{\mathrm{Split}}(\mathbb{C})$ is equivalent to the full subcategory of $\underline{\underline{\mathrm{Poset}}}$ determined by 1 and Σ.

(2) For $\mathbb{C}$ the monoid of continuous endomorphisms on $\overline{\omega}$ (the canonical ω-chain with its limit, $0 \leq 1 \leq \cdots \leq n \leq \cdots \leq \infty$) give an equivalent presentation of $\underline{\mathrm{Split}}(\mathbb{C})$ as a full subcategory of $\underline{\underline{\mathrm{Cpo}}}$.

We now move on to study the internal structure of presheaf categories.

Categories of presheaves are complete and cocomplete (with limits and colimits given pointwise — check it!), and hence bicartesian (with this structure given as follows:

$$0 \in |\hat{\mathbb{C}}| \quad , \quad 0(c) = 0 \in |\underline{\underline{\mathrm{Set}}}|$$

$$1 \in |\hat{\mathbb{C}}| \quad , \quad 1(c) = 1 \in |\underline{\underline{\mathrm{Set}}}|$$

$$P + Q \in |\hat{\mathbb{C}}| \quad , \quad (P+Q)(c) = P(c) + Q(c) \quad \in |\underline{\underline{\mathrm{Set}}}|$$

$$P \times Q \in |\hat{\mathbb{C}}| \quad , \quad (P \times Q)(c) = P(c) \times Q(c) \quad \in |\underline{\underline{\mathrm{Set}}}| \;)$$

Note that in $\hat{\mathbb{C}}$ colimits are stable under pullbacks (see pp 6 and 7), and that products distribute over sums:

$$(P \times Q) + (P \times Q') \xrightarrow{\cong} P \times (Q + Q') .$$

In fact, $\hat{\mathbb{C}}$ is cartesian closed. Indeed, if exponentials exist we should have

$$P^Q(c) \cong \hat{\mathbb{C}}(y(c), P^Q) \cong \hat{\mathbb{C}}(y(c) \times Q, P)$$

Hence we define

$$P^Q(-) \stackrel{def}{=} \hat{\mathbb{C}}(y(-) \times Q, P)$$

and equip it with an "evaluation" map

$$P^Q(c) \times Q(c) \xrightarrow{\underline{eval}_c} P(c)$$

$$\left(y(c) \times Q \xrightarrow[\varphi]{} P,\ y(c) \xrightarrow[x]{} Q\right) \longmapsto \left(y(c) \xrightarrow[\langle \mathrm{id}, x \rangle]{} y(c) \times Q \xrightarrow[\varphi]{} P\right)$$

which you should check (exercise) that defines a natural transformation $\underline{eval} : P^Q \times Q \to P$ with the required universal property.

Before looking at some examples here's an alternative (and, of course, isomorphic) description of exponentials:

$$P^Q(z) \stackrel{def}{=} \left\{ \varphi \in \prod_{\substack{h : y \to z \\ (y \in |\mathbb{C}|)}} Q(y) \Rightarrow P(y) \ \text{ s.t. } \ \forall\, x \xrightarrow{f} y \xrightarrow{h} z \text{ in } \mathbb{C},\ \begin{array}{ccc} Q(y) & \xrightarrow{\varphi_h} & P(y) \\ {\scriptstyle Qf}\downarrow & & \downarrow{\scriptstyle Pf} \\ Q(x) & \xrightarrow[\varphi_{hf}]{} & P(x) \end{array} \right\}$$

with action, for $f : a \to b$ in $\mathbb{C}$, given by

$$\begin{array}{ccc} P^Q(b) & \xrightarrow{P^Q(f)} & P^Q(a) \\ \varphi & \longmapsto & \langle \varphi_{fh} \rangle_{\substack{h : x \to a \\ (x \in |\mathbb{C}|)}} \end{array}$$

(check that this definition yields a presheaf) and with "evaluation" map

$$\begin{array}{ccc} P^Q(c) \times Q(c) & \xrightarrow{\underline{\mathrm{eval}}_c} & P(c) \\ \varphi \,,\; x & \longmapsto & \varphi_{id_c}(x) \end{array}$$

which, again, you should check (exercise) that defines a natural transformation $\underline{\mathrm{eval}} : P^Q \times Q \to P$ with the required universal property.

Example: Let $\mathbb{C}$ be the arrow category $\boxed{0 \to 1}$. For presheaves $X = (X_1 \to X_0)$ and $Y = (Y_1 \to Y_0)$ we have that

$$(X^Y) = \left((X^Y)_1 \longrightarrow (X_0)^{Y_0} \right)$$

is the projection from

$$(X^Y)_1 =_{\mathrm{def}} \left\{ (f_1, f_0) \;\middle|\; \begin{array}{ccc} X_1 & \xrightarrow{f_1} & Y_1 \\ \downarrow & & \downarrow \\ X_0 & \xrightarrow[f_0]{} & Y_0 \end{array} \right\}$$

In particular, notice that for $X = (X_1 \hookrightarrow X_0)$ and

$Y = (Y_1 \hookrightarrow Y_0)$ inclusions, we have that

$$X^Y \cong \left(\{ f \in X_0^{Y_0} \mid \forall x \in X_1.\ f(x) \in Y_1 \} \hookrightarrow X_0^{Y_0} \right).$$

Exercise: Describe exponentials in the presheaf category over $\underline{\omega}^{op} = \boxed{0 \leftarrow 1 \leftarrow \cdots \leftarrow n \leftarrow \cdots}$.

Exercise

(1) In the category $\underline{Set}^{\mathbb{F}}$ of presheaves over $\mathbb{F}^{op}$, where $\mathbb{F}$ is the category of finite cardinals and all functions, write V for $y(1)$ and show that

$$(X^V)(n) \cong X(n+1) \qquad n \in |\mathbb{F}|$$

for all $X \in |\underline{Set}^{\mathbb{F}}|$. What is the presheaf action of X^V?

(2) In the category $\underline{Set}^{\mathbb{I}}$ of presheaves over $\mathbb{I}^{op}$, where $\mathbb{I}$ is the category of finite cardinals and injections, write N for $y(1)$ and show that

$$(X^N)(n) \cong (X(n))^n \times X(n+1) \qquad n \in |\mathbb{F}|$$

for all $X \in |\underline{Set}^{\mathbb{I}}|$. What is the presheaf action of X^N?

The topos of presheaves We provide the last missing ingredient (subobject classifier) to see that the category of presheaves is a topos

Def A topos is a category with finite limits, exponentials, and subobject classifier.

Def In a category with finite limits, a subobject classifier is a (necessarily monomorphic) map $t: 1 \rightarrowtail \Omega$ such that for every mono $m: S \rightarrowtail X$ there exists a unique "characteristic map" $\chi_m : X \to \Omega$ such that the diagram

$$\begin{tikzcd} S \arrow[r] \arrow[d, tail, "m"'] & 1 \arrow[d, tail, "t"] \\ X \arrow[r, "\chi_m"'] & \Omega \end{tikzcd}$$

is a pullback.

Remark: Note that isomorphic monomorphisms are classified by the same characteristic map:

- for monomorphisms $m: S \rightarrowtail X$ and $m': S' \rightarrowtail X$, write $m \leq m'$ if m factors through m' [that is, if there exists a (necessarily unique and monomorphic) map $i: S \to S'$ such that $m = m'i$]; further write $m \simeq m'$ iff $m \leq m'$ and $m' \leq m$ [that is, if there exists a (necessarily unique) isomorphism i such that $m = m'i$].

With this notation we have that $m \simeq m' \Rightarrow \chi_m = \chi_{m'}$.
Indeed,

$$\begin{tikzcd} S' \arrow[r, phantom, "\cong"] \arrow[rr, bend left, ""] \arrow[dr, bend right, "m'"'] & S \arrow[r] \arrow[d, tail, "m"] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, tail, "t"] \\ & X \arrow[r, "\chi_m"'] & \Omega \end{tikzcd} \Rightarrow \begin{tikzcd} S' \arrow[r] \arrow[d, tail, "m'"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, tail, "t"] \\ X \arrow[r, "\chi_m"'] & \Omega \end{tikzcd} \Rightarrow \chi_m = \chi_{m'}.$$

Thus, the process of classifying monomorphisms by characteristic maps is an assignment

$$\underline{\mathrm{Sub}}(X) \longrightarrow [X, \Omega] \qquad (*)$$

where $\underline{\mathrm{Sub}}(X)$ is the collection of subobjects of X defined to be equivalence classes of monomorphisms under $\approx$; i.e., $\underline{\mathrm{Sub}}(X) = \{ [m]_{\approx} \mid m \text{ is a mono into } X \}$.

Exercise Show that in a category with finite limits and subobject classifier, $(*)$ establishes a bijective correspondence. (Is there a converse?)

Example In $\underline{\mathrm{Set}}$, the inclusion $\{1\} \hookrightarrow \{0,1\}$ is a subobject classifier. (Check it formally!) Indeed, to classify a subset of a set we need an Ω with two elements to indicate whether or not an element of the set belongs to the subset.

Example Let us consider the more sophisticated example of the category of graphs $\underline{\mathrm{Set}}^{e \rightrightarrows n}$ (see (4) on pg 14) A subgraph $S = (S_e \rightrightarrows S_n)$ of a graph $G = (G_e \rightrightarrows G_n)$

is given by a subset S_n of nodes of the graph G_n together with a subset S_e of the edges G_e of the graph such that every edge in the subgraph is between nodes that are also in the subgraph:

$$\begin{tikzcd} S_e \arrow[r, dashed] \arrow[d, hook] & S_n \arrow[d, hook] \\ G_e \arrow[r, "G_{dom}"'] & G_n \end{tikzcd} \qquad \begin{tikzcd} S_e \arrow[r, dashed] \arrow[d, hook] & S_n \arrow[d, hook] \\ G_e \arrow[r, "G_{cod}"'] & G_n \end{tikzcd}$$

Hence to classify subgraphs of a graph we need a graph Ω with

- Two nodes, say 1 and 0, to indicate whether or not a node in the graph is in the subgraph (cf. the previous example), and with edges
- 0^2 to classify edges in the graph between nodes that are not in the subgraph,
- $0 \to 1$ to classify edges in the graph from a node that is not in the subgraph to a node that is in the subgraph,
- $1 \to 0$ to classify edges in the graph from a node that is in the subgraph to a node that is not in the subgraph,
- 1^2 to classify edges in the graph but not in the subgraph between nodes in the subgraph, and

– $1 \circlearrowleft t$ to classify edges both in the graph and in the subgraph between nodes in the subgraph.

Thus,

$$\Omega = \boxed{\begin{tikzcd} 0 \arrow[loop left] \arrow[r, bend left] & 1 \arrow[l, bend left] \arrow[loop above, "t"] \arrow[loop below] \end{tikzcd}}$$

and the subobject classifier is the subgraph inclusion

$$\begin{tikzcd} \boxed{1 \circlearrowleft t} \arrow[d, hook] \\ \Omega \end{tikzcd}$$

(Check it formally!)

<u>Exercise</u>: Describe the subobject classifier in $\underline{Set}^{\bullet\to\bullet}$.

We proceed to describe subobject classifiers in arbitrary presheaf categories.

<u>**Proposition**</u> A (locally small) category $\mathcal{E}$ with finite limits has a subobject classifier if and only if it has an object Ω and an isomorphism

$$\chi_X : \underline{Sub}_{\mathcal{E}}(X) \cong \mathcal{E}(X,\Omega)$$

natural for $X \in |\mathcal{E}|$, where the action of $\underline{Sub}_{\mathcal{E}}$ on morphisms is given as follows: for $f : Y \to X$ in $\mathcal{E}$,

$$\underline{\mathrm{Sub}}_{\mathcal{E}}(X) \xrightarrow{\underline{\mathrm{Sub}}_{\mathcal{E}}(f)} \underline{\mathrm{Sub}}_{\mathcal{E}}(Y)$$

$$[m] \longmapsto [f^{-1}(m)]$$

with
$$\begin{tikzcd} f^{-1}(S) \arrow[r] \arrow[d, tail, "f^{-1}(m)"'] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[d, tail, "m"] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

a pullback square

Hence, for a subobject classifier Ω in $\hat{\mathbb{C}}$, we should have

$$\Omega(c) \cong \hat{\mathbb{C}}(y(c), \Omega) \cong \underline{\mathrm{Sub}}_{\hat{\mathbb{C}}}(y\,c) \qquad (c \in |\mathbb{C}|)$$

and so we proceed to give a more elementary description of subobjects.

First, note that, for $P \in |\hat{\mathbb{C}}|$, we have the isomorphism

$$\underline{\mathrm{Sub}}_{\hat{\mathbb{C}}}(P) \cong \underline{\mathrm{SubPresheaves}}(P) \stackrel{\mathrm{def}}{=} \left\{ S \in |\hat{\mathbb{C}}| : \begin{array}{c} \forall c \in |\mathbb{C}|, \\ S(c) \subseteq P(c) \end{array} \right\}$$

natural in P, where the action of $\underline{\mathrm{SubPresheaves}}$ on natural transformations is given as follows: for $\varphi : Q \to P$ in $\hat{\mathbb{C}}$,

$$\underline{\text{SubPresheaves}}(P) \xrightarrow{\underline{\text{SubPresheaves}}(\varphi)} \underline{\text{SubPresheaves}}(Q)$$

$$S \longmapsto \varphi^{-1}(S)$$

with

$$\varphi^{-1}(S)(C) =^{\text{df}} \{\, y \in Q(C) \mid \varphi_C(y) \in S(C) \,\}$$

$$\cong \Big\{\, y : C \to Q \;\Big|\; \begin{tikzcd} & & S \arrow[d, hook] \\ C \arrow[r, "y"'] \arrow[urr, dashed] & Q \arrow[r, "\varphi"'] & P \end{tikzcd} \,\Big\} \quad (*)$$

(Exercise Check that the presheaf action of a subpresheaf of a presheaf P is completely determined by the presheaf action of P.)

Exercise Check that the square

$$\begin{tikzcd} \varphi^{-1}(S) \arrow[r] \arrow[d, hook] & S \arrow[d, hook] \\ Q \arrow[r, "\varphi"'] & P \end{tikzcd}$$

is a pullback.

Further, we now characterise subpresheaves as "sieves".

Def A sieve S of a presheaf P in $\hat{\mathbb{C}}$ is a set of arrows with domain in $\mathbb{C}$ and codomain P satisfying the following down-closure property:

for every $f: B \to A$ in $\mathbb{C}$ and $a: A \to P$ in S,
the composite $B \xrightarrow{f} A \xrightarrow{a} P$ is in S.

(Exercise What are the sieves of a representable?)

<u>Prop</u> We have an isomorphism

$$(**) \quad \underline{\text{SubPresheaves}}(P) \cong \underline{\text{Sieves}}(P) \qquad (P \in |\hat{\mathbb{C}}|)$$

natural in P, where $\underline{\text{Sieves}}(P)$ is the set of sieves of P with action on natural transformations given as follows: for $\varphi: Q \xrightarrow{\cdot} P$ in $\hat{\mathbb{C}}$,

$$\underline{\text{Sieves}}(P) \xrightarrow{\underline{\text{Sieves}}(\varphi)} \underline{\text{Sieves}}(Q)$$

$$S \longmapsto \varphi^*(S) =_{\text{df}} \{A \xrightarrow{a} Q \mid (A \xrightarrow{\varphi a} P) \in S\}$$

(Compare this definition with (*) on pp 42)

The isomorphism (**) above is given as follows:

$$\underline{\text{SubPresheaves}}(P) \longrightarrow \underline{\text{Sieves}}(P)$$

$$\begin{tikzcd}[row sep=small] & & S \arrow[d, hook] \\ (S \hookrightarrow P) \longmapsto \{A \to P \mid & A \arrow[ur, dashed] \arrow[r] & P\} \end{tikzcd}$$

$$\left(\cong \bigcup_{A \in |\mathbb{C}|} S(A)\right)$$

and

$$\underline{\text{Sieves}}(P) \longrightarrow \underline{\text{SubPresheaves}}(P)$$

$$S \longmapsto S' \hookrightarrow P$$

such that

$$S'(A) = \{ a \in P(A) \mid (A \xrightarrow{a} P) \in S \}$$

$$\left(\cong S \cap \hat{\mathbb{C}}(A, P) \right)$$

From the above considerations, we set

$$\Omega = \left(\mathbb{C}^{op} \overset{y}{\hookrightarrow} \hat{\mathbb{C}}^{op} \xrightarrow{\underline{\text{Sieves}}} \underline{\underline{\text{Set}}} \right)$$

That is,

$$\Omega(C) = \underline{\text{Sieves}}(C)$$

with action given, for $f : B \to A$ in $\mathbb{C}$, by

$$\Omega(A) \xrightarrow{\Omega(f)} \Omega(B)$$

$$S \longmapsto f^*(S) = \{ X \to B \mid (X \to B \xrightarrow[f]{} A) \in S \}.$$

And we equip Ω with the mono given by the "maximal sieve":

$$t : 1 \hookrightarrow \Omega \quad , \quad t_C = \biguplus_{X \in |\mathbb{C}|} \mathbb{C}(X, C) .$$

As we sketch below this yields a subobject classifier for presheaves.

First, observe that (exercise) for every subpresheaf $S \hookrightarrow y(C)$ in $\hat{\mathbb{C}}$ $(C \in |\mathbb{C}|)$ there exists a unique map $yC \to \Omega$, namely the sieve induced by the subpresheaf, such that the square

$$\begin{tikzcd} S \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, hook, "t"] \\ y(C) \arrow[r] & \Omega \end{tikzcd}$$

is a pullback.

Second, suppose we have the pullback square

$$\begin{tikzcd} S \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, hook, "t"] \\ P \arrow[r, "\varphi"'] & \Omega \end{tikzcd}$$

in $\hat{\mathbb{C}}$. Then, for every $f : C \to P$ $(C \in |\mathbb{C}|)$, the outer diagram in

$$\begin{tikzcd} f^{-1}(S) \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, hook, "t"] \\ C \arrow[r, "f"'] & P \arrow[r, "\varphi"'] & \Omega \end{tikzcd}$$

is a pullback. Hence, by the first observation above,

$$\varphi(f) = f^*(S).$$

Thus, to conclude that $t: 1 \hookrightarrow \Omega$ is a subobject classifier in $\hat{\mathbb{C}}$, we need only show that for every subpresheaf $S \hookrightarrow P$ in $\hat{\mathbb{C}}$, the mapping

$$C \xrightarrow[f]{} P \overset{\chi_S}{\longmapsto} f^*(S)$$

defines a natural transformation $P \to \Omega$ (which amounts to checking that $h^*(f^*(S)) = (fh)^*(S)$ for all h in $\mathbb{C}$) such that the square

$$\begin{tikzcd} S \arrow[r] \arrow[d, hook] & 1 \arrow[d, hook, "t"] \\ P \arrow[r, "\chi_S"'] & \Omega \end{tikzcd}$$

is a pullback. (This is left as an instructive exercise.)

Exercise: Check that the above general construction for subobject classifiers in presheaf categories yields the subobject classifier of pp. 40 when $\mathbb{C}^{op} = \boxed{e \rightrightarrows n}$.

Example: For $\underline{eq}_P : P \times P \to \Omega$ the characteristic map of the diagonal map $\Delta_P : P \rightarrowtail P \times P$; that is,

$$\begin{tikzcd} P \arrow[r] \arrow[d, tail, "\Delta_P"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "t"] \\ P \times P \arrow[r] & \Omega \end{tikzcd}$$

$$\underline{eq}_P =^{def} \chi_{\Delta_P}$$

we have that, for $x, y : C \to P$ ($C \in |\mathbb{C}|$), the "truth value" of $\underline{eq}_P(x,y) \in \Omega(C)$ is the sieve of all possible ways in which x and y may be made equal. Formally,

$$\underline{eq}_P(x,y) = \{ w : X \to C \text{ in } \mathbb{C} \mid x \circ w = y \circ w \}.$$

<u>Exercise</u> Show that in $\widehat{\omega}$, where $\omega = (0 \leq 1 \leq \dots \leq n \leq \dots)$, Ω looks as follows:

$$\begin{array}{cllllll}
0 & \emptyset, & \{0\} & & & & \\
\wedge\!| & \uparrow & \uparrow & \nwarrow\!\!\searrow & & & \\
1 & \emptyset, & \{0\}, & \{0,1\} & & & \\
\wedge\!| & \uparrow & \uparrow & \uparrow & \nwarrow\!\!\searrow & & \\
2 & \emptyset, & \{0\}, & \{0,1\}, & \{0,1,2\} & & \\
\wedge\!| & \uparrow & \uparrow & \uparrow & \uparrow & \nwarrow\!\!\searrow & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \\
\wedge\!| & \uparrow & \uparrow & \uparrow & \uparrow & \nwarrow\!\!\searrow & \\
n & \emptyset, & \{0\}, & \{0,1\}, & \{0,1,2\}, & \dots\ \dots & \{0,1,2\dots n\} \\
\wedge\!| & \uparrow & \uparrow & \uparrow & \uparrow & \dots\ \dots & \nwarrow\!\!\searrow \\
\vdots & \vdots & \vdots & \vdots & \vdots & \dots\ \dots & \vdots \quad \ddots
\end{array}$$

<u>Preservation properties of the Yoneda embedding</u>

<u>Exercise</u> Show that the Yoneda embedding preserves limits and exponentials.

<u>Exercise</u> For $\mathbb{F}$ the topos of finite cardinals and all functions between them, show that $y : \mathbb{F} \hookrightarrow \widehat{\mathbb{F}}$

does not preserve finite colimits (and the subobject classifier).

<u>Remark</u>: One may impose preservation properties (like, for example, that of finite coproducts) on the Yoneda embedding by "cutting down" the presheaves using "orthogonality conditions" (Freyd & Kelly).