A 2-Categorical Note
On Day's Tensor Product

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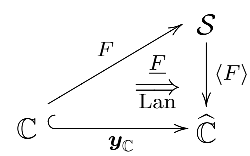
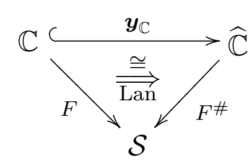
1. For a small category $\mathbb{C}$, let $\widehat{\mathbb{C}} \stackrel{\text{def}}{=} \boldsymbol{\mathcal{S}et}^{\mathbb{C}^{\text{op}}}$ and let $\boldsymbol{y}_{\mathbb{C}} : \mathbb{C} \hookrightarrow \widehat{\mathbb{C}}$ be the Yoneda embedding.

2. For a functor $F : \mathbb{C} \to \mathcal{S}$, where $\mathbb{C}$ is small, we have the following situation



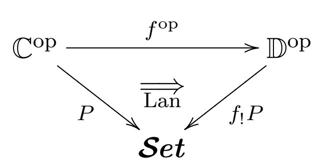
where $\langle F \rangle(S) = \mathcal{S}(F-, S)$ and $\underline{F}_c = \{\, F_{c,z} : \mathbb{C}(c, z) \to \mathcal{S}(Fc, Fz) \,\}_{z \in \mathcal{C}}$.

Furthermore, for $\mathcal{S}$ cocomplete, $\langle F \rangle$ has a left adjoint $F^{\#}$ given by

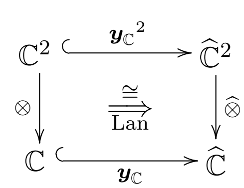


3. For a functor between small categories $f : \mathbb{C} \to \mathbb{D}$, let $f_! \dashv f^* : \widehat{\mathbb{D}} \to \widehat{\mathbb{C}}$ be given by $f_! \stackrel{\text{def}}{=} (\boldsymbol{y}_{\mathbb{D}}\, f)^{\#}$ and $f^* \stackrel{\text{def}}{=} \langle \boldsymbol{y}_{\mathbb{D}}\, f \rangle$.

Since $f^*Q \cong Q\, f^{\text{op}}$, we have that



4. Let $(\mathbb{C}, I, \otimes)$ be a small monoidal category. Day's monoidal structure on $\widehat{\mathbb{C}}$ has tensor unit $\boldsymbol{y}_{\mathbb{C}}(I)$ and tensor product $\widehat{\otimes}$ given by



Since

$$
\begin{array}{ccc}
 & & \widehat{\mathbb{C}}^2 \\
 & \overset{{\boldsymbol{y}_{\mathbb{C}}}^2}{\nearrow} \quad \underset{\mathrm{Lan}}{\overset{\cong}{\Longrightarrow}} & \Big\downarrow \langle {\boldsymbol{y}_{\mathbb{C}}}^2 \rangle \\
\mathbb{C}^2 & \xhookrightarrow[\boldsymbol{y}_{\mathbb{C}^2}]{} & \widehat{\mathbb{C}^2} \\
\otimes \Big\downarrow & \cong & \otimes_! \Big\downarrow \dashv \Big\uparrow \otimes^* \\
\mathbb{C} & \xhookrightarrow[\boldsymbol{y}_{\mathbb{C}}]{} & \widehat{\mathbb{C}}
\end{array}
$$

it follows that

$$\widehat{\otimes} \cong \otimes_! \langle {\boldsymbol{y}_{\mathbb{C}}}^2 \rangle$$

and hence that

$$
\begin{array}{ccc}
(\mathbb{C}^{\mathrm{op}})^2 \cong (\mathbb{C}^2)^{\mathrm{op}} & \xrightarrow{\otimes^{\mathrm{op}}} & \mathbb{C}^{\mathrm{op}} \\
P \times Q \Big\downarrow & \underset{\mathrm{Lan}}{\Longrightarrow} & \Big\downarrow P \widehat{\otimes} Q \\
\boldsymbol{\mathcal{S}et}^2 & \xrightarrow[\times]{} & \boldsymbol{\mathcal{S}et}
\end{array}
$$