Algebraic Theories and Equational Logics
(Rough Notes for an MFPS Tutorial Lecture)

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Abstract

The Tutorial is organised in two parts. In the first part, I will examine the basic theory of universal algebra; presenting the syntax and semantics of algebraic theories and their associated equational logic from the general perspective of categorical algebra. In the second part, I will show how to enrich the setting to provide an algebraic treatment of binding operators and substitution on top of which I will discuss second-order algebraic theories and their associated equational logic.

Algebraic Theories and Equational Logic.

M. Fiore (19 & 20.V.2008)

Part I: 1^{st} order (Universal algebra) — prequel to MFPS paper [FH]

Part II: 2^{nd} order (Simple Type Theory) — sequel to MFPS paper [FH]; prequel to LICS paper [F08]

Algebraic Theories	Equational Logic
model theory	logical theory
mathematical structures (algebras)	formal language
Semantics	Syntax

(dichotomy)

interpretation

soundness = formal manipulation makes sense in all models

completeness = what makes sense in all models is nothing but formal manipulation.

1. Universal algebra (algebraic mathematical structure).

Example: Groups

- carrier set: G
- operations: $e \in G$
  $m : G \times G \to G$
  $i : G \to G$
- axioms:
  such that
  $\forall x \in G .\ m(x,e) = x$
  $\dots$

2. Equational logic (formal language)

for reasoning about algebraic structure.

- sorts: $\underline{G}$
- operators: $\underline{e} : G$
  $\underline{m} : G, G \to G$
  $\underline{i} : G \to G$

[TERMS]

- axioms $x : G \vdash \underline{m}(x, \underline{e}) = x : G$
  $\dots$

unified as diagrams

$$\begin{tikzcd} G \times 1 \arrow[r, "id \times e"] \arrow[dr, "\pi_1"'] & G \times G \arrow[d, "m"] \\ & G \end{tikzcd}$$

Equational reasoning:

Rules for Equiv. Rel. + Axioms

+ Substitution

$$\frac{x_1 \dots x_n \vdash u \equiv v \qquad \Gamma \vdash u_i \equiv v_i \ (i = 1..n)}{\Gamma \vdash u[u_1 \dots u_n] \equiv v[v_1 \dots v_n]}$$

[SUBSTITUTION]

two readings

model in a category

formal diagram

3. Syntactic structure: Signatures

Specification of operators/operations

$$m : G, G \to G \qquad m : G \times G \to G$$

arity ⇝ mono-sorted case = a natural number indicating the number of arguments that the operator takes.

Signature:

sets of operators indexed by their arities.

$\{\Sigma(n)\}_{n \in \mathbb{N}}$

4. Syntactic structure: Terms.

(i) Concrete presentation

$$\frac{x \in V}{\lfloor x \rfloor \in TV} \qquad \frac{t_i \in TV}{f(t_1 \cdots t_n) \in TV} \ (f \in \Sigma(n)).$$

(An inductive definition ⇝ structural induction)

(ii) Model-theoretic universal characterisation.

Σ-algebras (Universal algebra).

- carriers: A
- operations: $\{ [\![f]\!]_A : A^n \to A \}_{f \in \Sigma(n), n \in \mathbb{N}}$.
- homomorphisms: maps that commute with interpretations.

⇝

- TV is canonically a Σ-algebra: purely formal/syntactic interpretation

$$[\![f]\!] : (TV)^n \to TV : t_1 \dots t_n \mapsto f(t_1 \dots t_n)$$

- $(TV, [\![-]\!])$ is the free Σ-algebra on V:

$$\begin{tikzcd} V \arrow[r, "\lfloor - \rfloor"] \arrow[dr, "\forall f"'] & TV \arrow[d, dashed, "f^\#"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} (TV, [\![-]\!]) \arrow[d, dashed, "\exists !\, f^\#"] \\ (A, [\![-]\!]_A) \end{tikzcd}$$

$\rightsquigarrow$ universal (implementation-independent) description.

$\rightsquigarrow$ structural induction principle.

$$f^\# \lfloor x \rfloor = f(x)$$

$$f^\#(o(t_1 \dots t_n)) = [\![o]\!]_A (f^\# t_1 \dots f^\# t_n)$$

5. Algebras (for an endofunctor).

Categorical (abstract) view:

$$\{ [\![f]\!]_A : A^n \to A \}_{f \in \Sigma(n),\, n \in \mathbb{N}}$$

$$\equiv \underbrace{\Big(\coprod_{n \in \mathbb{N}} \coprod_{f \in \Sigma(n)} A^n \Big)}_{\text{def}} \longrightarrow A$$

NB: Makes sense quite generally.

$$\Sigma(A) \Big(= \coprod_{n \in \mathbb{N}} \Sigma(n) \cdot X^n \Big)$$

$$\equiv \Sigma(A) \longrightarrow A$$ ~ algebra for an endofunctor.

- F-$\underline{\mathrm{Alg}}$: obj: $F(A) \xrightarrow{\alpha} A$

  morph: $(A,\alpha) \xrightarrow{h} (B,\beta)$ $\equiv$

$$\begin{tikzcd} FA \arrow[r, "Fh"] \arrow[d, "\alpha"'] & FB \arrow[d, "\beta"] \\ A \arrow[r, "h"'] & B \end{tikzcd}$$

$$F \circlearrowright \mathcal{C}$$

Exercise: Show that the two notions of Σ-algebra homomorphism coincide.

6. Free algebras

Thm:
$$\begin{tikzcd} (TV, [\![-]\!]) & \Sigma\text{-}\underline{\mathrm{Alg}} \arrow[d, bend left] \\ V \arrow[u, mapsto] & \underline{\mathrm{Set}} \arrow[u, bend left] \arrow[u, phantom, "\dashv"] \end{tikzcd}$$

Consider this situation generally:

$$\begin{tikzcd} & (TX, FTX \xrightarrow{\varphi_X} TX) & F\text{-}\underline{\mathrm{Alg}} \arrow[d, bend left] \\ X \arrow[ur, "\eta_X"'] & X \arrow[u, mapsto] & \mathcal{C} \arrow[u, bend left] \arrow[u, phantom, "\dashv"] \end{tikzcd}$$

$$\begin{tikzcd} FTX \arrow[r, "Ff^\#"] \arrow[d, "\varphi_X"'] & FA \arrow[d, "\alpha\ \forall"] \\ TX \arrow[r, dashed, "\exists!\ f^\#"] & A \\ X \arrow[u, "\eta_X"] \arrow[ur, "\forall f"'] & \end{tikzcd}$$

(Free)

7. Monads

$$\begin{tikzcd}
 & F\,\mathrm{Alg} \arrow[dr, "\cong"] \arrow[d, shift left] & \\
\mathcal{C}_\Sigma \arrow[ur] \arrow[r, shift left] & \mathcal{C} \arrow[u, shift left, "\dashv"] \arrow[l, shift left, "\top"] \arrow[loop below] \arrow[r, shift left] & \mathcal{C}^\Sigma \arrow[l, shift left, "\top"]
\end{tikzcd}$$

monadic

Kleisli categories (≈ free algebras)

Eilenberg-Moore algebras

$\Sigma \rightsquigarrow$ The monad freely generated by F

$\eta: X \to TX$

$\mu: TTX \to TX$

def:

$$\begin{tikzcd}
FTTX \arrow[r, "F\mu_X"] \arrow[d, "\varphi_{TX}"'] & FTX \arrow[d, "\varphi_X"] \\
TTX \arrow[r, "\exists!\,\mu_X"] & TX \\
TX \arrow[u, "\eta_{TX}"] \arrow[ur, "id"'] &
\end{tikzcd}$$

$\sim$ assoc.

$\sim$ one neutral element law

(Monad structure = Substitution structure).

8. Strengths [Kock]: allow the introduction of parameters.
(e.g. parametrised initiality).

st: $F(X) \times Y \to F(X \times Y)$
natural and coherent with the cartesian (tensor) structure

Example/Exercise: The endofunctor induced by a signature is canonically strong.

Prop: The strength on an endofunctor canonically extends to a monad strength on the free monad on the endofunctor.

parameterised initiality:

$$\begin{tikzcd}
F(TX)\times Y \arrow[r, "st"] \arrow[d, "\varphi_X\times id"'] & F(T(X)\times Y) \arrow[r, "Ff^\#"] & FA \arrow[d, "\forall\alpha"] \\
T(X)\times Y \arrow[rr, "\exists!", "f^\#"'] & & A \\
X\times Y \arrow[u, "\eta_X\times id"] \arrow[urr, "\forall f"'] & &
\end{tikzcd}$$

proof (of Prop.):

$$\begin{tikzcd}
F(TX)\times Y \arrow[r, "st"] \arrow[d, "\varphi_X\times id"'] & F(T(X)\times Y) \arrow[r, "F(st^\#)"] & FT(X\times Y) \arrow[d, "\varphi_{X\times Y}"] \\
T(X)\times Y \arrow[rr, "st^\#"] & & T(X\times Y) \\
X\times Y \arrow[u, "\eta_X\times id"] \arrow[urr, "\eta_{X\times Y}"'] & &
\end{tikzcd}$$

Exercise: Show that $st^\#$ is a monad strength, i.e. compatible with the monad structure.

9. Substitution

A strong monad (T, st, η, μ) yields substitution structure (see e.g. [ICMS])

$$\begin{cases} \eta_V : V \to TV \\ \sigma : T(V)\times (TU)^V \to TU \end{cases}$$

$$\begin{tikzcd}
T(V)\times (TU)^V \arrow[rr, "\sigma"] \arrow[d, "st"'] & \text{def} & TU \\
T(V\times (TU)^V) \arrow[rr, "T\,eval"'] & & TTU \arrow[u, "\mu_U"']
\end{tikzcd}$$

Example: Let T be the free monad on a signature endofunctor.

Exercise: State and establish the laws of substitution.

Exercise: Establish the equivalence between monad structure and substitution structure.

(Hint:

$$\begin{tikzcd}
TTX \arrow[r, phantom, "\cong"] \arrow[rrr, bend right, "\mu_X"', "\mathrm{def}"] & T(TX)\times 1 \arrow[r, "id \times \ulcorner id \urcorner"] & T(TX)\times (TX)^{TX} \arrow[r, "\sigma"] & TX
\end{tikzcd}$$
)

Exercise: Give a universal characterisation of substitution for free monads on endofunctors.

10. Equations/Axioms (see [FH])

$(V \vdash l \equiv r)$ $\leadsto$ stands for $l, r \in T_\Sigma(V)$.

Satisfaction relation:

- For (A, α) a Σ-algebra (equiv. a Π_Σ-algebra)

$(A, \alpha) \models l \equiv r$

iff

$$\begin{tikzcd}
& 1 \arrow[d, shift right, "l"'] \arrow[d, shift left, "r"] & \\
V \arrow[r] \arrow[dd, bend right, "\forall\rho"'] & TV \arrow[dd, "\rho^\#"] \arrow[dr, "T\rho"] & \\
& & TA \arrow[dl, "\alpha"] \\
& A &
\end{tikzcd}$$

NB: Generalises to an arbitrary strong monad.

(Exercise?)

iff $[\![l]\!]_A = [\![r]\!]_A : A^V \to A$ $\left(A^V \xrightarrow[\tau]{} TA^{TV} \underset{\alpha^r}{\overset{\alpha^l}{\rightrightarrows}} A \right)$

Thm

$$\begin{tikzcd} (\Sigma,E)\text{-}\underline{\mathrm{Alg}} \arrow[r, hook, "\perp"'] \arrow[dr, bend right] & \Sigma\text{-}\underline{\mathrm{Alg}} \arrow[l, bend right] \arrow[d] \\ & \underline{\mathrm{Set}} \arrow[ul, bend left, "\perp"] \arrow[u, bend right, "\perp"'] \arrow[loop right, "T"] \arrow[loop left, "T_E"] \end{tikzcd}$$

$$T \xrightarrow{q} T_E$$

$$\begin{tikzcd} \Sigma T X \arrow[r, "\Sigma q_X"] \arrow[d, "\varphi"'] & \Sigma T_E X \arrow[d, "\varphi_E"] \\ TX \arrow[r, "\exists!\, q_X"] & T_E X \\ X \arrow[u, "\eta"] \arrow[ur, "\eta_E"'] & \end{tikzcd}$$

A general construction of free algebras under equations: (see [FH*])

$$\begin{tikzcd} & \Sigma A \arrow[r, two heads, "\Sigma q_1"] \arrow[d] \arrow[dr] \arrow[dr, phantom, "\text{p.o.}"] & \Sigma A_1 \arrow[r, two heads] & \Sigma A_2 \arrow[d] & \cdots & \Sigma A_\infty \arrow[d, dashed, "\exists!"] \\ A^V \arrow[ur, phantom] \arrow[r, shift left, "\llbracket l \rrbracket_A"] \arrow[r, shift right, "\llbracket r \rrbracket_A"'] & A \arrow[r, two heads] \arrow[rrrr, bend right, "q"'] & A_1 \arrow[r, two heads] & A_2 & \cdots & A_\infty \end{tikzcd}$$

Σ preserves epis

A_∞ colim.

$\sim$ Σ preserves colims of ω-chains.

free (Σ,E)-$\underline{\mathrm{Alg}}$ on the Σ-Alg (A,α).

$\forall\,(V \vdash l = r) \in E$

q: formalizes the idea of quotienting by equations and congruence rules.

11. Completeness of equational logic.

$$\forall (A,\alpha) \in (\Sigma,E)\text{-}\underline{\underline{Alg}}.\ (A,\alpha) \models (V \vdash u \equiv v)$$

iff

$$(T_E V, [\![-]\!]) \models (V \vdash u \equiv v)$$

iff

$$1 \underset{v}{\overset{u}{\rightrightarrows}} TV \xrightarrow{q} T_E V$$

General abstract theorem [FH]

iff $V \vdash u \equiv v$ is derivable

idea: because the entailment relation is generated by axioms and closed under congruence rules (c.f. the construction of free (Σ, E)-algebras).

Hence: $T_{(\Sigma,E)}(\Gamma)$ describes the theory of the presentation (Σ, E) (in context Γ).

12. Theories

(i) Universal algebra: Abstract clone of operations.

Example: Clone of operations.

• $\mathcal{C}_E(n) = T_E(x_1 \ldots x_n)$

are subject to suitable axioms.

$$\left\{\begin{array}{ll} \bullet\ x_i \in \mathcal{C}_E(n) \quad \text{i} \leq n & (\text{cf } \eta) \\ \bullet\ \mathcal{C}_E(n) \times (\mathcal{C}_E m)^n \to \mathcal{C}_E m & (\text{cf } \sigma) \end{array}\right.$$

More generally:

- $\mathcal{O}(n)$ $n \in \mathbb{N}$ "operations in context $x_1 - x_n$"
- $v_i \in \mathcal{O}(n)$ $i = 1..n$ "variables"
- $\mathcal{O}(n) \times (\mathcal{O}m)^n \to \mathcal{O}(m)$ "substitution".

subject to suitable axioms

(vi) Categorical algebra: Lawvere Theories (classifying category)

Example: Lawvere Theory of an abstract clone of operations:

obj: $n \in \mathbb{N}$

maps: $\mathcal{L}(m,n) \overset{df}{=} \prod_{i=1}^{n} \mathcal{L}(m,1) = (\mathcal{L}(m,1))^n$

$$\mathcal{L}(m,1) = \mathcal{O}(m)$$

identities: $(x_1, \dots, x_n) \in \mathcal{L}(n,n) = (\mathcal{L}(n,1))^n = (\mathcal{O}(n))^n$

composition: $\mathcal{L}(n,\ell) \times \mathcal{L}(m,n) \to \mathcal{L}(m,\ell)$

$$\begin{tikzcd}
(\mathcal{O}n)^\ell \times (\mathcal{O}m)^n \arrow[r] \arrow[dr, phantom, "df"] \arrow[d] & (\mathcal{O}m)^\ell \\
(\mathcal{O}(n) \times (\mathcal{O}m)^n)^\ell \arrow[ur] &
\end{tikzcd}$$

Definition: $\mathcal{L}$

- obj $= \mathbb{N}$
- $m = \underbrace{1 \pi \cdots \pi 1}_{m \text{ times}}$

13. Finitary aspects:

presentations	Theories		
(Σ,E)	monads	clones $\simeq$	Lawvere theories.
	$T_{(\Sigma,E)}$	$\mathcal{C}_E$	$\mathcal{L}_E$

$T_{(\Sigma,E)} \rightsquigarrow \boxed{?}$

$\mathcal{L}_E$: full subcategory of $\underline{\mathrm{Set}}_{T_{(\Sigma,E)}}$ with objects $[n]$ $(n\in\mathbb{N})$.

Need to consider *finitary* aspects:

$$\Sigma(X) = \coprod_{n\in\mathbb{N}} \Sigma(n)\,.\,X^{n}$$

n finite $\Rightarrow$ Σ finitary = preserves filtered colimits

$$\Downarrow$$

$$T_\Sigma \text{ finitary} \sim T_\Sigma(X) = \bigcup_{\Gamma \subset_{fin} X} T_\Sigma(\Gamma)$$

in fact

$T_{(\Sigma,E)}$ finitary

- The monads associated to equational theories are determined by their values on finite sets:

$$\frac{\underline{\mathrm{Set}} \xrightarrow{fin} \underline{\mathrm{Set}}}{\underline{\mathrm{FinSet}} \to \underline{\mathrm{Set}}}$$

14. Finitary extensions:

$$\begin{tikzcd} \mathbb{F} \arrow[r, hook, "J"] \arrow[dr, "F"'] & \underline{\mathrm{Set}} \arrow[d, "F^{\#}"] \arrow[d, phantom, "\mathrm{Lan} \cong"', shift right=4] \\ & \underline{\mathrm{Set}} \end{tikzcd}$$

$$F^{\#}(X) = \int^{\Gamma \in \mathbb{F}} F(\Gamma) . X^{\Gamma}$$

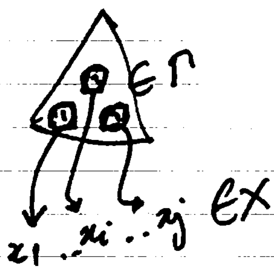
$$= \Big(\coprod_{\Gamma \in \mathbb{F}} F(\Gamma) . X^{\Gamma} \Big) \Big/ \simeq$$

equivalence relation generated by identifying the images

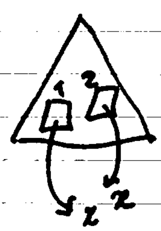
$$\begin{tikzcd} & F(\Gamma) \times X^{\Gamma} \arrow[dr] & \\ F(\Gamma) \times X^{\Delta} \arrow[ur, "\mathrm{id} \times X^{\rho}"] \arrow[dr, "F\rho \times \mathrm{id}"'] & & \int^{n} F(n) . X^{n} \\ & F(\Delta) \times X^{\Delta} \arrow[ur] & \end{tikzcd}$$

for all $\rho : \Gamma \to \Delta$

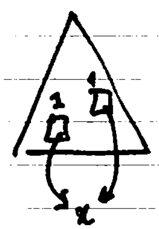
idea: an element of $F^{\#}(X)$ is given by a "structure"



The descriptions



and



are equivalent.

Prop: $[\mathbb{F}, \underline{\mathrm{Set}}] \simeq \underline{\mathrm{Fin}}[\underline{\mathrm{Set}}, \underline{\mathrm{Set}}]$

$$F \mapsto F^{\#}$$

$$TJ \leftarrow\!\shortmid T$$

substitution monoidal structure $\leadsto \boxed{?} \leftarrow\!\shortmid$ composition monoidal structure

15. <u>Substitution monoidal structure</u>: (see [FPT])

- <u>unit</u>: $V : \mathbb{F} \hookrightarrow \underline{Set}$

  <u>Exercise</u>: Show that $V^{\#} \cong Id$.

- <u>substitution tensor product</u>: (<u>NB</u>: It is not strict).

$$(F \bullet G)(\Gamma) = \int^{\Delta \in \mathbb{F}} F(\Delta) \cdot (G\Gamma)^{\Delta}$$

<u>Exercise</u>: Show that $(F \bullet G)^{\#} \cong F^{\#} G^{\#}$.

<u>Exercise</u>: Show that $(\underline{Set}^{\mathbb{F}}, V, \bullet)$ is a monoidal category.

<u>Exercise</u>: Show that $(\underline{Set}^{\mathbb{F}}, V, \bullet)$ and $(\underline{Fin}[\underline{Set}, \underline{Set}], Id, \circ)$ are equivalent as monoidal categories.

16. <u>Monoids in $\underline{Set}^{\mathbb{F}}$</u>

<u>Corollary</u>: $\underline{Mon}(Set^{\mathbb{F}}, V, \bullet) \simeq \underline{FinMonad}(Set)$

Theories

finitary monads on $\underline{Set}$ $\simeq$ monoids in $\underline{Set}^{\mathbb{F}}$ Abstract clones $\simeq$ Lawvere Theories

<u>Exercise</u>: $\boxed{\simeq}$

idea:

monad structure:

$$e : V \to F \quad \rightsquigarrow \quad \Gamma \xrightarrow{e_\Gamma} F(\Gamma) \quad \text{"variables"}$$

$$m : F \circ F \to F \quad \rightsquigarrow \quad \int^{\Delta} F(\Delta) \times (F\Gamma)^{\Delta} \longrightarrow F(\Gamma)$$

monad laws
= substitution laws.

$$F(\Delta) \times (F\Gamma)^{\Delta} \longrightarrow F(\Gamma)$$
compatible

"substitution"

17. Models.

Theorem finitary monads $\simeq$ $(V,\circ)$-monoids $\simeq$ Lawvere theories

models

Eilenberg-Moore algebras

$$\begin{tikzcd}
A \arrow[r, "\eta"] \arrow[dr, "id"'] & TA \arrow[d, "\alpha"] & TTA \arrow[l, "T\alpha"'] \arrow[d, "\mu"] \\
& A & TA \arrow[l, "\alpha"]
\end{tikzcd}$$

Actions

$$\begin{tikzcd}
V \cdot A \arrow[r, "e \cdot A"] \arrow[dr, phantom, "\cong"] & M \cdot A \arrow[d, "\alpha"] & M \cdot (M \cdot A) \arrow[l, "M \cdot \alpha"'] \arrow[d, phantom, "\cong"] \\
& A & (M \cdot M) \cdot A \arrow[d, "m \cdot A"] \\
& & M \cdot A \arrow[ul, "\alpha"]
\end{tikzcd}$$

operad view

$\mathrm{Mod}(\mathcal{L}) \hookrightarrow \mathrm{Set}^{\mathcal{L}}$

$\mathbf{A}$ s.t. $\mathbf{A}(n) \cong (\mathbf{A}1)^n$

Exercise: Show the equivalences.

18. <u>From $\underbrace{\text{1st-order models}}_{\substack{\text{algebraic} \\ \text{structure} \\ \text{in } \underline{\mathrm{Set}}}}$ to $\underbrace{\text{2nd-order models}}_{\substack{\text{algebraic structure} \\ \text{with substitution structure} \\ \text{in } \underline{\mathrm{Set}}^{\mathbb{F}}}}$</u>

(i) <u>1st-order models of syntax in the 2nd-order context.</u>

Σ signature $= \{\Sigma(n)\}_{n \in \mathbb{N}}$ (i.e. $\Sigma \in \underline{\mathrm{Set}}^{\mathbb{N}}$).

$$\begin{tikzcd}
\overline{\Sigma} & \underline{\mathrm{Set}}^{\mathbb{F}} \arrow[d, bend right] \arrow[d, phantom, "\dashv"] \\
\Sigma \arrow[u, leftrightarrow] & \underline{\mathrm{Set}}^{\mathbb{N}} \arrow[u, bend right]
\end{tikzcd}$$

$$\overline{\Sigma}(\Gamma) = \int^{n \in \mathbb{N}} \Sigma(n) . \Gamma^n = \coprod_{n \in \mathbb{N}} \Sigma(n) . \Gamma^n$$

<u>We have the following situation:</u>

$$\begin{tikzcd}
(\overline{\Sigma} \bullet (-))\text{-}\underline{\mathrm{Alg}} \arrow[dr] & & \underline{\mathrm{Mon}}(\underline{\mathrm{Set}}^{\mathbb{F}}) \arrow[dl] \\
& \underline{\mathrm{Set}}^{\mathbb{F}} \arrow[loop below, "\overline{\Sigma} \bullet (-)"] &
\end{tikzcd}$$

<u>Exercise</u>: Show that $\overline{\Sigma} \bullet X \cong \coprod_{n \in \mathbb{N}} \Sigma(n) \times X^n$.

<u>and extend it to consider algebras with substitution structure:</u>

$$\bar{\Sigma}\cdot X \xrightarrow{\chi} X$$

Σ-$\underline{\mathrm{Mon}}$ $\sim$ obj $V \xrightarrow{e} X \xleftarrow{m} X\cdot X$

$$\text{s.t.}\quad \begin{tikzcd} (\bar{\Sigma}\cdot X)\cdot X \cong \bar{\Sigma}\cdot(X\cdot X) \arrow[r, "\bar{\Sigma}\cdot m"] \arrow[d, "\chi\cdot X"'] & \bar{\Sigma}\cdot X \arrow[d, "\chi"] \\ X\cdot X \arrow[r, "m"'] & X \end{tikzcd}$$

$$\begin{tikzcd} \Sigma\text{-}\underline{\mathrm{Mon}} \arrow[d] \\ \underline{\underline{\mathrm{Set}}}^{\mathbb{F}} \arrow[u, bend left, "\dashv"] \end{tikzcd}$$

$$Z \mapsto \mu X.\, V + \bar{\Sigma}\cdot X + Z\cdot X$$

in particular the initial Σ-Mon $\mu X.\, V + \bar{\Sigma}\cdot X$ is the free $(\bar{\Sigma}\cdot(-))$-algebra on V and yields the syntactic Lawvere theory associated to the signature Σ.

(see [FO8])

19. Second-order syntax:

Example:

variables $\qquad$ variable binding / α-equivalence.

$$(\beta)\quad \underbrace{M[x],N[\,]}_{\text{meta variables}} \vdash (\lambda x.M[x])\,N[\,] = \underbrace{M[N[\,]]}_{\text{meta substitution}}$$

involves capture-avoiding substitution.

$$(\eta)\quad M[\,] \vdash \lambda x.M[\,]\,x = M[\,].$$

20. Algebraic treatment of variable binding: (see [FPT])

idea:

$$\frac{\Gamma,x \vdash t}{\Gamma \vdash \lambda x.t} \quad\approx\quad \begin{tikzcd} T(\Gamma,x) \arrow[d] \\ T(\Gamma) \end{tikzcd}$$

context extension.

$$\begin{tikzcd} \mathbb{F}^{op} \arrow[r, hook] \arrow[d, "+1"'] & Set^{\mathbb{F}} \arrow[d, "- \times V \; (\dashv \uparrow \dashv)"] \\ \mathbb{F}^{op} \arrow[r, hook] & Set^{\mathbb{F}} \end{tikzcd} \implies (X^V)(\Gamma) = X(\Gamma + 1)$$

The arity of variable binding.

Example: (λ-calculus) app : $(0,0)$, abs : (1) — one argument binding one variable.

two arguments and no binding

signature endofunctor : $\Sigma(X) = X^2 + X^V$

21. Algebraic models of variable binding with substitution. (see [FPT])

(1) Pointed strengths.

$$\underline{X^V \cdot Y \longrightarrow (X \cdot Y)^V}$$

$$\begin{tikzcd} (X^V \cdot Y) \times V \arrow[rr] \arrow[dr] & & X \cdot Y \\ & (X^V \times V) \cdot (Y \times V) \arrow[ur] & \end{tikzcd}$$

comes from a distribution law:

$$(X \cdot Y) \times Z \longrightarrow (X \times Z) \cdot (Y \times Z)$$

$\rightsquigarrow$ smooth counterpart of a syntactic mechanism for avoiding capturing free variables.

for pointed Y (equipped with $V \to Y$)

(provides "fresh" variables as needed

(ii) Algebras with substitution structure.

$\Sigma \circlearrowright \underline{\underline{Set}}^{\mathbb{F}}$ pointed strength

$$\Sigma(X)\bullet Y \longrightarrow \Sigma(X\bullet Y)$$
$$st_{X,\, V\to Y}$$

Σ-monoids: alg: $\Sigma(X) \xrightarrow{\xi} X$ s.t.

$$V \xrightarrow{e} X \xleftarrow{m} X\cdot X$$

$$\begin{array}{ccccc}
\Sigma(X)\bullet X & \xrightarrow{st_{X,e}} & \Sigma(X\cdot X) & \xrightarrow{\Sigma m} & \Sigma X \\
\xi\cdot X \downarrow & & & & \downarrow \xi \\
X\cdot X & & \xrightarrow{\quad m \quad} & & X
\end{array}$$

morphisms: Σ-alg & monoid homomorphism.

Thm:

$$\begin{array}{cc}
\Sigma\text{-}\underline{\underline{Mon}} & \left(\mu X.\, V+\Sigma(X)+Z\bullet X\right) = \mathcal{M}(Z) \\
\uparrow \dashv \downarrow & \uparrow \\
\underline{\underline{Set}}^{\mathbb{F}} & Z
\end{array}$$

Example: For $\Sigma(X) = X^2 + X^V$ the signature endofunctor for the λ-calculus, $\mathcal{M}\, 0$ yields the Lawvere theory of λ-terms ($\mathcal{L}(n) = \{ t \in \Lambda_\alpha \mid x_1 \ldots x_n \vdash t \}$) up to α-equivalence with capture-avoiding simultaneous substitution.

More generally, let $\mathcal{H} \in \underline{Set}^{\mathbb{N}}$ be an indexed family of term metavariables:

$$M \in \mathcal{X}(n) \rightsquigarrow M[-1,\ldots,-n]$$

Then,

$$\mathcal{M}(\overline{\mathcal{X}})$$

yields the Lawvere theory of λ-terms (up to α-equivalence) with metavariables in $\mathcal{X}$, equipped with capture-avoiding simultaneous substitution.

22. <u>Metasubstitution:</u> (see [F08])

$$\mathcal{M}(X) \times (\mathcal{M}Y)^X \xrightarrow[\varsigma]{} \mathcal{M}Y$$

comes from a <u>cartesian strength</u> and the monad structure (which embodies capture-avoiding substitution structure)

<u>Thm:</u> A cartesian strength

$$\Sigma(X) \times Y \to \Sigma(X \times Y)$$

compatible with the pointed tensorial strength

$$\Sigma(X) \cdot Y \to \Sigma(X \cdot Y)$$

canonically induces a cartesian strength

$$\mathcal{M}(X) \times Y \to \mathcal{M}(X \times Y)$$

[See his '08 paper [F08] for concrete syntactic theory.]

23. Second-order equational theories

From the Theory of Term Equational Systems

- Equations: $l \equiv r : \mathcal{Y}(\Gamma) \to \mathcal{M}(\mathcal{X})$

  terms with metavariables from $\mathcal{X}$ in context Γ

- Model Theory: for syntactically generated signatures

$$
\begin{tikzcd}
\Sigma\text{-}\underline{\mathrm{Mon}}/E \arrow[r, hook, bend right] \arrow[r, phantom, "\perp"] & \Sigma\text{-}\underline{\mathrm{Mon}} \arrow[l, bend right] \arrow[d, shift left] \arrow[d, phantom, "\dashv"] \\
 & \underline{\mathrm{Set}}^{\mathbb{F}} \arrow[u, shift left] \arrow[loop below, "\mathcal{M}"]
\end{tikzcd}
$$

is monadic

  Example: For $E = \beta\eta$, the initial object in $\Sigma\text{-}\underline{\mathrm{Mon}}/E$ is the Lawvere theory of $\alpha\beta\eta$-equivalence classes of λ-terms

- Logical theory:
  From the theory of Term Equational Logics

- Completeness:
  From the theory of free constructions for Equational systems.

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