A Proposal for Automating Diagrammatic Reasoning in Continuous Domains

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Abstract. This paper presents one approach to the formalisation of diagrammatic proofs as an alternative to algebraic logic. An idea of 'generic diagrams' is developed whereby one diagram (or rather, one sequence of diagrams) can be used to prove many instances of a theorem. This allows the extension of Jamnik's ideas in the Diamond system to continuous domains. The domain is restricted to non-recursive proofs in real analysis whose statement and proof have a strong geometric component. The aim is to develop a system of diagrams and redraw rules to allow a mechanised construction of sequences of diagrams constituting a proof. This approach involves creating a diagrammatic theory. The method is justified formally by (a) a diagrammatic axiomatisation, and (b) an appeal to analysis, viewing the diagram as an object in $\mathbb{R}^2$. The idea is to then establish an isomorphism between diagrams acted on by redraw rules and instances of a theorem acted on by rewrite rules. We aim to implement these ideas in an interactive prover entitled Rap (the Real Analysis Prover).

1 Introduction

There are some conjectures which people can prove by the use of geometric operations on diagrams, so called diagrammatic proofs. Insight is often more clearly perceived in these diagrammatic proofs than in the algebraic proofs.

It is not surprising that geometry (and geometric reasoning) was the original form of mathematics. For example, Pythagoras' Theorem was proved circa 500BC. The Pythagoreans' proof was lost, and as with anything relating to Pythagoras, it is impossible to know just what was done, when and by whom. However his proof would certainly have been geometric [9]. The elegant proof in Figure 1 is due to an unknown Chinese mathematician writing ∼200 BC [7]. By comparison, algebra is a recent invention, usually attributed to al-Khwarizmi in 830AD[1][8]. The modern algebraic formalism is barely a hundred years old, the

[1] One could argue it began with Diophantus c250AD, but this does not affect our argument.

M. Anderson, P. Cheng, and V. Haarslev (Eds.): Diagrams 2000, LNAI 1889, pp. 286–299, 2000.

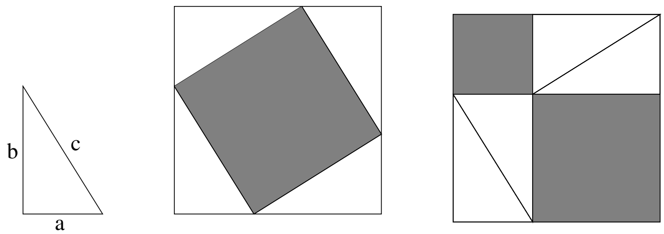

Fig. 1. The geometric proof of Pythagoras' Theorem is a classic example of diagrammatic reasoning. How can such a proof be formalised?

result of the axiomatisation project of Hilbert, Frege, Russell et al. A side-effect of this great project is that diagrams have fallen out of favour as acceptable methods of proof. Only algebra is regarded as formal. The current monopoly of algebraic formal mathematics is summed up by Tennant [2]:

> "[the diagram] is only an heuristic... it has no proper place in the proof as such... For the proof is a syntactic object consisting only of sentences arranged in a finite & inspectable array."

This insistence on algebraic formalism is a curious position, as Barker-Plummer observes: "most mathematicians deny that diagrams have any formal status, but on the other hand, diagrams are ubiquitous in mathematics texts" [1]. In the light of this, it is interesting to note that in al-Khwarizmi's work, the use of algebra is justified with geometric proofs [8].

2 Axiom: *n.* Received or Accepted Principle; Self Evident Truth

- *Collins Gem English Dictionary*

In spite of current doctrine, the concepts of axioms and proof are not inherently restricted to sentential reasoning. The only necessary criterion for rigorous proof is that all inferences are valid. That is, any conclusions reached are genuinely implicit in the hypothesis.

However, the problem remains that formally, drawings *cannot* prove anything about algebra, and vice versa. $\mathbb{R}^2$ is an abstract object[2] and not identical to an (infinite) sheet of paper, which is subject to physical laws and limitations, i.e. a line does not define a set in $\mathbb{R} \times \mathbb{R}$, nor does $\{(x, y)|y = ax + b\}$ define a line. People can make the connection and reason with visual representations, but syntactic manipulations only act on algebra. Since we wish to make inferences about (algebraic) objects beyond the diagram, an interpretation of the diagram is

[2] $\mathbb{R}$ denotes all real numbers, and hence $\mathbb{R}^2$ denotes the real plane.

needed. This can be justified by "homomorphic" mappings of diagrams to target domains, as described by Barwise & Etchemendy ("homomorphic mapping" is not a technical term, but it denotes any mapping where significant parts of diagram structure are preserved) [2].

Our basic framework will be to establish two isomorphisms:[3]

$$\text{diagrams} \longleftrightarrow \text{objects in } \mathbb{R}^2 \longleftrightarrow \text{mathematical target domain}$$

Only the second mapping can be proved to be an isomorphism. In spite of this we will talk interchangeably of concrete diagrams drawn on paper, blackboards, computer screens, etc. and diagrams 'drawn' on a bounded subset of $\mathbb{R}^2$. The universal use in mathematics of $\mathbb{R}^2$ to represent flat surfaces, and the obvious mapping between them, justify this laxness. Where we wish to distinguish, we will use 'pure diagrams' to refer to drawn diagrams, and '$\mathbb{R}$eal diagrams' for the equivalent objects in $\mathbb{R}^2$.

3 An Example Proof

Figure 2 gives an example proof demonstrating the use of diagrams with redraw rules in place of algebra with rewrite rules. The example is a typical analysis theorem (it assumes some familiarity with the subject).

Theorem 1 (Metric space continuity implies topological continuity[4]). *For X, Y metric spaces, ($\forall x \in X, \varepsilon > 0 \;\; \exists\, \delta > 0$ such that $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$) implies ($\forall S \subset Y, S$ open in $Y \Rightarrow f^{-1}(S)$ open in X).*

Our proposed approach is to give diagrammatic definitions for the mathematical concepts in the form of "redraw rules" – rewrite rules with diagrammatic pre- and post-conditions. An individual diagram can be suggestive and convey much to the viewer, but it is only the behaviour of diagrams that can produce rigorous proofs. We ignore the degenerate case $f^{-1}(S) = \emptyset$ – where the theorem is true by definition – by excluding it from the antecedent. The proof is not a single diagram but an ordered sequence of diagrams, in the same way that an algebraic proof is not a single statement but an ordered list of statements. The comments explain the steps taken, but are not necessary to the proof process. As we will show, the reasoning involved can be given entirely in diagrammatic inference rules. This approach requires defining diagrams, redraw rules and interpretations. The behaviour of the diagrams under the redraw rules can then be shown to be isomorphic (via the interpretation) to the behaviour of objects in the target domain.

[3] A function that is both one-to-one and onto is called an isomorphism.

[4] This theorem roughly says that if a function maps nearby points to nearby points, then its inverse will preserve open sets.

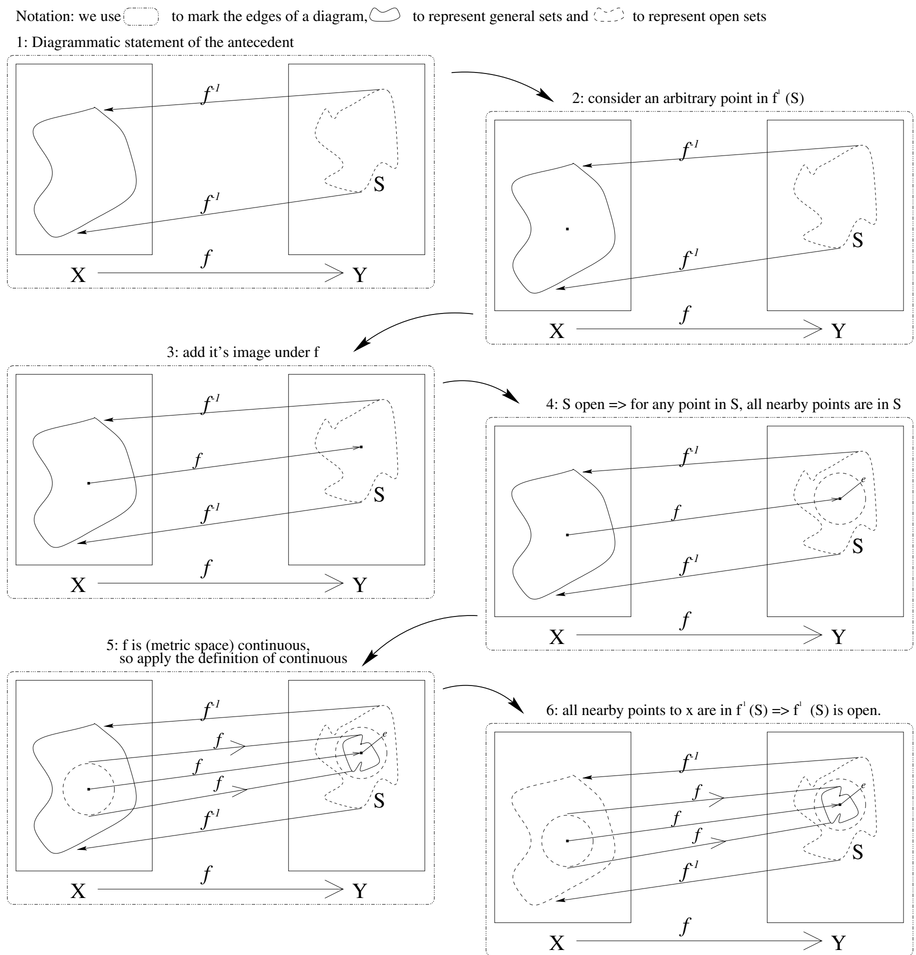

Fig. 2. An example diagrammatic proof: metric space continuity implies topological continuity

4 Related Work

Diagrammatic reasoning is a relatively new area of research and there is little directly related work. Barwise and Etechemendy's HyperProof system, which currently sets the standard for the educational applications of theorem provers [3], uses diagrams to great effect. It combines a first order logic prover with a visual representation for reasoning in a blocks world domain [2]. However, although HyperProof mixes diagrammatic logic with sentential (predicate) logic, it does not have any diagrammatic inference rules. To date, most systems have concentrated on using diagrams to guide an essentially algebraic proof [4]. For example, systems such as "&"/Grover use diagrams as heuristics [1].

The exception to this is the Diamond system, which proves theorems in natural number theory. It uses the constructive ω-rule to generalise from diagrammatic proofs of individual cases by showing that the given proof defines a uniform proof procedure for any instance [5]. The constructive ω rule works via meta-induction, and is therefore restricted to countable domains. However, we claim in this paper that the idea of using a uniform proof procedure to show that all cases can be proved (and are therefore true) can be adapted to continuous domains. We also draw on ideas from Shin's work for Venn Diagrams, where results from real analysis are used to show soundness of diagrammatic inference rules [11].

5 Some Formal Notation for Scribbling

Here we present our ideas for the formalisation of a diagrammatic system that reasons in continuous domains. Perhaps the main purpose of such formalisations is to avoid questions of human interpretation and intuition. This is analogous to the development of formal methods for sentential logic, allowing for truly rigorous proofs. Let $\mathcal{L}$ be a language. We now define a *drawing* and a *drawing function*.

Definition 1. *A drawing D is a set $\{(X_1, t_1, i_1), (X_2, t_2, i_2), ...\}$ where $X_j \subset \mathbb{R}^2$, $t_j \in \mathcal{L}$, $i_j \in \mathbb{N}$.*

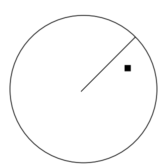

Fig. 3. An example drawing: $D = \{(\{(x, y) \mid x^2 + y^2 = 2\} \cup \{(x, y) \mid y = x,\ x \in [0, \sqrt{2}]\}, closed\_ball, 1),\ (\{(1, 1)\}, point, 2)\}$ Implicit in this conversion is that the scale of the diagram does not matter

Definition 2. *A graphic object or drawing function is a partial function $d_n : (D, P) \to D'$ where D, D' are drawings, P are parameters in $\mathbb{N}$, $\mathbb{R}$ or $\mathcal{L}$, and $D' = D \cup \{(X, n, i)\}$ such that $i = 1 + max\{j : (Y, m, j) \in D\}$. i is called the instance number. An instantiated or drawn object is a particular value for a drawing function.*

Often a construct depends upon a previously drawn part of the diagram. For example, in the definition of a continuous function:

$$\forall x, \varepsilon > 0 \exists\, \delta(x, \varepsilon) \text{ such that } |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$$

δ is dependent on ε and x. In our framework, these dependencies are handled implicitly: objects are assumed to be dependent on everything that was drawn before them, and independent of anything drawn afterwards. This information is 'stored' in the instance numbers of each object (e.g., see Figure 4).

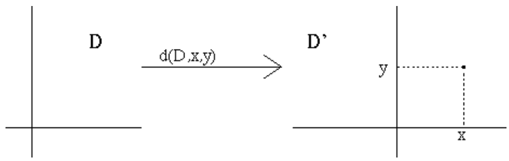

Fig. 4. One possible point drawing function $d_1(D, x, y) \to D \cup \{\{(x, y)\}, point, 1\}$

We now define a label and a diagram.

Definition 3. *A label is a partial function* $l : \{drawings\} \to \mathcal{L} \times \mathbb{R}$.

Let $d_1(p_1) \circ d_2(p_2)$ denote $d_2(d_1(\emptyset, p_1), p_2)$. Given a set of graphic objects and labels $P = d_1, d_2, ..l_1, l_2, ...$, we now define a *diagram type.*

Definition 4. *D_P is a diagram of type P if $D_P = \{d = d_1(p_1) \circ d_2(p_2) \circ .., l_1, .., l_m)$ where the l_i are label functions, such that $l_i(d) = (n, x), l_j(d) = (n, y) \Rightarrow x = y$ (we will refer to this as the labelling condition).*

Let $\mathbb{D} = \mathbb{D}(P)$ denote the set of all diagrams of type P. Or, to put it another way, D_P is a drawing of various objects in specific positions, marked according to object type and instance. Labels are used to show equal values for lengths or other such properties. The extra structure in the form of labels and diagram types on top of $\mathbb{R}^2$ prevents ambiguity such as in the diagram in Figure 5.

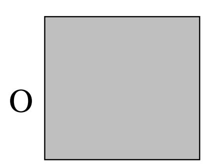

Fig. 5. Ambiguity in diagrams: is the 'O' a label for the area of the rectangle, a label for the length of a side, or just a passing circle?

Definition 5. *A diagrammatic theory is a tuple*

$$< \mathcal{L}, \{objects\}, \{labels\}, \sim, \{R_1, R_2, .., \} >$$

where the $R_i : \mathbb{D} \rightarrow \mathbb{D}$ *are called redraw rules.*

We say a diagram D is within a theory, if the theory contains redraw rules $R_1, R_2, ...$ such that $D = R_1 \circ \ldots \circ R_N(\emptyset)$.

5.1 Generic Diagrams

One big stumbling block in diagrammatic reasoning is the problem of universally quantified variables. Diagrams are inherently existentially quantified by the fact that they are drawn, and therefore specific objects. We cannot draw an abstract object. For any concept we wish to express there will usually be a continuum of different instances, and we can only ever draw a finite selection of these. For example, a theorem may mention a line of any length, but a drawn line must have a fixed length.

There are two solutions to this, and we will use a combination of both in this project. One is to let the interpretation do some of the work. The other is to define equivalence classes of diagrams, and work with proof processes which can be shown to be valid for all equivalent diagrams. In this way, each diagram is allowed to stand for a class of related diagrams.

Consider the two triangle diagrams in Figure 6. Are they equivalent? This

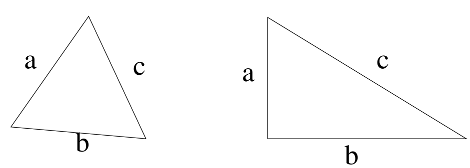

Fig. 6. Are these diagrams equivalent?

depends on what we are trying to show. The proof that the internal angles of a triangle add up to 180^o can be drawn with either. On the other hand, Pythagoras' Theorem is only true for the right angled triangle. It is envisioned that the automated theorem prover we propose here would have access to several related diagrammatic theories. It would choose an appropriate one to work in when given a conjecture.

A useful category of equivalence relations is one that includes equivalence relations induced by groups of geometric transformations. Given a group G of transformations of $\mathbb{R}^2$ we can define an equivalence relation $\sim$ over the diagrams.

Definition 6. *If $D = \{(X_1, t_1, 1), ...\}$, $D' = \{(X'_1, t'_1, 1), ...\}$ then $D \sim D'$ iff $\forall (X_j, t_j, i_j) \in D$, $t_j = t'_j$ and $\exists g \in G, g(X_j) = X'_j, g^{-1}(X'_j) = X_j$ or $\exists$ redraw rule R, diagrams C, C' such that $C \sim C', R(C) = D, R(C') = D'$.*

For example, in Figure 6, the diagrams would be equivalent if G contained rotations and stretches. We call G the group of unimportant deformations, and say a diagram D is a *generic diagram* for the equivalence class of D.

Definition 6 allows any equivalence relation, but in practice only a few are of interest. In an informal survey, students were presented with a collection of diagrams and asked to judge which ones should be considered equivalent. There was almost unanimous agreement as to which transformations should and should not be allowed in G. Translation, rotation, reflection are always considered valid. Transformations that do not preserve topological properties (i.e. inside/outside) are never valid. Those which do not preserve shapes, or affect one area of the diagram differently to another are accepted on diagrams containing 'amorphous blobs', but not on those composed of 'rigid' straight line shapes. These informal results can be summarised as: G should preserve the apparent properties (topology, shapes, etc.) of the diagram. It seems that diagrams are assumed to give all the relevant information: i.e. if a clearly recognisable shape is drawn, then this is an important feature. Otherwise, shape does not matter. Such cognitive issues do not affect the validity of a formalisation, but do affect it's usefulness.

We have used the redraw rules in defining $\sim$ in such a way as to ensure that $\forall D, D' \in \mathbb{D}$, R a redraw rule, $D \sim D' \Rightarrow R(D) \sim R(D')$. It is this property, which we call the *generic diagram property*, that allows us to generalise from a proof of a theorem for one instance, to a proof for all equivalent instances. A sequence of redraw rules must apply equally to all members of an equivalence class, thus guaranteeing equivalent proofs. If we define the relation $\sim$ differently – as we will need to for some areas – the generic diagram property must be proved. This should be possible using standard maths results and techniques.

6 Are Truth Values Relevant?

Sentential (predicate) logics do have many advantages, not least of which is the existence of well developed methods for checking their validity. Statements are associated with truth values and inference rules are valid if and only if they are sound. That is, P can be deduced from Q only if, in all models where Q is true, P is also true. This property of being sound can be tested quite simply using truth tables.

It is not clear that the values 'true' and 'false' have any meaning when applied to diagrams. Treating diagrammatic statements as predicate statements with a few spatial relations gives rise to 'diagrams' such as the one in Figure 7 taken from [6]. Here the diagram is viewed as an existentially quantified statement which can then be judged true or false. However this approach is quite unnatural, in that the objects considered are not diagrams themselves but sentential descriptions of diagrams. As such it is more an attempt to develop a predicate calculus with a visual interpretation. Our approach differs in that we do not

$B : box$	$\Rightarrow L_1, L_2, L_3, L_4 : line$ such that L_1 *connects_to* $L_2 \wedge L_2$ *connects_to* $L_3 \wedge L_3$ *connects_to* $L_4 \wedge L_4$ *connects_to* L_1 and set $B.lines = \{L_1, L_2, L_3, L_4\}$.
$A : and\_gate$	$\Rightarrow B : box, L : label$ such that L *inside* B with $L.text =$ "&" and set $A.frame = B.lines$.
$N : nand\_gate$	$\Rightarrow A : and\_gate, P : point$ such that P *on* $A.frame$ and set $N.frame = A.frame, N.out = P$.
$S : sr\_latch$	$\Rightarrow N_1, N_2 : nand\_gate, L_1, L_2, L_3, L_4 : line, L_5, L_6 : polyline$ such that L_1 *touches* $N_1.frame \wedge L_2$ *touches* $N_2.frame$ $\wedge L_3$ *touches* $N_1.out \wedge L_4$ *touches* $N_2.out$ $\wedge L_5$ *touches* $L_3 \wedge L_5$ *touches* $N_2.frame$ $\wedge L_6$ *touches* $L_4 \wedge L_6$ *touches* $N_1.frame$ and set $S.set = L_1, S.reset = L_2, S.out_1 = L_3, S.out_2 = L_4$.

Fig. 7. Ceci n'est pas une diagramme

try to interpret or parse diagrams. Instead we look at inference rules that act directly on diagrams.

A diagram cannot be true or false. How could we draw a false diagram? We can only talk of the truth and falsity of algebraic statements *associated* with the diagram by an 'interpretation' or 'logic mapping'. We therefore do not define soundness in diagrammatic reasoning at all. Hilbert would approve:

> "Mathematics is a game played according to certain simple rules with meaningless marks on paper."[10]

Instead we define *validity of interpretation*.

Definition 7. *Consider interpretations of the form $I : \mathbb{D} \to L$ where L is some conventional logic for the target domain and I is an injective function. We say I is valid iff $\forall D \in \mathbb{D}$, $D \xrightarrow{R} D' \Rightarrow I(D) \models I(D')$.*

Proofs of validity will vary. For example, if we wish to prove a theorem about a property p (e.g., area), then it suffices to show that the relevant redraw rules preserve property p.

If we do not have 'false' diagrams then we cannot use proof by contradiction. Such proofs can often be reformulated as proofs of the contrapositive without explicitly using contradiction.[5] It is hoped that using proof of the contrapositive will eliminate the need for proof by contradiction.

7 The Constructive $\aleph_1$ Rule

The *constructive ω-rule* allows us to deduce $\forall x P(x)$ by providing a uniform procedure to generate proofs for every x. In practice, this involves meta-induction:

[5] If we have a proof of $P \Rightarrow Q$ by $[P, \neg Q] \Rightarrow R$, $[P, \neg Q] \Rightarrow \neg R$. Then we can prove $\neg Q \Rightarrow (P \Rightarrow R)$ and $\neg Q \Rightarrow (P \Rightarrow \neg R)$. So $\neg Q \Rightarrow (\neg P \wedge R)$, $\neg Q \Rightarrow (\neg P \wedge \neg R)$ and by resolution (that is, a case split rather than contradiction) $\neg Q \Rightarrow \neg P$.

instead of induction on n, induction is carried out on the *proof of* n to show a valid proof exists for every case. This was introduced by Baker, and was used in Jamnik's DIAMOND system to generate general proofs from diagrammatic base case proofs [5].

We wish to extend this idea to prove theorems with a continuum (i.e. an uncountable number) of cases. This gives the inference rule:

$$\frac{\{P(x)|x \in X\}}{\forall x P(x)}$$

where X *may be of any cardinality.* We are interested in the case where X has cardinality $\aleph_1$, which we call the *constructive* $\aleph_1$ *rule*. Whilst very similar in concept, it leads to completely different proofs. Instead of using meta-induction, the existence of a uniform proof procedure for all cases is proved by showing that a specific proof for x_0 defines a valid proof for an arbitrary case $x \in X$. In the formalism outlined above, this amounts to showing that a proof from one diagram defines a valid proof from any other diagram in it's equivalence class. Intuitively, it can be seen that the generic diagram property and validity of interpretation as defined above are exactly what is needed to do this. Future work on this project should include a rigorous treatment of this.

8 Another Example: Pythagoras' Theorem

The generality of the proof of Theorem 1 given in Figure 2 (see §3) relied on multiple interpretations of diagrams. Here we demonstrate the use of equivalence classes for generalisation. Let G be the group of stretches along an axis, 90^o rotations, and translations. Assume that we have definitions for the following redraw rules:

1. draw_right_angled_triangle(a,b) (draw the triangle $\{(0,0),(a,0),(0,b)\}$ and add length labels to the short sides),
2. translate_triangle,
3. rotate_triangle_90°,
4. copy_triangle (draw a copy of specified triangle object with identical labels),
5. label_square_area (add an area label to a square area, i.e. recognise an emergent property),
6. subtract_triangle.

We only define one interpretation – each triangle is interpreted as itself. By stretching, one triangle can become any right-angled triangle, and so all right angled triangles are in the same equivalence class. Thus the correct generalisation from the specific traingle used in the diagrams to the general theorem is set by the equivalence class.The use of labels keeps the copied triangles identical to the original (since if a stretch breaks the identical size of two triangles, the result will fail the labelling condition and therefore will not be a valid diagram). By applying these rules (*in order:* 1,4,3,2,4,3,3,2,4,3,3,3,2,5) we draw the diagram in Figure 8. We then use rule 2 to transform this diagram to the diagram in Figure 9.

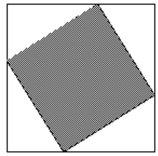

Fig. 8. Pythagoras' Theorem I

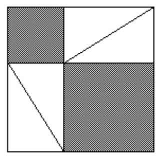

Fig. 9. Pythagoras' Theorem II

Rule 6 strips away the extra triangles. Pythagoras' Theorem, as presented here, is an area theorem and so to show the validity of our interpretation, we need to show that the translate and rotate redraw rules preserve the area, and the copy and subtract rules cancel each other out. In the purely diagrammatic theory, this is true by definition. In the $\mathbb{R}$eal diagrams, it is trivial for translation, copy and subtract, but not for rotation. In general, the fact that rotation preserves area is a corollary of Pythagoras' Theorem. However, for 90° rotations, Pythagoras' Theorem is not necessary: we only require $\left|\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right| = 1$. Thus the proof is valid for the instance shown. Also, the $\sim$ relation was induced from a group G as set out in §5.1. Hence the generic diagram property holds. Therefore the proof carries over from the proved instance (triangle $\{(0,0),(1,0),(0,2)\}$) to all equivalent instances. The equivalence class contains all right angled triangles,[6] so we have proved the theorem.

9 Concept Overview for Real Analysis

Historically, real analysis was developed to justify the use of calculus. $\mathbb{R}^3$ was meant to be the real world, and the definitions were supposed to capture how the universe works. Arguably most of the work went into coming up with the right definitions.

As the universe is a notoriously geometric place, it is not surprising that many of the concepts are best understood geometrically. This is why we choose the theory of real analysis. It is an area whose algebra is often confusing to students who meet it for the first time. Therefore it gives a good demonstration

[6] The trivial cases, where the 'triangle' is a line or a point are in separate equivalence classes and so must be considered separately.

of these ideas, taking complex algebraic formulations and replacing them with the geometric concepts they represent.

Working in Euclidean plane geometry, as with Pythagoras' Theorem above,[7] there is a canonical interpretation: that of each diagram as itself. Unfortunately there can be no such canonical interpretation for diagrams applying to $\mathbb{R}^n$, since $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for all $n \neq 2$.[8] Worse still, there are finite collections of open connected sets $S_1, .., S_N$ in $\mathbb{R}^n$, such that there do not exist $S'_1, ..., S'_N$ in $\mathbb{R}^2$ which are connected and have the same intersection relations as the S_i. These results mean that it is not possible to represent even finite collections of sets in $\mathbb{R}^n$ with sets in $\mathbb{R}^2$ without potentially losing some topological property. This is not a problem – it is actually quite convenient to ignore properties irrelevant to a theorem – but does introduce design choices and the need for several diagrammatic theories.

Analysis proofs often require reasoning about countably large sets of objects (e.g., infinite sequences or open covers). Whilst the framework we have outlined here does allow us to 'draw' countable sets of objects, they cannot physically be drawn. We therefore represent countable sets of objects by a single graphical object whose behaviour is correct with regard to the properties we are interested in (e.g., see Figure 10).

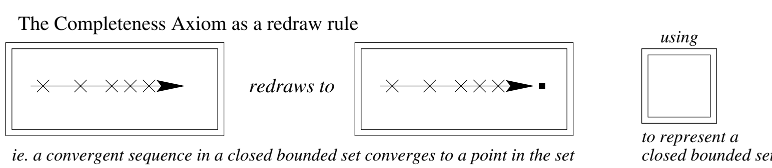

Fig. 10. Representing countable sets of objects: convergent sequences as illustrated by the (diagrammatic) completeness axiom

9.1 Example Proof for Theorem 1 Revisited

We can now begin to see how the example of a proof of Theorem 1 from §3 can be formalised. Drawing objects are needed for general sets, open sets, open balls $B_\varepsilon(x) = \{y| \, |x - y| < \varepsilon\}$, points, and function application arrows, plus a redraw rule for each step (e.g., see Figure 11). Here we define the equivalence relation $\sim$ by $D \sim D'$ iff there is a bijection $f : D \to D'$ such that $f(X, t, i) = (X', t', i') \Rightarrow t = t', i = i'$ and f is an isomorphism with respect to the relations `inside` and `intersects`. Part of our future work is to prove that the redraw rules used obey the generic diagram property.

[7] Pythagoras' Theorem is also an analysis result, and part of this project's remit.

[8] Bizarrely, for all n there are continuous surjective maps of $\mathbb{R}^2$ *onto* $\mathbb{R}^n$, but these are not injective.

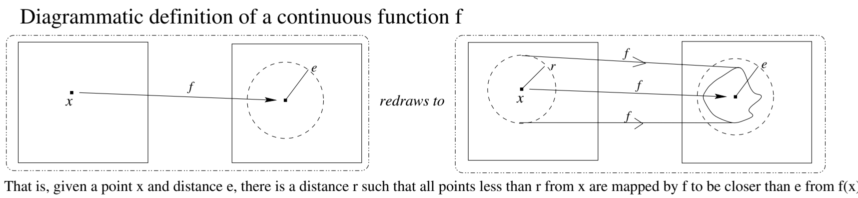

Fig. 11. Redraw rules for proof of Theorem 1 from §3: a diagrammatic definition for continuity

10 Limitations and Future Work

The next step in this project is to complete the formalisation of the ideas presented in this paper. We will then implement them in a prototype interactive theorem prover (called the Real Analysis Prover – RAP). Finally, we aim to implement an automated theorem prover for analysis theorems using diagrammatic reasoning. This will require developing heuristics appropriate to diagrammatic inference for guiding the proof search.

Working with actual diagrams, whilst possible, would be computationally very inefficient. The RAP system will therefore use a visual language of predicates with spatial relations.

Our current research and development of the formal structure is incomplete. One interesting question is whether the equivalence relation between diagrams can be relaxed to include so-called 'degenerate' or 'trivial' cases (such as empty sets or identical points). Currently these must be treated as separate cases. However, it is often possible to transform 'normal' diagrams into degenerate versions, but not vice versa. Ideally, the 'normal' proof would then carry over to degenerate cases.

So far, the proposed framework does not cover proofs of a recursive nature. In the future we hope to extend our framework to include them, perhaps by using a method used in DIAMOND, namely, generalisation via meta-induction.

Our aim in this project is to demonstrate the potential for applying diagrammatic reasoning in mathematical systems. The software we develop should also have a practical application in mathematics teaching, where we hope it will complement conventional methods.

The system outlined is only capable of incorporating algebraic manipulations in a crude way (by 'hiding' the algebra in redraw rules). Hybrid proofs, fluently combining diagrammatic and algebraic reasoning, are clearly a desirable goal. Such systems might finally get close to reproducing the reasoning methods of real-life mathematicians.

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