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Abstract

We present a straightforward embedding of quantified multimodal logic in simple type theory and prove its soundness and completeness. Modal operators are replaced by quantification over a type of possible worlds. We present simple experiments, using existing higher-order theorem provers, to demonstrate that the embedding allows automated proofs of statements in these logics, as well as meta properties of them.

1 Motivation

There are two approaches to automate reasoning in modal logics. The *direct* approach [7, 33] develops specific calculi and tools for the task; the *translational* approach [35, 36] transforms modal logic formulas into first-order logic and applies standard first-order tools.

In previous work [10, 8, 11] we have picked up and extended the embedding of multimodal logics in simple type theory as proposed by Brown [16]. The starting point is a characterization of multimodal logic formulas as particular λ-terms in simple type theory. A characteristic of the encoding is that the definiens of the $\Box_r$ operator λ-abstracts over the accessibility relation r. We have proved that this encoding is sound and complete [8, 11] and we have illustrated that this encoding supports the formulation of meta properties of encoded multimodal logics such as the correspondence between certain axioms and properties of the accessibility relation [10]. Some of these meta properties can even be effectively automated within our higher-order theorem prover LEO-II [13].

In this paper we extend our previous work to quantified multimodal logics. Multimodal logics with quantification for propositional variables have been studied by others before, including Kripke [30], Bull [17], Fine [19, 20], Kaplan [28], and Kremer [29]. Also first-order modal logics [23, 26] have been studied in numerous publications. We are interested here in multimodal logics with quantification over both propositional and first-order variables, a combination investigated, for example, by Fitting [21]. In contrast to Fitting we here pursue the translational approach and study the embedding of quantified multimodal logic in simple type theory. This approach has several advantages:

- The syntax and semantics of simple type theory is well understood [1, 2, 9, 25]. Studying (quantified) multimodal logics as fragments of simple type theory can thus help to better understand semantical issues.
- For simple type theory, various automated proof tools are available, including Isabelle/HOL [34], HOL [24], LEO-II [13], and TPS [5]. Employing the transformation presented in this paper, these systems become immediately applicable to quantified multimodal logics or fragments of them.
- Even meta properties of quantified modal logics can be formalized and mechanically analyzed within these provers.
- The systematic study of embeddings of multimodal logics in simple type theory can identify fragments of simple type theory that have interesting computational properties (such as the detection of the guarded fragment). This can foster improvements to proof tactics in interactive proof assistants.

Our paper is organized as follows. In Section 2 we briefly review simple type theory and adapt Fitting's [21] notion of quantified multimodal logics. In Section 3 we extend our previous work [8, 10, 11] and present an embedding of quantified multimodal logic in simple type theory. This embedding is shown sound and complete in Section 4. In Section 5 we present some simple experiments with the automated theorem provers LEO-II, TPS, and IsabelleP and the model finder IsabelleM. These experiments exploit the new TPTP THF infrastructure [12].

2 Preliminaries

2.1 Simple Type Theory

Classical higher-order logic or *simple type theory* $\mathcal{STT}$ [3, 18] is built on top of the simply typed λ-calculus. The set $\mathcal{T}$ of simple types is usually freely generated from a set of basic types $\{o, \iota\}$ (where o is the type of Booleans and ι is the type of individuals) using the function type constructor $\rightarrow$. Instead of $\{o, \iota\}$ we here consider a set of base types $\{o, \iota, \mu\}$, providing an additional base type μ (the type of possible worlds).

The simple type theory language $\mathcal{STT}$ is defined by (α, $\beta \in \mathcal{T}$):

$$\begin{aligned} s,t \quad ::= \quad & p_\alpha \mid X_\alpha \mid (\lambda X_\alpha . s_\beta)_{\alpha\rightarrow\beta} \mid (s_{\alpha\rightarrow\beta}\, t_\alpha)_\beta \mid (\neg_{o\rightarrow o}\, s_o)_o \mid \\ & (s_o \vee_{o\rightarrow o\rightarrow o} t_o)_o \mid (s_\alpha =_{\alpha\rightarrow\alpha\rightarrow o} t_\alpha)_o \mid (\Pi_{(\alpha\rightarrow o)\rightarrow o}\, s_{\alpha\rightarrow o})_o \end{aligned}$$

p_α denotes typed constants and X_α typed variables (distinct from p_α). Complex typed terms are constructed via abstraction and application. Our logical connectives of choice are $\neg_{o\rightarrow o}$, $\vee_{o\rightarrow o\rightarrow o}$, $=_{\alpha\rightarrow\alpha\rightarrow o}$ and $\Pi_{(\alpha\rightarrow o)\rightarrow o}$ (for each type α). From these connectives, other logical connectives can be defined in the usual way. We often use binder notation $\forall X_\alpha . s$ for $\Pi_{(\alpha\rightarrow o)\rightarrow o}(\lambda X_\alpha . s_o)$. We denote *substitution* of a term A_α for a variable X_α in a term B_β by $[A/X]B$. Since we consider α-conversion implicitly, we assume the bound variables of B avoid variable capture. Two common relations on terms are given by β-reduction and η-reduction. A β-redex has the form $(\lambda X. s)t$ and β-reduces to $[t/X]s$. An η-redex has the form $(\lambda X. sX)$ where variable X is not free in s; it η-reduces to s. We write $s =_\beta t$ to mean s can be converted to t by a series of β-reductions and expansions. Similarly, $s =_{\beta\eta} t$ means s can be converted to t using both β and η. For each $s \in L$ there is a unique β-*normal form* and a unique $\beta\eta$-*normal form*.

The semantics of $\mathcal{STT}$ is well understood and thoroughly documented in the literature [1, 2, 9, 25]; our summary below is adapted from Andrews [4].

A *frame* is a collection $\{D_\alpha\}_{\alpha\in\mathcal{T}}$ of nonempty sets D_α, such that $D_o = \{T, F\}$ (for truth and falsehood). The $D_{\alpha\rightarrow\beta}$ are collections of functions mapping D_α into D_β. The members of D_ι are called *individuals*. An *interpretation* is a tuple $\langle\{D_\alpha\}_{\alpha\in\mathcal{T}}, I\rangle$ where function I maps each typed constant c_α to an appropriate element of D_α, which is called the *denotation* of c_α (the logical symbols $\neg$, $\vee$, Π^α, and $=_{\alpha\rightarrow\alpha\rightarrow o}$ are always given the standard denotations). A *variable assignment* ϕ maps variables X_α to elements in D_α. An interpretation $\langle\{D_\alpha\}_{\alpha\in\mathcal{T}}, I\rangle$ is a *Henkin model* (equivalently, a *general model*) if and only if there is a binary function $\mathcal{V}$ such that $\mathcal{V}_\phi\, s_\alpha \in D_\alpha$ for each variable assignment ϕ and term $s_\alpha \in L$, and the following conditions are satisfied for all ϕ and all $s, t \in L$: (a) $\mathcal{V}_\phi X_\alpha = \phi X_\alpha$, (b) $\mathcal{V}_\phi\, p_\alpha = I p_\alpha$, (c) $\mathcal{V}_\phi(s_{\alpha\rightarrow\beta}\, t_\alpha) = (\mathcal{V}_\phi\, s_{\alpha\rightarrow\beta})(\mathcal{V}_\phi t_\alpha)$, and (d) $\mathcal{V}_\phi(\lambda X_\alpha . s_\beta)$ is that function from D_α into D_β whose value for each argument $z \in D_\alpha$ is $\mathcal{V}_{[z/X_\alpha]\phi} s_\beta$, where $[z/X_\alpha]\phi$ is that variable assignment such that $([z/X_\alpha]\phi)X_\alpha = z$ and $([z/X_\alpha]\phi)Y_\beta = \phi Y_\beta$ if $Y_\beta \neq X_\alpha$. (Since $I\neg$, $I\vee$, $I\Pi$, and $I\!=$ always denote the standard truth functions, we have $\mathcal{V}_\phi\,(\neg s) = T$ if and only if $\mathcal{V}_\phi\, s = F$, $\mathcal{V}_\phi\,(s \vee t) = T$ if and only if $\mathcal{V}_\phi\, s = T$ or $\mathcal{V}_\phi\, t = T$, $\mathcal{V}_\phi\,(\forall X_\alpha . s_o) = \mathcal{V}_\phi\,(\Pi^\alpha(\lambda X_\alpha . s_o)) = T$ if and only if for all $z \in D_\alpha$ we have $\mathcal{V}_{[z/X_\alpha]\phi}\, s_o = T$, and $\mathcal{V}_\phi\,(s = t) = T$ if and only if $\mathcal{V}_\phi\, s = \mathcal{V}_\phi\, t$. Moreover, we have $\mathcal{V}_\phi\, s = \mathcal{V}_\phi\, t$ whenever $s =_{\beta\eta} t$.) It is easy to verify that Henkin models obey the rule that everything denotes, that is, each term t_α always has a denotation $\mathcal{V}_\phi\, t_\alpha \in D_\alpha$. If an interpretation $\langle\{D_\alpha\}_{\alpha\in\mathcal{T}}, I\rangle$ is a Henkin model, then the function $\mathcal{V}_\phi$ is uniquely determined.

We say that formula $A \in L$ is *valid in a model* $\langle\{D_\alpha\}_{\alpha\in\mathcal{T}}, I\rangle$ if and only if $\mathcal{V}_\phi A = T$ for every variable assignment ϕ. A model for a set of formulas H is a model in which each formula of H is valid. A formula A is Henkin-valid if and only if A is valid in every Henkin model. We write $\models^{\mathcal{STT}} A$ if A is Henkin-valid.

2.2 Quantified Multimodal Logic

First-order quantification can be constant domain or varying domain. Below we only consider the constant domain case: every possible world has the same domain. We adapt the presentation of syntax and semantics of quantified modal logic from Fitting [21]. In contrast to Fitting we are not interested in **S5** structures but in the more general case of **K**.

Let $\mathcal{IV}$ be a set of first-order (individual) variables, $\mathcal{PV}$ a set of propositional variables, and $\mathcal{SYM}$ a set of predicate symbols of any arity. Like Fitting, we keep our definitions simple by not having function or constant symbols. While Fitting [21] studies quantified monomodal logic, we are interested in quantified multimodal logic. Hence, we introduce multiple $\Box_r$ operators for symbols r from an index set S. The grammar for our quantified multimodal logic $\mathcal{QML}$ is thus

$$s,t \;::=\; P \mid k(X^1,\ldots,X^n) \mid \neg\, s \mid s \,\vee\, t \mid \forall X\mathbf{.}\, s \mid \forall P\mathbf{.}\, s \mid \Box_r\, s$$

where $P \in \mathcal{PV}$, $k \in \mathcal{SYM}$, and $X, X^i \in \mathcal{IV}$.

Further connectives, quantifiers, and modal operators can be defined as usual. We also obey the usual definitions of free variable occurrences and substitutions.

Fitting introduces three different notions of semantics: **QS5**π^-, **QS5**π, and **QS5**π^+. We study related notions **QK**π^-, **QK**π, and **QK**π^+ for a modal context **K**, and we support multiple modalities.

A **QK**π^- *model* is a structure $M = (W, (R_r)_{r\in S}, D, P, (I_w)_{w\in W})$ such that $(W, (R_r)_{r\in S})$ is a multimodal frame (that is, W is the set of possible worlds and the R_r are accessibility relations between worlds in W), D is a non-empty set (the first-order domain), P is a non-empty collection of subsets of W (the propositional domain), and the I_w are interpretation functions mapping each n-place relation symbol $k \in \mathcal{SYM}$ to some n-place relation on D in world w.

A *variable assignment* $g = (g^{iv}, g^{pv})$ is a pair of maps $g^{iv} : \mathcal{IV} \longrightarrow D$ and $g^{pv} : \mathcal{PV} \longrightarrow P$, where g^{iv} maps each individual variable in $\mathcal{IV}$ to a an object in D and g^{pv} maps each propositional variable in $\mathcal{PV}$ to a set of worlds in P.

Validity of a formula s for a model $M = (W, (R_r)_{r\in S}, D, P, I_w)$, a world $w \in W$, and a variable assignment $g = (g^{iv}, g^{pv})$ is denoted as $M, g, w \models s$ and defined as follows, where $[a/Z]g$ denotes the assignment identical to g except that $([a/Z]g)(Z) = a$:

$$
\begin{array}{rll}
M, g, w \models k(X^1, \ldots, X^n) & \text{if and only if} & \langle g^{iv}(X^1), \ldots, g^{iv}(X^n)\rangle \in I_w(k) \\
M, g, w \models P & \text{if and only if} & w \in g^{pv}(P) \\
M, g, w \models \neg\, p & \text{if and only if} & M, g, w \not\models p \\
M, g, w \models p \,\vee\, q & \text{if and only if} & M, g, w \models p \text{ or } M, g, w \models q \\
M, g, w \models \forall X\textbf{.}\, p & \text{if and only if} & M, ([d/X]g^{iv}, g^{pv}), w \models p \text{ for all } d \in D \\
M, g, w \models \forall Q\textbf{.}\, p & \text{if and only if} & M, (g^{iv}, [v/Q]g^{pv}), w \models p \text{ for all } v \in P \\
M, g, w \models \Box_r\, p & \text{if and only if} & M, g, v \models p \text{ for all } v \in W \\
 & & \text{ with } \langle w, v\rangle \in R_r
\end{array}
$$

A $\mathbf{QK}\pi^-$ model $M = (W, (R_r)_{r\in S}, D, P, (I_w)_{w\in W})$ is a $\mathbf{QK}\pi$ *model* if for every variable assignment g and every formula $s \in \mathcal{QML}$, the set of worlds $\{w \in W \mid M, g, w \models s\}$ is a member of P.

A $\mathbf{QK}\pi$ model $M = (W, (R_r)_{r\in S}, D, P, (I_w)_{w\in W})$ is a $\mathbf{QK}\pi^+$ *model* if every world $w \in W$ is member of an atom in P. The *atoms* of P are minimal non-empty elements of P: no proper subsets of an atom are also elements of P.

A $\mathcal{QML}$ formula s is *valid in model* M *for world* w if $M, g, w \models s$ for all variable assignments g. A formula s is *valid in model* M if $M, g, w \models s$ for all g and w. Formula s is $\mathbf{QK}\pi$-*valid* if s is valid in all $\mathbf{QK}\pi$ models, when we write $\models^{\mathbf{QK}\pi} s$; we define $\mathbf{QK}\pi^-$-valid and $\mathbf{QK}\pi^+$-valid analogously.

In the remainder we mainly focus on $\mathbf{QK}\pi$ models. These models naturally correspond to Henkin models, as we shall see in Sect.4.

3 Embedding Quantified Multimodal Logic in $\mathcal{STT}$

The idea of the encoding is simple. We choose type ι to denote the (non-empty) set of individuals and we reserve a second base type μ to denote the (non-empty) set of possible worlds. The type o denotes the set of truth values. Certain formulas of type $\mu \to o$ then correspond to multimodal logic expressions. The multimodal connectives $\neg$, $\vee$, and $\Box$, become λ-terms of types $(\mu \to o) \to (\mu \to o)$, $(\mu \to o) \to (\mu \to o) \to (\mu \to o)$, and $(\mu \to \mu \to o) \to (\mu \to o) \to (\mu \to o)$ respectively.

Quantification is handled as usual in higher-order logic by modeling $\forall X. p$ as $\Pi(\lambda X. p)$ for a suitably chosen connective Π, as we remarked in Section 2. Here we are interested in defining two particular modal $\mathbf{\Pi}$-connectives: $\mathbf{\Pi}^\iota$, for quantification over individual variables, and $\mathbf{\Pi}^{\mu\to o}$, for quantification over modal propositional variables that depend on worlds, of types $(\iota \to (\mu \to o)) \to (\mu \to o)$ and $((\mu \to o) \to (\mu \to o)) \to (\mu \to o)$, respectively.

In previous work [10] we have discussed first-order and higher-order modal logic, including a means of explicitly excluding terms of certain types. The idea was that no proper subterm of $t_{\mu\to o}$ should introduce a dependency on worlds. Here we skip this restriction. This leads to a simpler definition of a quantified multimodal language $\mathcal{QML}^{STT}$ below, and it does not affect our soundness and completeness results.

Definition 3.1 (Modal operators)
The modal operators $\neg$, $\vee$, $\Box$, $\mathbf{\Pi}^\iota$, and $\mathbf{\Pi}^{\mu\to o}$ are defined as follows:

$$\begin{aligned}
\neg_{(\mu\to o)\to(\mu\to o)} &= \lambda\phi_{\mu\to o}. \lambda W_\mu. \neg(\phi\, W)\\
\vee_{(\mu\to o)\to(\mu\to o)\to(\mu\to o)} &= \lambda\phi_{\mu\to o}. \lambda\psi_{\mu\to o}. \lambda W_\mu. \phi\, W \vee \psi\, W\\
\Box_{(\mu\to\mu\to o)\to(\mu\to o)\to(\mu\to o)} &= \lambda R_{\mu\to\mu\to o}. \lambda\phi_{\mu\to o}. \lambda W_\mu. \forall V_\mu. \neg(R\, W\, V) \vee \phi\, V\\
\mathbf{\Pi}^\iota_{(\iota\to(\mu\to o))\to(\mu\to o)} &= \lambda\phi_{\iota\to(\mu\to o)}. \lambda W_\mu. \forall X_\iota. \phi\, X\, W\\
\mathbf{\Pi}^{\mu\to o}_{((\mu\to o)\to(\mu\to o))\to(\mu\to o)} &= \lambda\phi_{(\mu\to o)\to(\mu\to o)}. \lambda W_\mu. \forall P_{\mu\to o}. \phi\, P\, W
\end{aligned}$$

Further operators can be introduced, for example,

$$\begin{aligned}
\top_{(\mu\to o)\to(\mu\to o)} &= \forall P_{\mu\to o}. P \vee \neg P\\
\bot_{(\mu\to o)\to(\mu\to o)} &= \neg \top\\
\wedge_{(\mu\to o)\to(\mu\to o)\to(\mu\to o)} &= \lambda\phi_{\mu\to o}. \lambda\psi_{\mu\to o}. \neg(\neg\phi \vee \neg\psi)\\
\supset_{(\mu\to o)\to(\mu\to o)\to(\mu\to o)} &= \lambda\phi_{\mu\to o}. \lambda\psi_{\mu\to o}. \neg\phi \vee \psi\\
\Diamond_{(\mu\to\mu\to o)\to(\mu\to o)\to(\mu\to o)} &= \lambda R_{\mu\to\mu\to o}. \lambda\phi_{\mu\to o}. \neg(\Box\, R\, (\neg\phi))\\
\mathbf{\Sigma}^\iota_{(\iota\to(\mu\to o))\to(\mu\to o)} &= \lambda\phi_{\iota\to(\mu\to o)}. \neg(\mathbf{\Pi}^\iota(\lambda X_\iota. \neg(\phi\, X)))\\
\mathbf{\Sigma}^{\mu\to o}_{((\mu\to o)\to(\mu\to o))\to(\mu\to o)} &= \lambda\phi_{(\mu\to o)\to(\mu\to o)}. \neg(\mathbf{\Pi}^{\mu\to o}(\lambda P_{\mu\to o}. \neg(\phi\, P)))
\end{aligned}$$

We could also introduce further modal operators, such as the difference modality D, the global modality E, nominals with !, or the @ operator (consider the recent work of Kaminski and Smolka

[27] in the propositional hybrid logic context; they also adopt a higher-order perspective):

$$
\begin{aligned}
D_{(\mu\to o)\to(\mu\to o)} &= \lambda\phi_{\mu\to o}\text{.}\,\lambda W_{\mu}\text{.}\,\exists V_{\mu}\text{.}\, W \neq V \wedge \phi V \\
E_{(\mu\to o)\to(\mu\to o)} &= \lambda\phi_{\mu\to o}\text{.}\,\phi \,\vee\, D\,\phi \\
!_{(\mu\to o)\to(\mu\to o)} &= \lambda\phi_{\mu\to o}\text{.}\,E\,(\phi \,\wedge\, \neg\,(D\,\phi)) \\
@_{\mu\to(\mu\to o)\to(\mu\to o)} &= \lambda W_{\mu}\text{.}\,\lambda\phi_{\mu\to o}\text{.}\,\phi\, W
\end{aligned}
$$

This illustrates the potential of our embedding for encoding quantified hybrid logic, an issue that we might explore in future work.

For defining $\mathcal{QML}^{STT}$-propositions we fix a set $\mathcal{IV}^{STT}$ of individual variables of type ι, a set $\mathcal{PV}^{STT}$ of propositional variables of type $\mu \to o$, and a set $\mathcal{SYM}^{STT}$ of k-ary (curried) predicate constants of types $\underbrace{\iota \to \ldots \to \iota}_{n} \to (\mu \to o)$. The latter types will be abbreviated as $\iota^n \to (\mu \to o)$ in the remainder. Moreover, we fix a set $\mathcal{S}^{STT}$ of accessibility relation constants of type $\mu \to \mu \to o$.

Definition 3.2 ($\mathcal{QML}^{STT}$-propositions) $\mathcal{QML}^{STT}$-propositions are defined as the smallest set of simply typed λ-terms for which the following hold:

- Each variable $P_{\mu\to o} \in \mathcal{PV}^{STT}$ is an atomic $\mathcal{QML}^{STT}$-proposition, and if $X_\iota^j \in \mathcal{IV}^{STT}$ (for $j = 1, \ldots, n$) and $k_{\iota^n\to(\mu\to o)} \in \mathcal{SYM}^{STT}$, then the term $(k\,X^1 \,\ldots\, X^n)_{\mu\to o}$ is an atomic $\mathcal{QML}^{STT}$-proposition.
- If ϕ and ψ are $\mathcal{QML}^{STT}$-propositions, then so are $\neg\,\phi$ and $\phi \,\vee\, \psi$.
- If $r_{\mu\to\mu\to o} \in \mathcal{S}^{STT}$ is an accessibility relation constant and if ϕ is an $\mathcal{QML}^{STT}$-proposition, then $\Box\, r\, \phi$ is a $\mathcal{QML}^{STT}$-proposition.
- If $X_\iota \in \mathcal{IV}^{STT}$ is an individual variable and ϕ is a $\mathcal{QML}^{STT}$-proposition then $\mathbf{\Pi}^\iota(\lambda X_\iota\text{.}\,\phi)$ is a $\mathcal{QML}^{STT}$-proposition.
- If $P_{\mu\to o} \in \mathcal{PV}^{STT}$ is a propositional variable and ϕ is a $\mathcal{QML}^{STT}$-proposition then $\mathbf{\Pi}^{\mu\to o}(\lambda P_{\mu\to o}\text{.}\,\phi)$ is a $\mathcal{QML}^{STT}$-proposition.

We write $\Box_r\,\phi$, $\forall X_\iota\text{.}\,\phi$, and $\forall P_{\mu\to o}\text{.}\,\phi$ for $\Box\, r\,\phi$, $\mathbf{\Pi}^\iota(\lambda X_\iota\text{.}\,\phi)$, and $\mathbf{\Pi}^{\mu\to o}(\lambda P_{\mu\to o}\text{.}\,\phi)$, respectively.

Because the defining equations in Definition 3.1 are themselves formulas in simple type theory, we can express proof problems in a higher-order theorem prover elegantly in the syntax of quantified multimodal logic. Using rewriting or definition expanding, we can reduce these representations to corresponding statements containing only the basic connectives $\neg$, $\vee$, $=$, Π^ι, and $\Pi^{\mu\to o}$ of simple type theory.

Example 3.3 The following $\mathcal{QML}^{STT}$ proof problem expresses that in all accessible worlds there exists truth:

$$\Box_r\,\exists P_{\mu\to o}\text{.}\; P$$

The term rewrites into the following $\beta\eta$-normal term of type $\mu \to o$

$$\lambda W_\mu\text{.}\,\forall Y_\mu\text{.}\,\neg(r\,W\,Y) \vee (\neg\forall P_{\mu\to o}\text{.}\,\neg(P\,Y))$$

Next, we define validity of $\mathcal{QML}^{STT}$ propositions $\phi_{\mu\to o}$ in the obvious way: a $\mathcal{QML}$-proposition $\phi_{\mu\to o}$ is valid if and only if for all possible worlds w_μ we have $w_\mu \in \phi_{\mu\to o}$, that is, if and only if $\phi_{\mu\to o}\, w_\mu$ holds.

Definition 3.4 (Validity)
Validity is modeled as an abbreviation for the following simply typed λ-term:

$$\text{valid} \quad = \quad \lambda\phi_{\mu\to o}.\, \forall W_\mu.\, \phi\, W$$

Example 3.5 We analyze whether the proposition $\Box_r\, \exists P_{\mu\to o}.\, P$ is valid or not. For this, we formalize the following proof problem

$$\text{valid}\ (\Box_r\, \exists P_{\mu\to o}.\, P)$$

Expanding this term leads to

$$\forall W_\mu.\forall Y_\mu.\neg(r\, W\, Y) \vee (\neg\forall X_{\mu\to o}.\neg(X\, Y))$$

It is easy to check that this term is valid in Henkin semantics: put $X = \lambda Y_\mu.\top$.

An obvious question is whether the notion of quantified multimodal logics we obtain via this embedding indeed exhibits the desired properties. In the next section, we prove soundness and completeness for a mapping of $\mathcal{QML}$-propositions to $\mathcal{QML}^{STT}$-propositions.

4 Soundness and Completeness of the Embedding

In our soundness proof, we exploit the following mapping of $\mathbf{QK}\pi$ models into Henkin models. We assume that the $\mathcal{QML}$ logic L under consideration is constructed as outlined in Section 2 from a set of individual variables $\mathcal{IV}$, a set of propositional variables $\mathcal{PV}$, and a set of predicate symbols $\mathcal{SYM}$. Let $\Box_{r^1}, \ldots, \Box_{r^n}$ for $r^i \in S$ be the box operators of L.

Definition 4.1 ($\mathcal{QML}^{STT}$ logic $L^{\mathcal{STT}}$ for $\mathcal{QML}$ logic L)
Given an $\mathcal{QML}$ logic L, define a mapping $\dot{\_}$ as follows:

$$\begin{aligned}
\dot{X} &= X_\iota \text{ for every } X \in \mathcal{IV} \\
\dot{P} &= P_{\mu\to o} \text{ for every } P \in \mathcal{PV} \\
\dot{k} &= k_{\iota^n\to(\mu\to o)} \text{ for n-ary } k \in \mathcal{SYM} \\
\dot{r} &= r_{\mu\to\mu\to o} \text{ for every } r \in S
\end{aligned}$$

The $\mathcal{QML}^{STT}$ logic $L^{\mathcal{STT}}$ is obtained from L by applying Def. 3.2 with $\mathcal{IV}^{STT} = \{\dot{X} \mid X \in \mathcal{IV}\}$, $\mathcal{PV}^{STT} = \{\dot{P} \mid P \in \mathcal{PV}\}$, $\mathcal{SYM}^{STT} = \{\dot{k} \mid k \in \mathcal{SYM}\}$, and $\mathcal{S}^{STT} = \{\dot{r} \mid r \in S\}$. Our construction obviously induces a one-to-one correspondence $\dot{\_}$ between languages L and $L^{\mathcal{STT}}$.

Moreover, let $g = (g^{iv} : \mathcal{IV} \longrightarrow D,\ g^{pv} : \mathcal{PV} \longrightarrow P)$ be a variable assignment for L. We define the corresponding variable assignment

$$\dot{g} = (\dot{g}^{iv} : \mathcal{IV}^{STT} \longrightarrow D = D_\iota,\ \dot{g}^{pv} : \mathcal{PV}^{STT} \longrightarrow P = D_{\mu\to o})$$

for $L^{\mathcal{STT}}$ so that $\dot{g}(X_\iota) = \dot{g}(\dot{X}) = g(X)$ and $\dot{g}(P_{\mu\to o}) = \dot{g}(\dot{P}) = g(P)$ for all $X_\iota \in \mathcal{IV}^{STT}$ and $P_{\mu\to o} \in \mathcal{PV}^{STT}$.

Finally, a variable assignment $\dot{g}$ is lifted to an assignment for variables Z_α of arbitrary type by choosing $\dot{g}(Z_\alpha) = d \in D_\alpha$ arbitrarily, if $\alpha \neq \iota, \mu \to o$.

We assume below that L, $L^{\mathcal{STT}}$, g and $\dot{g}$ are defined as above.

Definition 4.2 (Henkin model H^Q for $\mathbf{QK}\pi$ model Q)
Given a $\mathbf{QK}\pi$ model $Q = (W, (R_r)_{r\in S}, D, P, (I_w)_{w\in W})$ for L, a Henkin model $H^Q = \langle \{D_\alpha\}_{\alpha\in\mathcal{T}}, I\rangle$ for $L^{\mathcal{STT}}$ is constructed as follows. We choose

- the set D_μ as the set of possible worlds W,
- the set D_ι as the set of individuals D (cf. definition of $\dot{g}^{iv}$),
- the set $D_{\mu\to o}$ as the set of sets of possible worlds P (cf. definition of $\dot{g}^{pv}$),[1]
- the set $D_{\mu\to\mu\to o}$ as the set of relations $(R_r)_{r\in S}$,
- and all other sets $D_{\alpha\to\beta}$ as (not necessarily full) sets of functions from D_α to D_β; for all sets $D_{\alpha\to\beta}$ the rule that everything denotes must be obeyed, in particular, we require that the sets $D_{\iota^n\to(\mu\to o)}$ contain the elements $Ik_{\iota^n\to(\mu\to o)}$ as characterized below.

[1] To keep things simple, we identify sets with their characteristic functions.

The interpretation I is as follows:

- Let $k_{\iota^n\to(\mu\to o)} = \dot{k}$ for $k \in \mathcal{SYM}$ and let $X^i_\iota = \dot{X}^i$ for $X^i \in \mathcal{IV}$. We choose $Ik_{\iota^n\to(\mu\to o)} \in D_{\iota^n\to(\mu\to o)}$ such that
$$(I\,k)(\dot{g}(X^1_\iota), \ldots, \dot{g}(X^n_\iota), w) = T$$
for all worlds $w \in D_\mu$ such that $Q, g, w \models k(X^1, \ldots, X^n)$; that is, if $\langle g(X^1), \ldots, g(X^n)\rangle \in I_w(k)$. Otherwise $(I\,k)(\dot{g}(X^1_\iota), \ldots, \dot{g}(X^n_\iota), w) = F$.
- Let $r_{\mu\to\mu\to o} = \dot{r}$ for $r \in S$. We choose $Ir_{\mu\to\mu\to o} \in D_{\mu\to\mu\to o}$ such that $(Ir_{\mu\to\mu\to o})(w, w') = T$ if $\langle w, w'\rangle \in R_r$ in Q and $(Ir_{\mu\to\mu\to o})(w, w') = F$ otherwise.

It is not hard to verify that $H^Q = \langle \{D_\alpha\}_{\alpha\in\mathcal{T}}, I\rangle$ is a Henkin model.

Lemma 4.3
Let $Q = (W, (R_r)_{r\in S}, D, P, (I_w)_{w\in W})$ be a $\mathbf{QK}\pi$ model and let $H^Q = \langle \{D_\alpha\}_{\alpha\in\mathcal{T}}, I\rangle$ be a Henkin model for Q. Furthermore, let $s_{\mu\to o} = \dot{s}$ for $s \in L$.
Then for all worlds $w \in W$ and variable assignments g we have $Q, g, w \models s$ in Q if and only if $\mathcal{V}_{[w/W_\mu]\dot{g}}\,(s_{\mu\to o} W_\mu) = T$ in H^Q.

Proof: The proof is by induction on the structure of $s \in L$.

Let $s = P$ for $P \in \mathcal{PV}$. By construction of Henkin model H^Q and by definition of $\dot{g}$, we have for $P_{\mu\to o} = \dot{P}$ that $\mathcal{V}_{[w/W_\mu]\dot{g}}\,(P_{\mu\to o}\, W_\mu) = \dot{g}(P_{\mu\to o})(w) = T$ if and only if $Q, g, w \models P$, that is, $w \in g(P)$.

Let $s = k(X^1, \ldots, X^n)$ for $k \in \mathcal{SYM}$ and $X^i \in \mathcal{IV}$. By construction of Henkin model H^Q and by definition of $\dot{g}$, we have for $\dot{k}(\dot{X}^1, \ldots, \dot{X}^n) = (k_{\iota^n\to(\mu\to o)}\, X^1_\iota\, \ldots\, X^n_\iota)$ that

$$\mathcal{V}_{[w/W_\mu]\dot{g}}\,((k_{\iota^n\to(\mu\to o)} X^1_\iota \ldots X^n_\iota)\, W_\mu) = (I\,k)(\dot{g}(X^1_\iota), \ldots, \dot{g}(X^n_\iota), w) = T$$

if and only if $Q, g, w \models k(X^1, \ldots, X^n)$, that is, $\langle g(X^1), \ldots, g(X^n)\rangle \in I_w(k)$.

Let $s = \neg\, t$ for $t \in L$. We have $Q, g, w \models \neg s$ if and only $Q, g, w \not\models s$, which is equivalent by induction to $\mathcal{V}_{[w/W_\mu]\dot{g}}\,(t_{\mu\to o}\, W_\mu) = F$ and hence to $\mathcal{V}_{[w/W_\mu]\dot{g}}\,\neg(t_{\mu\to o}\, W_\mu) =_{\beta\eta} \mathcal{V}_{[w/W_\mu]\dot{g}}\,((\neg\; t_{\mu\to o})\, W_\mu) = T$.

Let $s = (t \vee l)$ for $t, l \in L$. We have $Q, g, w \models (t \vee l)$ if and only if $Q, g, w \models t$ or $Q, g, w \models l$. The latter condition is equivalent by induction to $\mathcal{V}_{[w/W_\mu]\dot{g}}\,(t_{\mu\to o}\; W_\mu) = T$ or $\mathcal{V}_{[w/W_\mu]\dot{g}}\,(l_{\mu\to o}\; W_\mu) = T$ and therefore to $\mathcal{V}_{[w/W_\mu]\dot{g}}\,(t_{\mu\to o}\; W_\mu) \vee (l_{\mu\to o}\, W_\mu) =_{\beta\eta} \mathcal{V}_{[w/W_\mu]\dot{g}}\,(t_{\mu\to o} \;\vee\; l_{\mu\to o}\; W_\mu) = T$.

Let $s = \Box_r\, t$ for $t \in L$. We have $Q, g, w \models \Box_r\, t$ if and only if for all u with $\langle w, u\rangle \in R_r$ we have $Q, g, u \models t$. The latter condition is equivalent by induction to this one: for all u with $\langle w, u\rangle \in R_r$ we have $\mathcal{V}_{[u/V_\mu]\dot{g}}\,(t_{\mu\to o}\, V_\mu) = T$. That is equivalent to

$$\mathcal{V}_{[u/V_\mu][w/W_\mu]\dot{g}}\,(\neg(r_{\mu\to\mu\to o}\, W_\mu\, V_\mu) \vee (t_{\mu\to o}\, V_\mu)) = T$$

and thus to

$$\mathcal{V}_{[w/W_\mu]\dot{g}}\,(\forall Y_\mu\centerdot (\neg(r_{\mu\to\mu\to o}\, W_\mu\, Y_\mu) \vee (t_{\mu\to o}\, Y_\mu))) =_{\beta\eta} \mathcal{V}_{[w/W_\mu]\dot{g}}\,(\Box_r\, t\, W_\mu) = T.$$

Let $s = \forall X_\centerdot\, t$ for $t \in L$ and $X \in \mathcal{IV}$. We have $Q, g, w \models \forall X_\centerdot\, t$ if and only if $Q, [d/X]g, w \models t$ for all $d \in D$. The latter condition is equivalent by induction to $\mathcal{V}_{[d/X_\iota][w/W_\mu]\dot{g}}\,(t_{\mu\to o}\, W_\mu) = T$ for all $d \in D_\iota$. That condition is equivalent to $\mathcal{V}_{[w/W_\mu]\dot{g}}\,(\Pi^\iota_{(\iota\to o)\to o}(\lambda X_{\iota\centerdot}\, t_{\mu\to o}\, W_\mu)) =_{\beta\eta} \mathcal{V}_{[w/W_\mu]\dot{g}}\,((\lambda V_{\mu\centerdot}\,(\Pi^\iota_{(\iota\to o)\to o}\,(\lambda X_{\iota\centerdot}\, t_{\mu\to o}\, V_\mu)))\, W_\mu) = T$ and so by definition of $\mathbf{\Pi}^\iota$ to $\mathcal{V}_{[w/W_\mu]\dot{g}}\,((\mathbf{\Pi}^\iota_{(\iota\to(\mu\to o))\to(\mu\to o)}\,(\lambda X_{\iota\centerdot}\, t_{\mu\to o}))\, W_\mu) = \mathcal{V}_{[w/W_\mu]\dot{g}}\,((\forall X_{\iota\centerdot}\, t_{\mu\to o})\, W_\mu) = T$.

The case for $s = \forall P_\centerdot\, t$ where $t \in L$ and $P \in \mathcal{PV}$ is analogous to $s = \forall X_\centerdot\, t$. $\Box$

We exploit this result to prove the soundness of our embedding.

Theorem 4.4 (Soundness for $\mathbf{QK}\pi$ semantics) *Let $s \in L$ be a $\mathcal{QML}$ proposition and let $s_{\mu\to o} = \dot{s}$ be the corresponding $\mathcal{QML}^{STT}$ proposition. If $\models^{\mathcal{STT}}$ (valid $s_{\mu\to o}$) then $\models^{\mathbf{QK}\pi} s$.*

Proof: By contraposition, assume $\not\models^{\mathbf{QK}\pi} s$: that is, there is a $\mathbf{QK}\pi$ model $Q = (W, (R_r)_{r\in S}, D, P, (I_w)_{w\in W})$, a variable assignment g and a world $w \in W$, such that $Q, g, w \not\models s$. By Lemma 4.3, we have $\mathcal{V}_{[w/W_\mu]\dot{g}}\,(s_{\mu\to o}\, W_\mu) = F$ in a Henkin model H^Q for Q. Thus, $\mathcal{V}_{\dot{g}}\,(\forall W_{\mu\centerdot}\,(s_{\mu\to o}\, W)) =_{\beta\eta} \mathcal{V}_{\dot{g}}$ (valid $s_{\mu\to o}$) $= F$. Hence, $\not\models^{\mathcal{STT}}$ (valid $s_{\mu\to o}$). $\Box$

In order to prove completeness, we reverse our mapping from Henkin models to $\mathbf{QK}\pi$ models.

Definition 4.5 ($\mathcal{QML}$ logic $L^{\mathcal{QML}}$ for $\mathcal{QML}^{STT}$ logic L) The mapping $\bar{\_}$ is defined as the reverse map of $\dot{\_}$ from Def. 4.1.

The $\mathcal{QML}$ logic $L^{\mathcal{QML}}$ is obtained from $\mathcal{QML}^{STT}$ logic L by choosing $\mathcal{IV} = \{\bar{X}_\iota \mid X_\iota \in \mathcal{IV}^{STT}\}$, $\mathcal{PV} = \{\bar{P}_{\mu\to o} \mid P_{\mu\to o} \in \mathcal{PV}^{STT}\}$, $\mathcal{SYM} = \{\bar{k}_{\iota^n\to(\mu\to o} \mid k_{\iota^n\to(\mu\to o)} \in \mathcal{SYM}^{STT}\}$, and $S = \{\bar{r}_{\mu\to\mu\to o} \mid r_{\mu\to\mu\to o} \in \mathcal{S}^{STT}\}$.

Moreover, let $g : \mathcal{IV}^{STT} \cup \mathcal{PV}^{STT} \longrightarrow D \cup P$ be a variable assignment for L. The corresponding variable assignment $\bar{g} : \mathcal{IV} \cup \mathcal{PV} \longrightarrow D \cup P$ for $L^{\mathcal{QML}}$ is defined as follows: $\bar{g}(X) = \bar{g}(\bar{X}_\iota) = g(X_\iota)$ and $\bar{g}(P) = \bar{g}(\bar{P}_{\mu\to o}) = g(P_{\mu\to o})$ for all $X \in \mathcal{IV}$ and $P \in \mathcal{PV}$.

We assume below that L, $L^{\mathcal{QML}}$, g and $\bar{g}$ are defined as above.

Definition 4.6 ($\mathbf{QK}\pi^-$ model Q^H for Henkin model H) Given a Henkin model $H = \langle\{D_\alpha\}_{\alpha\in\mathcal{T}}, I\rangle$ for $\mathcal{QML}^{STT}$ logic L, we construct a $\mathcal{QML}$ model $Q^H = (W, (R_r)_{r\in S}, D, P, (I_w)_{w\in W})$ for $L^{\mathcal{QML}}$ by choosing $W = D_\mu$, $D = D_\iota$, $P = D_{\mu\to o}$[2], and $(R_r)_{r\in S} = D_{\mu\to\mu\to o}$. Let $k = \bar{k}_{\iota^n\to(\mu\to o)}$ and let $X^i = \bar{X}^i_\iota$. We choose $I_w(k)$ such that $\langle\bar{g}(X^1), \ldots, \bar{g}(X^n)\rangle \in I_w(k)$ if and only if

$$(I\,k)(g(X^1_\iota), \ldots, g(X^n_\iota), w) = T.$$

Finally, let $r = \bar{r}_{\mu\to\mu\to o}$. We choose R_r such that $\langle w, w'\rangle \in R_r$ if and only if $(I r_{\mu\to\mu\to o})(w, w') = T$.

It is not hard to verify that $Q^H = (W, (R_r)_{r\in S}, D, P, (I_w)_{w\in W})$ meets the definition of $\mathbf{QK}\pi^-$ models. Below we will see that it also meets the definition of $\mathbf{QK}\pi$ models.

[2] Again, we identify sets with their characteristic functions.

Lemma 4.7 *Let $Q^H = (W, (R_r)_{r \in S}, D, P, (I_w)_{w \in W})$ be a $\mathbf{QK}\pi^-$ model for a given Henkin model $H = \langle \{D_\alpha\}_{\alpha \in \mathcal{T}}, I \rangle$. Furthermore, let $s = \bar{s}_{\mu \to o}$.*

For all worlds $w \in W$ and variable assignments g we have $\mathcal{V}_{[w/W_\mu]g}\,(s_{\mu \to o} W_\mu) = T$ in H if and only if $Q^H, \bar{g}, w \models s$ in Q^H.

Proof: The proof is by induction on the structure of $s_{\mu \to o} \in L$ and it is analogous to the proof of Lemma 4.3.

$\square$

With the help of Lemma 4.7, we now show that the $\mathbf{QK}\pi^-$ models we construct in Def. 4.6 are in fact always $\mathbf{QK}\pi$ models. Thus, Henkin models never relate to $\mathbf{QK}\pi^-$ models that do not already fulfill the $\mathbf{QK}\pi$ criterion.

Lemma 4.8 *Let $Q^H = (W, (R_r)_{r \in S}, D, P, (I_w)_{w \in W})$ be a $\mathbf{QK}\pi^-$ model for a given Henkin model $H = \langle \{D_\alpha\}_{\alpha \in \mathcal{T}}, I \rangle$. Then Q^H is also a $\mathbf{QK}\pi$ model.*

Proof: We need to show that for every variable assignment $\bar{g}$ and formula $s = \bar{s}_{\mu \to o}$ the set $\{w \in W \mid Q^h, \bar{g}, w \models s\}$ is a member of P in Q^H. This is a consequence of the rule that everything denotes in the Henkin model H. To see this, consider $\mathcal{V}_g s_{\mu \to o} = \mathcal{V}_g(\lambda V_\mu . s_{\mu \to o} V)$ for variable V_μ not occurring free in $s_{\mu \to o}$. By definition of Henkin models this denotes that function from $D_\mu = W$ to truth values $D_o = \{T, F\}$ whose value for each argument $w \in D_\mu$ is $\mathcal{V}_{[w/V_\mu]g}(s\,V)$, that is, $s_{\mu \to o}$ denotes the characteristic function $\lambda w \in W . \mathcal{V}_{[w/V_\mu]g}\,(s_{\mu \to o} V_\mu) = T$ which we identify with the set $\{w \in W \mid \mathcal{V}_{[w/V_\mu]g}\,(s_{\mu \to o} V_\mu) = T\}$. Hence, we have $\{w \in W \mid \mathcal{V}_{[w/V_\mu]g}\,(s_{\mu \to o} V_\mu) = T\} \in D_{\mu \to o}$. By the choice of $P = D_{\mu \to o}$ in the construction of Q^H we know $\{w \in W \mid \mathcal{V}_{[w/V_\mu]g}\,(s_{\mu \to o} V_\mu) = T\} \in P$. By Lemma 4.7 we get $\{w \in W \mid Q^h, \bar{g}, w \models s\} \in P$.

$\square$

Theorem 4.9 (Completeness for $\mathbf{QK}\pi$ models) *Let $s_{\mu \to o}$ be a $\mathcal{QML}^{STT}$ proposition and let $s = \bar{s}_{\mu \to o}$ be the corresponding $\mathcal{QML}$ proposition. If $\models^{\mathbf{QK}\pi} s$ then $\models^{\mathcal{STT}}$ (valid $s_{\mu \to o}$).*

Proof: By contraposition, assume $\not\models^{\mathcal{STT}}$ (valid $s_{\mu \to o}$): there is a Henkin model $H = \langle \{D_\alpha\}_{\alpha \in \mathcal{T}}, I \rangle$ and a variables assignment g such that $\mathcal{V}_g\,(\text{valid}\, s_{\mu \to o}) = F$. Hence, for some world $w \in D_\mu$ we have $\mathcal{V}_{[w/W_\mu]g}\,(s_{\mu \to o} W_\mu) = F$. By Lemma 4.7 we then get $Q^H, \bar{g}, w \not\models^{\mathbf{QK}\pi^-} s$ for $s = \bar{s}_{\mu \to o}$ in $\mathbf{QK}\pi^-$ model Q^H for H. By Lemma 4.8 we know that Q^H is actually a $\mathbf{QK}\pi$ model. Hence, $\not\models^{\mathbf{QK}\pi} s$. Box

Our soundness and completeness results obviously also apply to fragments of $\mathcal{QML}$ logics.

Corollary 4.10 *The reduction of our embedding to propositional quantified multimodal logics (which only allow quantification over propositional variables) is sound and complete.*

Corollary 4.11 *The reduction of our embedding to first-order multimodal logics (which only allow quantification over individual variables) is sound and complete.*

Corollary 4.12 *The reduction of our embedding to propositional multimodal logics (no quantification) is sound and complete.*

5 Applying the Embedding in Practice

In this section, we illustrate the practical benefits of our embedding with the help of some simple experiments. We employ off-the-shelf automated higher theorem provers and model generators for simple type theory to solve problems in quantified multimodal logic. Future work includes the encoding of a whole library of problems for quantified multimodal logics and the systematic evaluation of the strengths of these provers to reason about them.

In our case studies, we have employed the simple type theory automated reasoners LEO-II, TPS [5], IsabelleM and IsabelleP.[3] These systems are available online via the SystemOnTPTP tool and they exploit the new TPTP infrastructure for typed higher-order logic [12].

The formalization of the modal operators (Def. 4.1) and the notion of validity (Def. 3.4) in THF syntax [12] is presented in Appendix A. As secured by the theoretical results of this paper, these few lines of definitions are all we need to make simple type theory reasoners applicable to quantified multimodal logic.

If we call the theorem provers LE0-II and IsabelleP with this file, then they try to find a refutation from these equations: they try to prove their inconsistency. As expected, none of the systems reports success. The model finder IsabelleM, however, answers in 0.6 seconds that a model has been found. IsabelleM employs the SAT solver zChaff.

When applying our systems to Example 3.5, we get the following results (where $+/t$ represents that a proof has been found in t seconds and $-/t$ reports that no proof has been found within t seconds): IsabelleP: $+/1.0$, LEO-II: $+/0.0$, TPS: $+/0.3$. IsabelleM does not find a model (this also holds for the examples below).

We also tried the Barcan formula and its converse:

$$\begin{aligned} BF: &\quad \text{valid } (\forall X_\iota \centerdot \Box_r\, (p_{\iota\to(\mu\to o)}\, X)) \supset (\Box_r\, \forall X_\iota \centerdot (p_{\iota\to(\mu\to o)}\, X)) \\ BF^{-1}: &\quad \text{valid } (\Box_r\, \forall X_\iota \centerdot (p_{\iota\to(\mu\to o)}\, X)) \supset (\forall X_\iota \centerdot \Box_r\, (p_{\iota\to(\mu\to o)}\, X)) \end{aligned}$$

The results for BF and BF^{-1} are IsabelleP: $+/0.7$, LEO-II and LEO-IIP: $+/0.0$, TPS: $+/0.2$. This confirms that our first-order quantification is constant domain.

The next example analyzes the equivalence of two quantified multimodal logic formula schemes (which can be read as “if it is possible for everything to be P, then everything is potentially P”):

$$\begin{aligned} &\forall R_{\iota\to\mu\to o} \centerdot \forall P_{\iota\to(\mu\to o)} \centerdot \\ &\quad (\text{valid } (\Diamond_R\, \forall X_\iota \centerdot (P\, X)) \supset (\forall X_\iota \centerdot \Diamond_R\, (P\, X))) \\ &\qquad \Leftrightarrow \\ &\quad (\text{valid } (\exists X_\iota \centerdot \Box_R\, (P\, X)) \supset (\Box_R\, \exists X_\iota \centerdot (P\, X))) \end{aligned}$$

The results are: IsabelleP: $+/2.0$, LEO-II: $+/0.0$, TPS:$+/0.2$.

---

[3] IsabelleM is a model finder in Isabelle that has been made available in batch mode, while IsabelleP applies a series of Isabelle proof tactics in batch mode.

An interesting meta property is the correspondence between axiom

$$\text{valid } \forall P_{\iota\to(\mu\to o)}\mathbf{.}\ (\Diamond_i \Box_j P) \supset \Box_k \Diamond_l P$$

and the (i, j, k, l)-confluence property:

$$\forall A_{\mu}\mathbf{.}\forall B_{\mu}\mathbf{.}\forall C_{\mu}\mathbf{.}(((i\,A\,B) \wedge (k\,A\,C)) \Rightarrow \exists D_{\mu}\mathbf{.}((j\,B\,D) \wedge (l\,C\,D)))$$

The results are: IsabelleP: $+/3.7$, LEO-II: $+/0.3$, TPS:$+/0.2$. The problem encoding is presented in Appendix C.

Future work will investigate how well this approach scales for more challenging problems. We therefore invite potential users to encode their problems in the THF syntax and to submit them to the THF TPTP library.

6 Conclusion

We have presented a straightforward embedding of quantified multimodal logics in simple type theory and we have shown that this embedding is sound and complete for $\mathbf{QK}\pi$ semantics. This entails further soundness and completeness results of our embedding for fragments of quantified multimodal logics. We have formally explored the natural correspondence between $\mathbf{QK}\pi$ models and Henkin models and we have shown that the weaker $\mathbf{QK}\pi^-$ models do not enjoy such a correspondence.

Non-quantified and quantified (normal) multimodal logics can thus be uniformly seen as natural fragments of simple type theory and their semantics (except some weak notions such as $\mathbf{QK}\pi^-$ models) can be studied from the perspective of the well understood semantics of simple type theory. Vice versa, via our embedding we can characterize some computationally interesting fragments of simple type theory, which in turn may lead to some powerful proof tactics for higher-order proof assistants.

Future work includes further extensions of our embedding to also cover quantified hybrid logics [14, 15] and full higher-order modal logics [22, 31]. A first suggestion in direction of higher-order modal logics has already been made [10]. This proposal does however not yet address intensionality aspects. However, combining this proposal with non-extensional notions of models for simple type theory [9, 32] appears a promising direction.

References

[1] Peter B. Andrews. General models and extensionality. *Journal of Symbolic Logic*, 37:395–397, 1972.

[2] Peter B. Andrews. General models, descriptions, and choice in type theory. *Journal of Symbolic Logic*, 37:385–394, 1972.

[3] Peter B. Andrews. *An Introduction to Mathematical Logic and Type Theory: To Truth Through Proof*. Kluwer Academic Publishers, second edition, 2002.

[4] Peter B. Andrews. Church's type theory. In E. N. Zalta, editor, *The Stanford Encyclopedia of Philosophy*. Center for the Study of Language and Information, Stanford University, 2008. http://plato.stanford.edu/archives/fall2008/entries/type-theory-church/.

[5] Peter B. Andrews and Chad E. Brown. TPS: A hybrid automatic-interactive system for developing proofs. *Journal of Applied Logic*, 4(4):367–395, 2006.

[6] Alessandro Armando, Peter Baumgartner, and Gilles Dowek, editors. *Automated Reasoning, 4th International Joint Conference, IJCAR 2008, Sydney, Australia, August 12-15, 2008, Proceedings*, volume 5195 of *LNCS*. Springer, 2008.

[7] Matteo Baldoni, Laura Giordano, and Alberto Martelli. A framework for a modal logic programming. In *Joint International Conference and Symposium on Logic Programming*, pages 52–66, 1996.

[8] Christoph Benzmüller. Automating access control logic in simple type theory with LEO-II. In *Proceedings of The IFIP 24th International Information Security Conference, Pafos, Cyprus*, IFIP. Springer, 2009. In press.

[9] Christoph Benzmüller, Chad E. Brown, and Michael Kohlhase. Higher-order semantics and extensionality. *Journal of Symbolic Logic*, 69:1027–1088, 2004.

[10] Christoph Benzmüller and Lawrence Paulson. Exploring properties of normal multimodal logics in simple type theory with leo-ii. In Christoph Benzmüller, Chad E. Brown, Jörg Siekmann, and Richard Statman, editors, *Festschrift in Honor of Peter B. Andrews on His 70th Birthday*, Studies in Logic, Mathematical Logic and Foundations, chapter Exploring Properties of Normal Multimodal Logics in Simple Type Theory with LEO-II. College Publications, 2008. ISBN 978-1-904987-70-3.

[11] Christoph Benzmüller and Lawrence C. Paulson. Multimodal and intituitionistic logics in simple type theory, 2009. Submitted.

[12] Christoph Benzmüller, Florian Rabe, and Geoff Sutcliffe. THF0 — The Core TPTP Language for Classical Higher-Order Logic. In P. Baumgartner, A. Armando, and D. Gilles, editors, *Proceedings of the 4th International Joint Conference on Automated Reasoning*, number 5195 in Lecture Notes in Artificial Intelligence, pages 491–506, 2008.

[13] Christoph Benzmüller, Frank Theiss, Larry Paulson, and Arnaud Fietzke. LEO-II — a cooperative automatic theorem prover for higher-order logic. In Armando et al. [6], pages 162–170.

[14] Patrick Blackburn and Maarten Marx. Tableaux for quantified hybrid logic. In Uwe Egly and Christian G. Fermüller, editors, *Automated Reasoning with Analytic Tableaux and Related Methods, International Conference, TABLEAUX 2002, Copenhagen, Denmark, July 30 - August 1, 2002, Proceedings*, volume 2381 of *Lecture Notes in Computer Science*, pages 38–52. Springer, 2002.

[15] Torben Braüner. Natural deduction for first-order hybrid logic. *Journal of Logic, Language and Information*, 14(2):173–198, 2005.

[16] Chad E. Brown. Encoding hybrid logic in higher-order logic. Unpublished slides from an invited talk presented at Loria Nancy, France, April 2005. `http://mathgate.info/cebrown/papers/hybrid-hol.pdf`.

[17] Robert A. Bull. On modal logic with propositional quantifiers. *Journal of Symbolic Logic*, 34(2):257–263, 1969.

[18] Alonzo Church. A formulation of the simple theory of types. *Journal of Symbolic Logic*, 5:56–68, 1940.

[19] Kit Fine. Propositional quantifiers in modal logic. *Theoria*, 36:336–346, 1970.

[20] Kit Fine. Failures of the interpolation lemma in quantified modal logic. *Journal of Symbolic Logic*, 44(2):201–206, 1979.

[21] Melvin Fitting. Interpolation for first-order S5. *Journal of Symbolic Logic*, 67(2):621–634, 2002.

[22] Melvin Fitting. *Types, Tableaus, and Gödel's God*. Kluwer Academic Publishers, 2002.

[23] Melvin Fitting and Richard L. Mendelsohn. *First-Order Modal Logic*. Kluwer, 1998.

[24] Mike J. Gordon and Tom F. Melham. *Introduction to HOL: A Theorem-Proving Environment for Higher-Order Logic*. Cambridge University Press, 1993.

[25] Leon Henkin. Completeness in the theory of types. *Journal of Symbolic Logic*, 15:81–91, 1950.

[26] G. Hughes and M. Cresswell. *A New Introduction to Modal Logic*. Routledge, London, 1996.

[27] Mark Kaminski and Gert Smolka. Terminating tableaux for hybrid logic with the difference modality and converse. In Armando et al. [6], pages 210–225.

[28] David Kaplan. S5 with quantifiable propositional variables. *Journal of Symbolic Logic*, 35:355, 1970.

[29] Philip Kremer. On the complexity of propositional quantification in intuitionistic logic. *Journal of Symbolic Logic*, 62(2):529–544, 1997.

[30] Saul Kripke. A completeness theorem in modal logic. *Journal of Symbolic Logic*, 24(1):1–14, 1959.

[31] Reinhard Muskens. Higher-order modal logic. In Patrick Blackburn, Johan F. A. K. van Benthem, and Frank Wolter, editors, *Handbook of Modal Logic, Volume 3 (Studies in Logic and Practical Reasoning)*. Elsevier Science Inc., New York, NY, USA, 2006.

[32] Reinhard Muskens. Intensional models for the theory of types. *Journal of Symbolic Logic*, 72(1):98–118, 2007.

[33] Linh Anh Nguyen. A fixpoint semantics and an SLD-resolution calculus for modal logic programs. *Fundamenta Informaticae*, 55(1):63–100, 2003.

[34] Tobias Nipkow, Lawrence C. Paulson, and Markus Wenzel. *Isabelle/HOL: A Proof Assistant for Higher-Order Logic*. Number 2283 in Lecture Notes in Computer Science. Springer-Verlag, 2002.

[35] Andreas Nonnengart. How to use modalities and sorts in Prolog. In Craig MacNish, David Pearce, and Luís Moniz Pereira, editors, *Logics in Artificial Intelligence, European Workshop, JELIA '94*, volume 838 of *Lecture Notes in Computer Science*, pages 365–378. Springer, 1994.

[36] Hans Jürgen Ohlbach. A resolution calculus for modal logics. In Ewing L. Lusk and Ross A. Overbeek, editors, *9th International Conference on Automated Deduction, Argonne, Illinois, USA, May 23-26, 1988, Proceedings*, volume 310 of *Lecture Notes in Computer Science*, pages 500–516. Springer, 1988.

A THF Formalization of Quantified Multi-Modal Logic in Simple Type Theory

```
%------------------------------------------------------------------------
% File     : QML.ax
% Domain   : Quantified multimodal logic
% Problems :
% Version  :
% English  : Embedding of quantified multimodal logic in
%            simple type theory
% Refs     :
% Source   : Formalization in THF by C. Benzmueller
% Names    :
% Status   :
% Rating   :
% Syntax   :
% Comments :
%------------------------------------------------------------------------
%---- declaration of additional base type mu
thf(mu,type,(
    mu: $tType )).

%---- modal operators not, or, box, Pi (for types mu and $i>$o)
thf(mnot,definition,
    ( mnot
    = ( ^ [Phi: $i > $o,W: $i] :
            ~ ( Phi @ W ) ) )).

thf(mor,definition,
    ( mor
    = ( ^ [Phi: $i > $o,Psi: $i > $o,W: $i] :
            ( ( Phi @ W )
            | ( Psi @ W ) ) ) )).

thf(mbox,definition,
    ( mbox
    = ( ^ [R: $i > $i > $o,Phi: $i > $o,W: $i] :
          ! [V: $i] :
            ( ~ ( R @ W @ V )
            | ( Phi @ V ) ) ) )).

thf(mall_ind,definition,
    ( mall_ind
    = ( ^ [Phi: mu > $i > $o,W: $i] :
          ! [X: mu] :
            ( Phi @ X @ W ) ) )).

thf(mall_prop,definition,
    ( mall_prop
    = ( ^ [Phi: ( $i > $o ) > $i > $o,W: $i] :
          ! [P: $i > $o] :
            ( Phi @ P @ W ) ) )).

%---- further modal operators
thf(mtrue,definition,
```

```
    ( mtrue
    = ( mall_prop
      @ ^ [P: $i > $o] :
          ( mor @ P @ ( mnot @ P ) ) ) )).

thf(mtrue,definition,
    ( mfalse
    = ( mall_prop
      @ ^ [P: $i > $o] :
          ( mnot @ mtrue ) ) )).

thf(mand,definition,
    ( mand
    = ( ^ [Phi: $i > $o,Psi: $i > $o] :
          ( mnot @ ( mor @ ( mnot @ Phi ) @ ( mnot @ Psi ) ) ) ) )).

thf(mimpl,definition,
    ( mimpl
    = ( ^ [Phi: $i > $o,Psi: $i > $o] :
          ( mor @ ( mnot @ Phi ) @ Psi ) ) )).

thf(mdia,definition,
    ( mdia
    = ( ^ [R: $i > $i > $o,Phi: $i > $o] :
          ( mnot @ ( mbox @ R @ ( mnot @ Phi ) ) ) ) )).

thf(mexi_ind,definition,
    ( mexi_ind
    = ( ^ [Phi: mu > $i > $o] :
          ( mnot
          @ ( mall_ind
            @ ^ [X: mu] :
                ( mnot @ ( Phi @ X ) ) ) ) ) )).

thf(mexi_prop,definition,
    ( mexi_prop
    = ( ^ [Phi: ( $i > $o ) > $i > $o] :
          ( mnot
          @ ( mall_prop
            @ ^ [P: $i > $o] :
                ( mnot @ ( Phi @ P ) ) ) ) ) )).

%---- definition of validity
thf(mvalid,definition,
    ( mvalid
    = ( ^ [Phi: $i > $o] :
        ! [W: $i] :
          ( Phi @ W ) ) )).
```

B THF Example: In all Worlds exists Truth

```
%------------------------------------------------------------------------
% File     : ex1.p
% Domain   : Quantified multimodal logic
% Problems :
% Version  :
% English  : In all accessible worlds exists truth.
% Refs     :
% Source   : Formalization in THF by C. Benzmueller
% Names    :
% Status   :
% Rating   :
% Syntax   :
% Comments :
%------------------------------------------------------------------------
%---- include the definitions for qunatified multimodal logic
include('QML.ax').

%---- provide a consant for accesibility relation r
thf(r,type,r:$i>$i>$o).

%---- conjecture statement
thf(ex1,conjecture,
    (mvalid @ (mbox @ r @ (mexi_prop @ (^[P:$i>$o]: P))))).
```

C THF Example: Confluence Property of Accessibility Relations

```
%------------------------------------------------------------------------
% File     : ex9.p
% Domain   : Quantified multimodal logic
% Problems :
% Version  :
% English  : Confluence property of accessibility relations
% Refs     :
% Source   : Formalization in THF by C. Benzmueller
% Names    :
% Status   :
% Rating   :
% Syntax   :
% Comments :
%------------------------------------------------------------------------
%---- include the definitions for qunatified multimodal logic
include('QML.ax').

%---- constants for accesibility relations
thf(i,type,(
    i: $i > $i > $o )).

thf(j,type,(
    j: $i > $i > $o )).

thf(k,type,(
    k: $i > $i > $o )).

thf(l,type,(
    l: $i > $i > $o )).

%---- definition of confluence property
thf(confluence,definition,
    ( confluence
    = ( ^ [I: $i > $i > $o,J: $i > $i > $o,
           K:  $i > $i > $o,L: $i > $i > $o] :
        ! [A: $i,B: $i,C: $i] :
          ( ( ( I @ A @ B )
            & ( K @ A @ C ) )
         => ? [D: $i] :
              ( ( J @ B @ D )
              & ( L @ C @ D ) ) ) ) )).

%---- correspondence between axiom and confluence property
thf(conj,conjecture,
    ( ( mvalid
      @ ( mall_prop
        @ ^ [P: $i > $o] :
            ( mimpl @ ( mdia @ i @ ( mbox @ j @ P ) )
                    @ ( mbox @ k @ ( mdia @ l @ P ) ) ) ) )
  <=> ( confluence @ i @ j @ k @ l ) )).
```