Alexandria: Libraries and Tools for Research Mathematicians

Mathematicians are fallible. Announcements of results are too frequently followed by the discovery of serious errors. We've seen this with proofs of Fermat's Last Theorem, the Riemann hypothesis and the abc conjecture. Proofs are getting larger, up to hundreds of pages.

Automated proof assistants, originally developed for program verification, have already made contributions to mathematics. The four colour theorem, whose original proof required trusting a computer program, has been verified independently by George Gonthier using Coq. The Kepler conjecture, concerning the optimal packing of spheres and similarly reliant on a computer proof, has been verified using HOL Light and Isabelle.

Proof assistants are also computer programs, but carefully designed for soundness and with increasing attention towards preserving mathematical intuition. Our aim is not merely to check proofs but to support the process of doing mathematics.

We work with professional research mathematicians to build comprehensive libraries of formalised mathematics, hence the name "Alexandria". We have done a lot in our first year:

- Projective Geometry
- Quaternions and Octonions
- Irrational rapidly convergent series
- Effectively counting real and complex roots of polynomials

While formalising such material, we look for gaps and deficiencies in Isabelle and its libraries. We try to remain faithful to the spirit of the original mathematics. Even machine-formalised proofs should be clear and readable in their own right, to provide further confidence of their correctness. It's also valuable to observe the learning process as mathematicians get to grips with Isabelle.

The Mean Value Theorem: by GH Hardy, then using Isabelle

THEOREM. *If $\phi(x)$ is continuous in the closed interval (a, b), and differentiable in the open interval, then there is a value ξ of x between a and b, such that*

$$\phi(b)-\phi(a)=(b-a)\,\phi'(\xi).$$

It is easy to give a strict proof. Consider the function

$$\phi(b)-\phi(x)-\frac{b-x}{b-a}\{\phi(b)-\phi(a)\},$$

which vanishes when $x=a$ and $x=b$. It follows from Theorem B of § 122 that there is a value ξ for which its derivative vanishes. But this derivative is

$$\frac{\phi(b)-\phi(a)}{b-a}-\phi'(x);$$

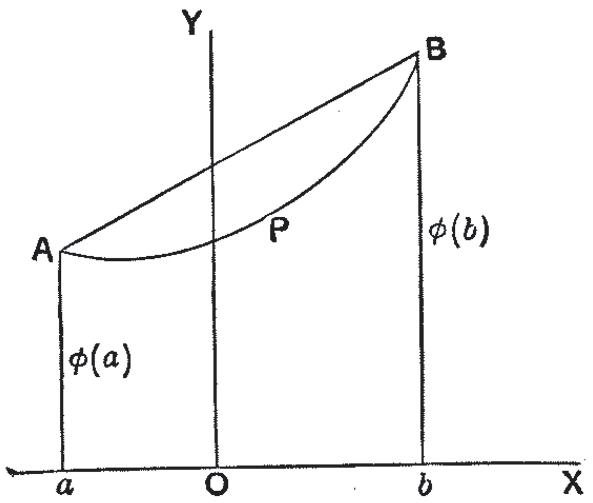

Fig. 41

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theorem mvt:
  fixes f :: "real ⇒ real"
  assumes "a < b"
    and contf: "continuous_on {a..b} f"
    and derf: "⋀x. ⟦a < x; x < b⟧ ⟹ (f has_derivative f' x) (at x)"
  obtains ξ where "a < ξ" "ξ < b" "f b - f a = (f' ξ) (b - a)"
proof -
  have "∃x. a < x ∧ x < b ∧ (λy. f' x y - (f b - f a) / (b - a) * y) = (λv. 0)"
  proof (intro Rolle_deriv[OF ‹a < b›])
    fix x
    assume x: "a < x" "x < b"
    show "((λx. f x - (f b - f a) / (b - a) * x)
              has_derivative (λy. f' x y - (f b - f a) / (b - a) * y)) (at x)"
      by (intro derivative_intros derf[OF x])
  qed (use assms in ‹auto intro!: continuous_intros simp: field_simps›)
  then obtain ξ where
    "a < ξ" "ξ < b" "(λy. f' ξ y - (f b - f a) / (b - a) * y) = (λv. 0)"
    by metis
  then show ?thesis
    by (smt pos_le_divide_eq pos_less_divide_eq that)
qed
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The next phase of research will include using machine learning to guide proof construction, driven by the 2 million lines of text stored in the Archive of Formal Proofs. We'd like to be able to recognise "proof idioms" — common patterns of reasoning that have already been identified in the Archive — so that they can be automatically suggested to the user. Also on the agenda: a variety of advanced decision procedures.

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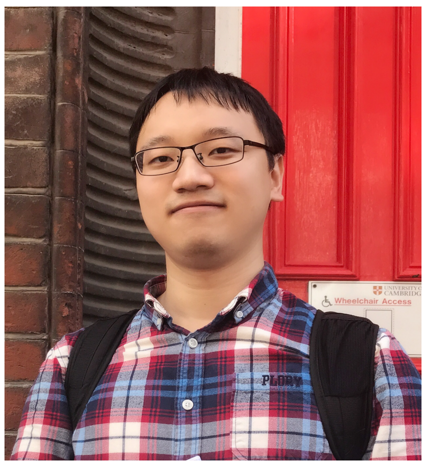

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