Computational Models of Higher Categories
Lecture 4

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Plan for this lecture

Today we will discuss some more phenomena that arise in higher categories, and explore them with homotopy.io:

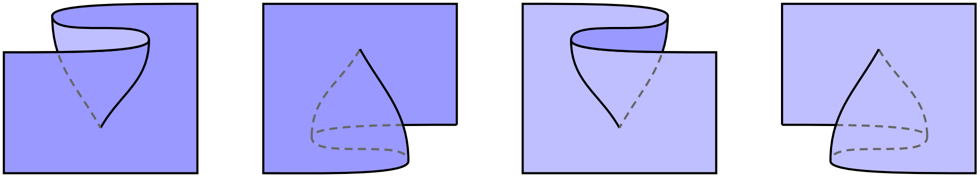
- Higher adjunctions and coherence
- Catastrophes in higher categories
- Knot theory and the Reidemeister moves
- Homotopy types of the spheres
- Combinatorics of homotopy.io

The theory and implementation of homotopy.io is the work of many people, including Nathan Corbyn, Lukas Heidemann, Nick Hu, David Reutter, Chiara Sarti and Calin Tataru.

Adjunction of 1-morphisms in a 3-category

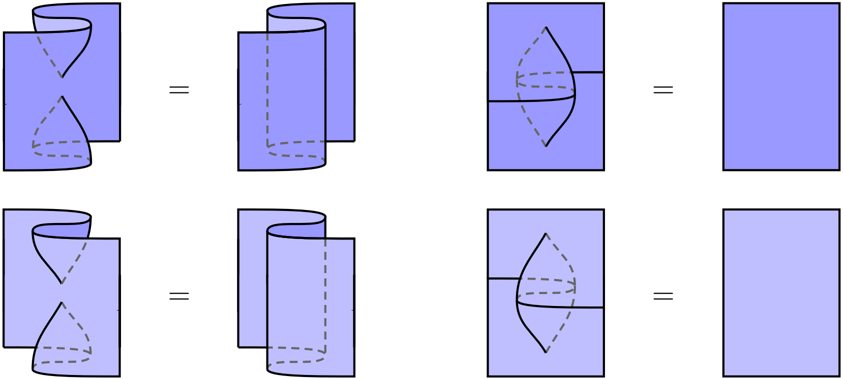
Definition. In a 3-category, a 1-morphism A has a *right dual* B when it can be equipped with 2-morphisms ϵ, η called *folds*

and invertible 3-morphisms called *cusps*:



Adjunction of 1-morphisms in a 3-category

The invertibility equations look like this:



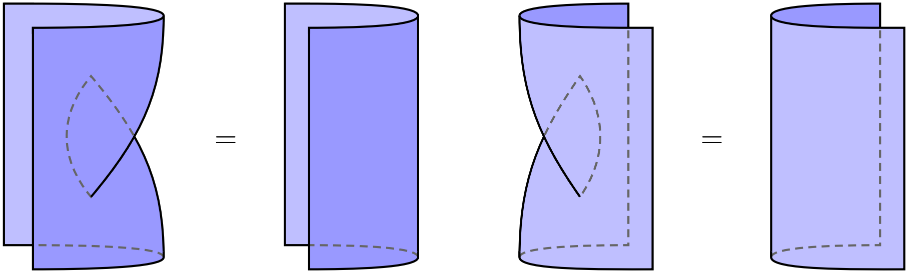
It's just like deforming a piece of fabric! These equations say *ironing* is possible.

This is the same definition that we saw for an adjunction of 1-morphisms in a 2-category, with equations replaced by isomorphisms.

Coherent adjunctions in a 3-category

Dual objects in a monoidal category sometimes satisfy the following equations, which say that they behave in a simple way.

Definition. An adjunction of 1-morphisms in a 3-category is *coherent* when the *swallowtail equations* are satisfied, along with their vertically-flipped variants:



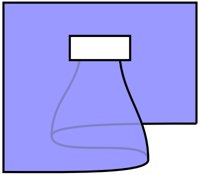
Note the *interchange 3-morphisms* on the left-hand sides of each equation.

Promoting an adjunction in a 3-category

Not every adjunction in a 3-category is coherent. But adjunctions can be *promoted*.

Theorem. In a 3-category, an adjunction of 1-morphisms gives rise to a coherent adjunction.

Proof. We redefine the following cusp 3-morphism as follows:



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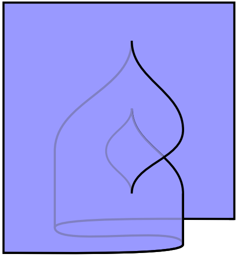
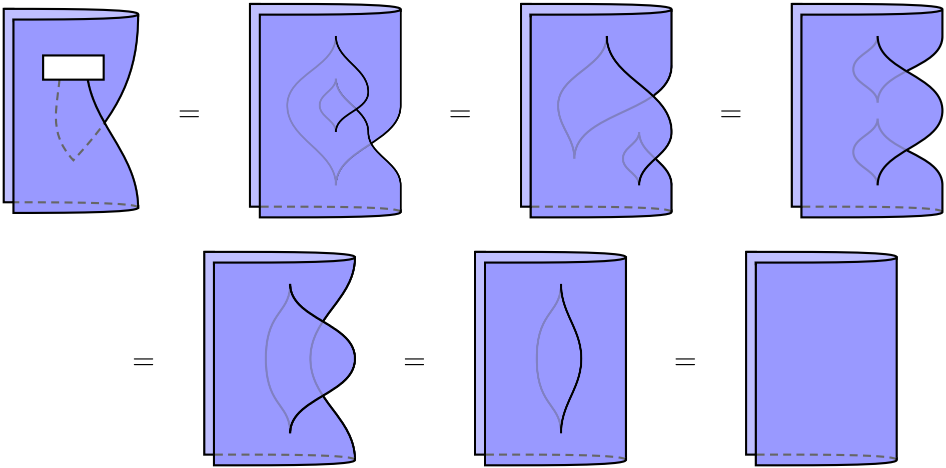


Promoting an adjunction in a 3-category

We can prove one of the swallowtail equations like this:

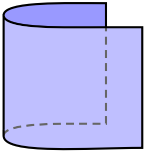


The other swallowtail equations follow similarly.

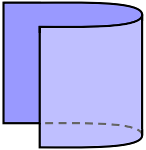
Surface structure

These diagrams look a lot like surfaces, but we don't yet have all surface behaviour.

To capture full manifold behaviour, our fold 2-morphisms must *themselves* have duals:

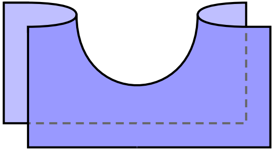


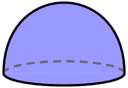
$\dashv$



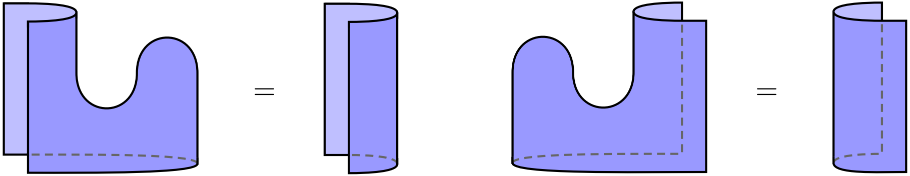
Surface structure

This adjunction has a unit and counit, which we draw like this:





The snake equations for the duality then look like this:

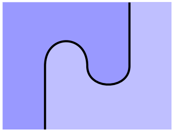


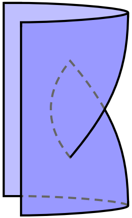
Again, this makes sense in terms of isotopy of surfaces!

Higher catastrophes

In the theory of manifolds, *catastrophes* are critical points in their local structure, where their projection changes abruptly.

The first two examples are the snake in 2d, and the swallowtail in 3d.





A new singularity arises in each dimension, with the *butterfly* in 4d, and the *wigwam* in 5d.

These catastrophes have applications in the theory of dynamical systems, where they describe sudden changes in the configuration space.

We can illustrate the butterfly singularity in homotopy.io.

Manifolds and higher categories

We have seen how duality in higher categories gives rise to manifold-like structures.

This leads to the *cobordism hypothesis*: the n-category of n-dimensional manifolds is equivalent to the free n-category with all symmetric adjoints on the following signature:

$$\Gamma = \{x : \star, p : x \to x\}$$

This is an amazing idea, because higher-dimensional manifolds are wild.

Surfaces are easy to classify (by genus and connectedness), but this is not the case for manifolds of dimension 3 and higher.

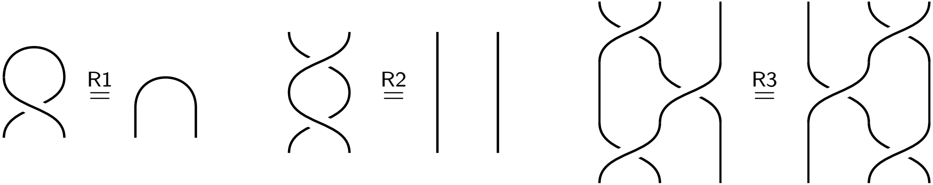
The cobordism hypothesis is originally due to Baez and Dolan, and an outline proof was recently put forward by Lurie.

A good way to explore this in *homotopy.io* is via the free ∞-groupoid on Γ, as long as we take care not to use "too much" invertibility.

The manifolds generated by this process are *normal framed*. To get oriented manifolds we must add additional equations.

Knot theory

The dynamics of knots is governed by the three *Reidemeister moves*:



To investigate this, we can use a 1-degenerate signature with p invertible:

$$\{x : \star,\ p : \mathrm{id}(x) \to \mathrm{id}(x)\}$$

Equations R2 and R3 are normal framed isotopies, so *homotopy.io* knows about them.

Equation R1 is only an *oriented* isotopy, so we have to define p as an *oriented generator*.

Homotopy groups of spheres

The *homotopy type* of the n-sphere is the free ∞-groupoid generated by this context:

$$S^n := \{x : \star,\ a : \mathrm{id}^{n-1}(x) \to \mathrm{id}^{n-1}(x)\}$$

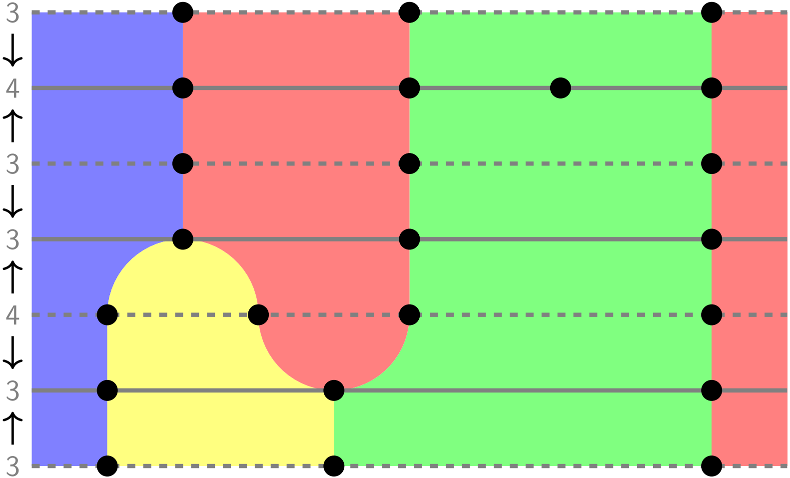
We can obtain the homotopy groups as $\pi_k(S^n) = \mathrm{Hom}(\mathrm{id}^{k-1}(x), \mathrm{id}^{k-1}(x))$ (up to equiv).

- $S^1 = \{x : \star,\ a : x \to x\}$
  - $\pi_1(S^1) = \mathbb{Z}$. Powers of a.
  - $\pi_2(S^1) = 0$. Only the identity $\mathrm{id}^2(x)$.
- $S^2 = \{x : \star,\ a : \mathrm{id}(x) \to \mathrm{id}(x)\}$
  - $\pi_1(S^2) = 0$. Only the identity $\mathrm{id}(x)$.
  - $\pi_2(S^2) = \mathbb{Z}$. Powers of a.
  - $\pi_3(S^2) = \mathbb{Z}$. Powers of the figure-8.
  - $\pi_4(S^2) = \mathbb{Z}/2$. Figure-8 of figure-8.
- $S^3 = \{x : \star,\ a : \mathrm{id}^2(x) \to \mathrm{id}^2(x)\}$
  - $\pi_1(S^3) = 0$. Only the identity $\mathrm{id}(x)$.
  - $\pi_2(S^3) = 0$. Only the identity $\mathrm{id}^2(x)$.
  - $\pi_3(S^3) = \mathbb{Z}$. Powers of a.
  - $\pi_4(S^2) = \mathbb{Z}/2$. Powers of the figure-8.

Combinatorial structure of homotopy.io

Consider the following string diagram in a 2-category.

It gives rise to an alternating sequence of monotone functions.



Zigzags

Definition. For a category $\mathcal{C}$, we define its *zigzag category* $Z(\mathcal{C})$ as follows:

- an object X is a sequence of cospans in $\mathcal{C}$;
- a morphism $f : X \to X'$ is:
  - a monotone function between singular levels,
  - with identities between regular levels,
  - built from morphisms of $\mathcal{C}$,
  - such that all squares commute.

$$
\begin{tikzcd}
X \arrow[d, "f"'] \\
X'
\end{tikzcd}
\qquad
\begin{tikzcd}
& & R_0 \arrow[rr] \arrow[dll, equal] & & S_0 \arrow[dll] & & R_1 \arrow[ll] \arrow[rr] \arrow[dll, equal] \arrow[drr, equal] & & S_1 \arrow[drr] & & R_2 \arrow[ll] \arrow[rr] & & S_2 \arrow[dll] & & R_3 \arrow[ll] \arrow[dll, equal] \arrow[drr, equal] & & \\
R'_0 \arrow[rr] & & S'_0 & & R'_1 \arrow[ll] \arrow[rr] & & S'_1 & & R'_2 \arrow[ll] \arrow[rr] & & S'_2 & & R'_3 \arrow[ll] \arrow[rr] & & S'_3 & & R'_4 \arrow[ll]
\end{tikzcd}
$$

Definition. Write Δ for the category of nonempty finite total orders and monotone functions, Δ_+ when including the empty set, and $\Delta_=$ for the subcat preserving endpoints. Clearly $\Delta_= \hookrightarrow \Delta \hookrightarrow \Delta_+$.

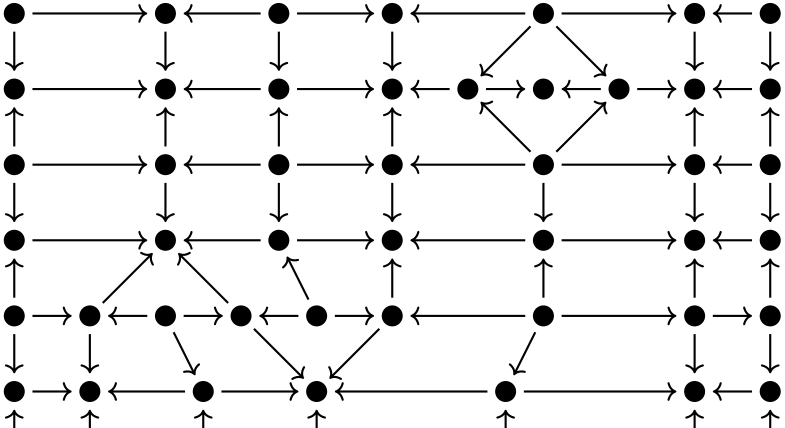
Lemma. There is an equivalence of categories $\Delta_+ \simeq (\Delta_=)^{\mathrm{op}}$.

Thus we obtain $S : Z(\mathcal{C}) \to \Delta_+$ and $R : Z(\mathcal{C})^{\mathrm{op}} \to \Delta_=$.

Diagrams from zigzags

Definition. An *untyped n-diagram* is an object of $Z^n(1) := Z(Z(...(Z(1))))$.

Here are 53 examples of untyped 0-diagrams, all objects of 1: Here are 7 examples of untyped 1-diagrams, all objects of $Z(1) = \Delta_+$: Here is 1 example of an untyped 2-diagram, an object of $Z(\Delta_+)$:



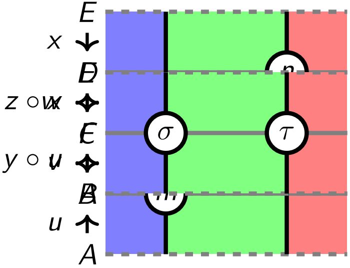
High-level methods

Zigzags are too unwieldy for direct construction by hand.

High-level methods are needed to build nontrivial homotopies.

Consider the problem of constructing the homotopy that *contracts* this composite:

$$\begin{tikzcd}
 & & F & & \\
 & B \arrow[ur, "y"] \arrow[rr, phantom, "\vee"] & & D \arrow[ul, "z"'] & \\
A \arrow[ur, "u"] & & C \arrow[ul, "v"'] \arrow[ur, "w"] & & E \arrow[ul, "x"']
\end{tikzcd}$$



We build the contraction as a pushout of cospans in Z_1^n, and the homotopy itself as an associated zigzag map.

Colimit algorithm

(With David Reutter.)

Theorem. For a category $\mathcal{C}$, the following correctly constructs colimits in $Z(\mathcal{C})$, or correctly fails:

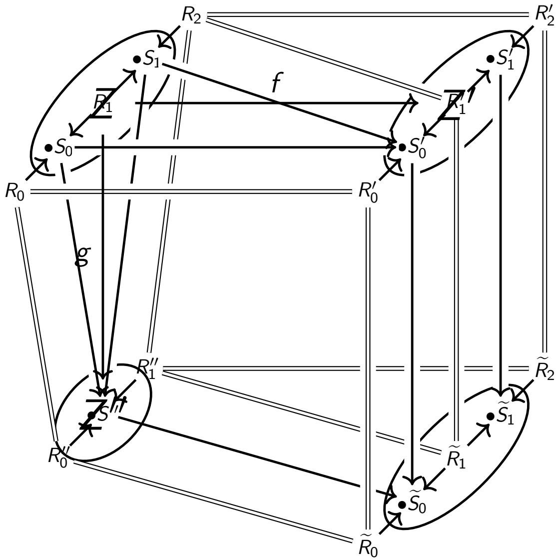
(1) Project to Δ_+.

(2) Take colimit there, or fail.

(3) Label with colimits of underlying $\mathcal{C}$-morphisms, or fail.

(4) Commutativity conditions are automatically satisfied.

(5) Type check the result.



Class 4 – Activities

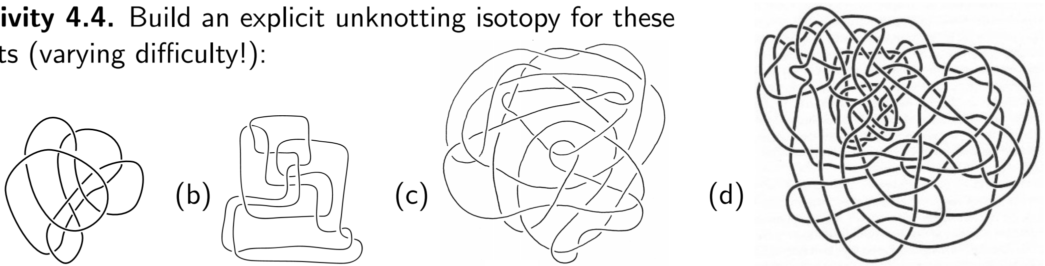
Activity 4.1. Use the periodic table to construct a free symmetric monoidal category as a 3-degenerate 4-category, and show that the braiding is equivalent to its inverse.

Activity 4.2. Formalize a coherent adjunction of 1-morphisms in a 3-category, and verify the 3d renderings of manifold structures.

Activity 4.3. Show that in $\pi_3(S^2)$ there is only one figure-8 generator.

Activity 4.4. Build an explicit unknotting isotopy for these knots (varying difficulty!):

(a) (b) (c) (d)



(a) Kauffman's Unknot, http://homepages.math.uic.edu/~kauffman/IntellUnKnot.pdf, Figure 4
(b) Thistlethwaite's Unknot, https://en.wikipedia.org/wiki/Unknot
(c) Ochai's Second Unknot, https://arxiv.org/abs/1110.2871, https://www.youtube.com/watch?v=HJuPMMJHOlg
(d) Haken's Gordian Knot, https://mickburton.co.uk/2015/06/05/how-do-you-construct-hakens-gordian-knot/