Computational Models of Higher Categories
Lecture 3

Jamie Vicary
University of Cambridge

Midland Graduate School in the Foundations of Computing Science
University of Birmingham
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Paths modulo units

Suppose we have these cells in a 2-category:

$x : \star$ $\qquad$ $y : \star$

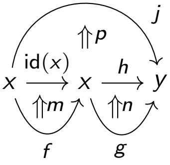
$f : x \to x$ $\qquad$ $h : x \to y$

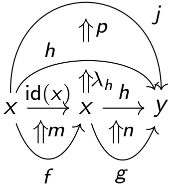
$g : x \to y$ $\qquad$ $j : x \to y$

$m : f \Rightarrow \mathrm{id}(x)$ $\qquad$ $p : h \Rightarrow j$

$n : g \Rightarrow h$

We might try to compose them as follows:

A :=


B :=


Simple, but invalid $\qquad$ Verbose, but correct

But wait—the unit composite seems trivial. We would like the following:

- The proof assistant should accept A as valid. Yes!
- Automatic "deflation" of B to give A, removing trivial structure. Yes!
- Concept of "equality up to units" such that "A = B" is validated. Yes!
- Automatic "inflation" of A to give B, inserting missing coherences. Research.

Semistrictness

The graphical calculus suggests *homotopy* as a basic mechanism with which to manipulate terms and execute computations.

The 'strictest' definitions of higher category do not allow these manipulations, and are known to be insufficiently general.

At the opposite end of the spectrum, the 'weakest' definitions allow not only these manipulations, but far more besides.

A model of higher categories is *semistrict* if it is as strict as possible, while still preserving equivalence to fully weak higher categories. However, yields long proofs!

For proof construction, better to be *semiweak*: as weak as possible, except strictly associative and unital. Yields unique composites.

The system homotopy.io is semiweak. Today we will see how to give Catt the same property.

Plan for this lecture

- Describe **Catt** in a more formal way.
  (Finster & Mimram, arXiv:1706.02866)
- See how models of **Catt** are weak ∞-categories.
- Give a reduction relation on **Catt** terms which "removes unit structure".
  (Finster, Reutter, Rice & Vicary, arXiv:2007.08307)
- Define the theory $\mathbf{Catt_{su}}$, by using reduction to generate definitional equality.
- Models of $\mathbf{Catt_{su}}$ are *strictly unital ∞-categories*, and we explore their properties.
- We can also make associators strict using similar methods, giving a theory $\mathbf{Catt_{sua}}$.
  (Finster, Rice & Vicary, arXiv:2302.05303)
- Investigate nontrivial examples, including Eckmann-Hilton.
- Learn new syntax allowing us to try this out in the proof assistant.

Catt as a type theory

Contexts $\Gamma, \Delta, \ldots$ are lists of variables-with-types:

$$x : A,\ y : B, \ldots, z : C$$

$\Gamma \vdash$ — "*Γ is the generating data for a free weak ∞-category $\widetilde{\Gamma}$*"

Types $A, B, C, \ldots$ are trivial, or pairs of parallel terms:

$$\star \qquad u \to v$$

$\Gamma \vdash A$ — "*in $\widetilde{\Gamma}$, there is a hom-set A*"

Terms $t, u, v, \ldots$ are variables or coherences:

$$x \quad \mathsf{coh}(\Gamma : A)[\sigma]$$

$\Gamma \vdash t : A$ — "*in $\widetilde{\Gamma}$, there is a morphism t in the hom-set A*"

Substitutions $\sigma : \Gamma \to \Delta$ are functions $\sigma : \mathsf{var}(\Gamma) \to \mathsf{tm}(\Delta)$

$\Delta \vdash \sigma : \Gamma$ — "*there is a strict ∞-functor $\sigma : \widetilde{\Gamma} \to \widetilde{\Delta}$*"

No definitional equality except renaming bound variables—"**Catt** does not compute".

Catt term construction – quick reminder

"in a pasting context, parallel full terms can be filled"

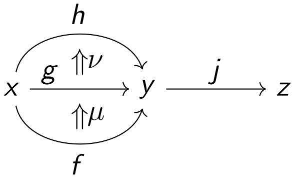
We can construct terms as follows, when Γ is a pasting context:

$$\frac{\partial^-(\Gamma) \vdash u : A \quad \partial^+(\Gamma) \vdash v : A}{\Gamma \vdash \mathsf{coh}(\Gamma, u, v) : u \to v}$$

Side conditions: fullness, dimension, globularity.

$\partial^+(\Gamma)$ $\qquad x \xrightarrow{h} y \xrightarrow{j} z$ $\qquad \delta^+(\Gamma) \vdash u : A$

Γ



$\boxed{\Gamma \vdash \mathsf{coh}(\Gamma, u, v) : u \to_A v}$

$\partial^-(\Gamma)$ $\qquad x \xrightarrow{f} y \xrightarrow{j} z$ $\qquad \delta^-(\Gamma) \vdash u : A$

Catt term construction – quick reminder

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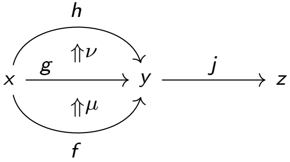
Side conditions: fullness, dimension, globularity.

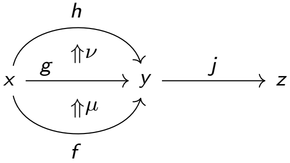
Γ



$\Gamma \vdash v : A$

$$\boxed{\Gamma \vdash \mathsf{coh}(\Gamma, u, v) : u \to_A v}$$

Γ



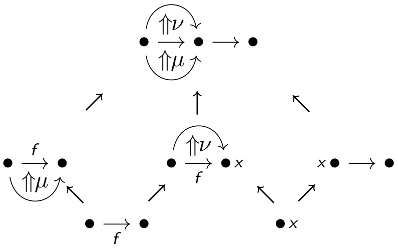
$\Gamma \vdash u : A$

Globular sums

For pasting schemes P, S, a *substitution* $\sigma : P \to S$ is a function $\mathrm{var}(P) \to \mathrm{word}(S)$, preserving dimension and source/target relationships.

Definition. **Catt** has pasting schemes as objects, and substitutions as morphisms.

Theorem. *In* **Catt***, every object is a colimit of locally-maximal disks ("globular sums").*



Definition. An ∞-*category* is a presheaf $\mathbf{Catt}^{\mathrm{op}} \to \mathbf{Set}$ preserving globular sums.

Known to agree with the definition of *contractible* ∞*-category.*

Recent work of Dmitri Ara, Thibaut Benjamin and John Bourke has shown equivalence to previous definitions by Grothendieck, Maltsiniotis, Batanin, Leinster, Brunerie.

This is a lightweight approach:

- no globular extension technology (Grothendieck/Maltsiniotis)
- no globular operad technology (Batanin/Leinster)

Understanding the definition of ∞-category

Definition. **Catt** has pasting schemes as objects, and substitutions as morphisms.

Definition. An ∞-*category* is a presheaf $C : \mathbf{Catt}^{\mathrm{op}} \to \mathbf{Set}$ preserving globular sums.

Let's see why a presheaf C like this gives an ∞-category:

- $C(x)$ gives the objects.
- $C(x(f)y)$ gives the 1-cells.
- $C(x(f)y(g)z)$ gives the pairs of composable 1-cells. (Need globular sum preservation)
- There is a substitution $\sigma : (x(f)y) \to (x(f)y(g)z)$ that acts as follows:
  $$\sigma(x) := x \quad \sigma(f) = f \circ g \quad \sigma(y) = z$$
- $C(\sigma) : C(x(f)y(g)z) \to C(x(f)y)$ is the horizontal composition operation, mapping pairs of composable 1-cells into their actual composite in the ∞-category.
- Similarly, $C(x(f(m)g)y)$ gives the set of 2-cells.
- There is a substitution $\tau : (x(f(m)g)y) \to (x(f)y(g)z(h)w)$ that acts as $\tau(m) = \alpha_{f,g,h}$.
- $C(\tau) : C(x(f)y(g)z(h)w) \to C(x(f(m)h)y)$ sends a triple of composable 1-cells to their actual associator in the ∞-category.

Strictness via reduction

Our goal is to modify this theory to achieve *strict units* and *strict associators*.

We will do this with a *reduction relation* on terms of the theory, which simplifies the syntax.

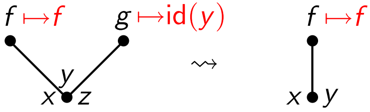
Let's start with units. In the context $(x(f)y)$, we would like the following reduction:

$$f \circ \mathrm{id}(y) \rightsquigarrow f$$

We can write that out with its full syntax:

$$(\mathrm{coh}(x(f)y(g)z, x, z))[f, \mathrm{id}(y)] \rightsquigarrow f$$

Let's draw the tree representation of the head context, with arguments in place:



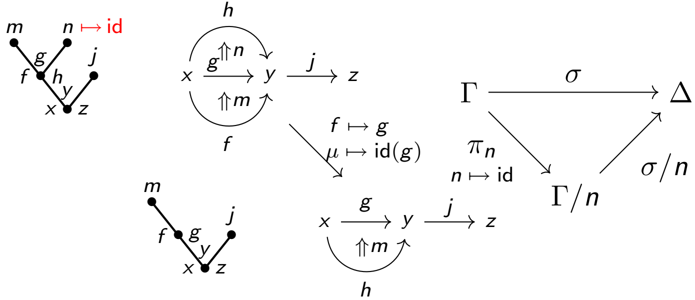
This gives us an idea: since g is sent to an identity, *remove the entire* g *branch*.

P Reduction

"prune identity arguments"

Suppose $n \in \mathsf{var}(\Gamma)$ is locally maximal, with $n[\sigma]$ an identity.

Then σ factorizes via Γ/n, with $n[\pi_n] = \mathsf{id}$:



The intuition is that the leaf at n has been collapsed, or "pruned".

We define the reduction as follows:

$$\mathsf{coh}(\Gamma, u, v)[\sigma] \rightsquigarrow_{\mathsf{P}} \mathsf{coh}(\Gamma/n, u[\pi_n], v[\pi_n])[\sigma/n]$$

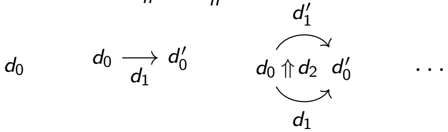
D Reduction

"simplify unary composites"

We define the *n-sphere type* S^n and the *n-disk context* D^n recursively:

$$\mathsf{D}^0 := \{d_0 : S^{-1}\}$$
$$\mathsf{D}^{n+1} := \{\mathsf{D}^n, d'_n : S^{n-1}, d_{n+1} : S^n\}$$

$$S^{-1} := \star$$
$$S^n := d_n \to d'_n$$



Then for any n-cell u with $n > 0$, we can build its *unary composite:*

$$\mathsf{coh}(D^n, d_{n-1}, d'_{n-1})[u] \rightsquigarrow_{\mathsf{D}} u$$

This reduces to u itself.

L Reduction

"eliminate loops"

Consider a term as follows:

$$\mathsf{coh}(\Gamma, u, u)[\sigma] : u[\sigma] \to u[\sigma]$$

This "coherence law" says "$u[\sigma] = u[\sigma]$".

But this is obvious, and already has a canonical witness:

$$\mathsf{id}(u[\sigma]) : u[\sigma] \to u[\sigma]$$

So it seems reasonable to eliminate these terms:

$$\mathsf{coh}(\Gamma, u, u)[\sigma] \rightsquigarrow_{\mathsf{L}} \mathsf{id}(u[\sigma])$$

Examples

$$\mathsf{coh}(\Gamma, u, v)[\sigma] \rightsquigarrow_{\mathsf{P}} \mathsf{coh}(\Gamma/\mu, u[\pi_\mu], v[\pi_\mu])[\sigma/\mu]$$
$$\mathsf{coh}(D^n, d_{n-1}, d'_{n-1})[u] \rightsquigarrow_{\mathsf{D}} u$$
$$\mathsf{coh}(\Gamma, u, u) \rightsquigarrow_{\mathsf{L}} \mathsf{id}(u[\sigma])$$

To get normalizing reductions, we extend $\rightsquigarrow_{\mathsf{P}}$, $\rightsquigarrow_{\mathsf{D}}$ and $\rightsquigarrow_{\mathsf{L}}$ to subterms, and add a single additional rule: never reduce the head of an identity.

- *Identity composite.*

$$\begin{aligned} f \circ \mathsf{id}(y) &\equiv \mathsf{coh}(x(f)y(g)z, x, z)[f, \mathsf{id}(y)] \\ &\rightsquigarrow_{\mathsf{P}} \mathsf{coh}(x(f)y, x, y)[f] \\ &\rightsquigarrow_{\mathsf{D}} f \quad \checkmark \end{aligned}$$

- *Left unitor.*

$$\begin{aligned} &\mathsf{coh}(x(f)y, \mathsf{id}(x) \circ f, f) \\ &\rightsquigarrow_{\mathsf{P}} \mathsf{coh}(x(f)y, (f), f) \\ &\rightsquigarrow_{\mathsf{D}} \mathsf{coh}(x(f)y, f, f) \equiv \mathsf{id}(f) \quad \checkmark \end{aligned}$$

- *Associator with identity.*

$$\begin{aligned} \alpha_{f,\mathsf{id}(y),g} &\equiv \mathsf{coh}(x(f)y(g)z(h)w, (f \circ g) \circ h, f \circ (g \circ h))[f, \mathsf{id}(y), g] \\ &\rightsquigarrow_{\mathsf{P}} \mathsf{coh}(x(f)y(g)z, (f \circ \mathsf{id}(y)) \circ g, f \circ (\mathsf{id}(y) \circ g))[f, g] \\ &\rightsquigarrow_{\mathsf{P}}\rightsquigarrow_{\mathsf{P}} \mathsf{coh}(x(f)y(g)z, (f) \circ g, f \circ (g))[f, g] \\ &\rightsquigarrow_{\mathsf{D}}\rightsquigarrow_{\mathsf{D}} \mathsf{coh}(x(f)y(g)z, f \circ g, f \circ g)[f, g] \\ &\rightsquigarrow_{\mathsf{L}} \mathsf{id}(\mathsf{coh}(x(f)y(g)z, x, z)[f, g]) \equiv \mathsf{id}(f \circ g) \quad \checkmark \end{aligned}$$

Coherence towers

In weak higher categories, low-dimensional coherences generate higher-dimensional coherences, a process that continues in all dimensions to produce *coherence towers*:

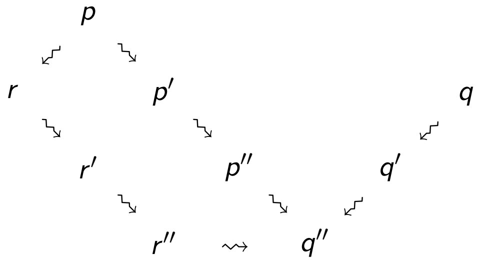
	$\vdots$
$\vdots$	$\pi_{f,g,h,j} = \mathrm{id}(\mathrm{id}(f \circ g \circ h \circ j))$
$\lambda_f = \mathrm{id}(f) = \rho_f$	$\alpha_{f,g,h} = \mathrm{id}(f \circ g \circ h)$
$\mathrm{id} \circ f = f = f \circ \mathrm{id}$	$f \circ (g \circ h) = (f \circ g) \circ h$
$\mathbf{Catt}_{\mathbf{su}}$	$\mathbf{Catt}_{\mathbf{sua}}$

The strictly unital/associative theories will collapse the *entire towers*, not just the base.

Equality

Theorem. Reduction is terminating and has unique normal forms.

Definition. The theory $\mathbf{Catt_{su}}$ is obtained by extending **Catt** with definitional equality "$=$", defining $p = q$ just when p, q have the same normal form under the reductions P, D, L:

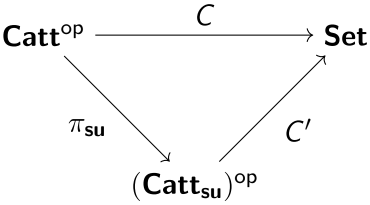


The definition of semistrict ∞-category

There is an obvious functor $\pi_{\mathbf{su}} : \mathbf{Catt} \to \mathbf{Catt}_{\mathbf{su}}$ sending terms to their equivalence class.

This functor is full and essentially surjective, but not faithful.

Definition. A *strictly unital ∞-category* is an ∞-category $\mathbf{Catt}^{\mathrm{op}} \to \mathbf{Set}$, which factors through $\pi_{\mathbf{su}}$:



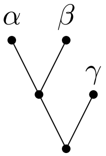
Such a factorization must be unique if it exists.

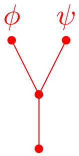
For an ∞-category, being strictly unital is therefore a *property*.

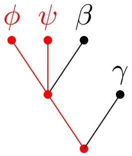
Insertion

Pruning, disk removal and loop removal give us strict units.

To get strict associators, we need an extra process called *insertion*.

$T :=$ 

, $\mathrm{arg}(\alpha) :=$ 

$\rightsquigarrow_I$ $T' :=$ 

This allows us to "insert" argument contexts into head contexts, giving strict associators.

We define the theory $\mathbf{Catt}_{\mathbf{sua}}$ by defining equality using I, D, L reductions.

Strict units and associators

Definition. A *strictly unital and associative ∞-category* is an ∞-category $\mathbf{Catt}^{\mathsf{op}} \to \mathbf{Set}$, which factors through $\pi_{\mathbf{sua}}$:

$$
\begin{array}{ccc}
\mathbf{Catt}^{\mathsf{op}} & \xrightarrow{\quad C \quad} & \mathbf{Set} \\
& {\scriptstyle \pi_{\mathbf{sua}}} \searrow \qquad \nearrow {\scriptstyle C'} & \\
& (\mathbf{Catt}_{\mathbf{sua}})^{\mathsf{op}} &
\end{array}
$$

The following historical conjecture can now be precisely stated.

Conjecture. Every ∞-category is equivalent to a strictly unital and associative ∞-category.

General contexts

So far we have focused on contexts given by pasting schemes.

There is a more general notion of context, given by a list of variables, whose source and targets are words over the earlier variables. (Side conditions: globularity, dimension.)

Here are some examples:

- $x : \star,\ y : \star, z : \star, f : x \to y, g : x \to z$
- $x : \star,\ y : \star, z : \star, f : x \to y, g : y \to z, h : x \to z, m : f \circ g \to h.$
- $x : \star, f : x \to y, y : \star$ — Not a context
- $x : \star, a : \mathrm{id}(x) \to \mathrm{id}(x), b : \mathrm{id}(x) \to \mathrm{id}(x)$ — 1-degenerate
- $x : \star, a : \mathrm{id}(\mathrm{id}(x)) \to \mathrm{id}(\mathrm{id}(x)), b : \mathrm{id}(\mathrm{id}(x)) \to \mathrm{id}(\mathrm{id}(x))$ — 2-degenerate

Contexts give us a *universal perspective* on ∞-categories.

Theorem. Every ∞-category C is equivalent to a free ∞-category on some context $\widetilde{C}$.

$\widetilde{C}$ is the *cofibrant replacement* of C. Weak ∞-functors $C \to D$ are strict ∞-functors $\widetilde{C} \to D$.

Eckmann-Hilton

Let's consider again the 1-degenerate context:

$$x : \star, \;\; a : \mathrm{id}(x) \to \mathrm{id}(x), \;\; b : \mathrm{id}(x) \to \mathrm{id}(x)$$

Then it's easy to see from the graphical calculus that a 3-cell equivalence must exist:

$$\mathrm{EH}_{a,b} : a \bullet b \to b \bullet a$$

We can illustrate this as follows:

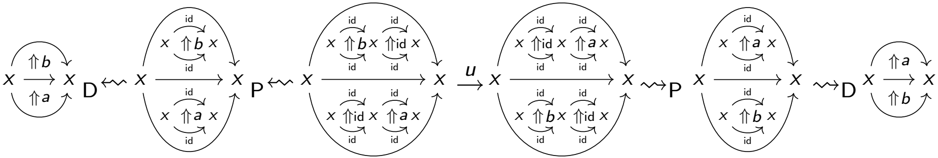
Eckmann-Hilton

$\Gamma := \{x : \star,\ a : \mathsf{id}(x) \to \mathsf{id}(x),\ b : \mathsf{id}(x) \to \mathsf{id}(x)\}$

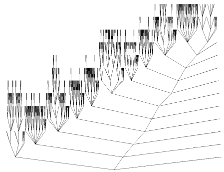
In context Γ, the *Eckmann-Hilton 3-cell* has the following type:

$$\mathsf{EH}_{a,b} : a \bullet b \to b \bullet a$$

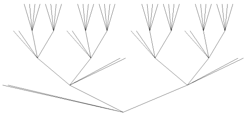
In $\mathbf{Catt_{su}}$ we can construct it as an interchanger u:



We can also formalize it in $\mathbf{Catt}$. We can visualize the syntax trees of the two proofs:



$=$



Catt (1224 vertices) **Catt$_{\mathbf{su}}$** (60 vertices)

$\Delta := \{x : \star,\ a : \mathrm{id}(\mathrm{id}(x)) \to \mathrm{id}(\mathrm{id}(x)),$
$b : \mathrm{id}(\mathrm{id}(x)) \to \mathrm{id}(\mathrm{id}(x))\}$

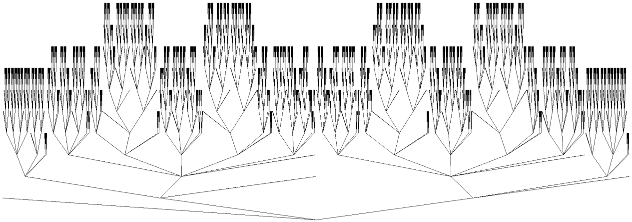
Syllepsis

In Δ, the *Syllepsis 5-cell* has the following type:

$$\mathsf{SY}_{s,t} : \mathsf{EH}_{a,b} \bullet_3 \mathsf{EH}^{-1}_{b,a} \to \mathrm{id}(a \bullet b)$$

Geometrically, it says "the double braid is isotopic to the identity".

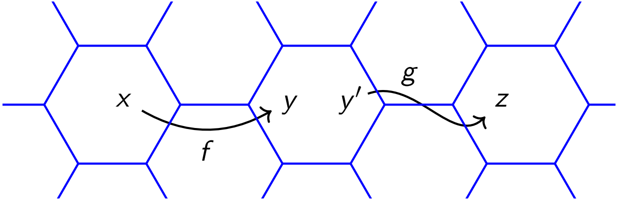
We can construct it in $\mathbf{Catt_{sua}}$. Its syntax tree has 2,713 vertices:



Cannot yet construct $\mathsf{SY}_{s,t}$ in **Catt**. Estimated **Catt** proof size $\sim$ 100,000 vertices.

Research frontier

Path types are *not* contractible . . .



. . . but they *can* be carved into contractible pieces.

Can we gain this advantage for Martin-Löf identity types?

Could this go some way to alleviate the burden of proof-relevance?

Could this approach to semistrictness apply beyond path types?

Catt additional syntax

We saw this syntax already:

- Coherence construction: `coh name (pasting) : source => target`
- Coherence application: `...  (name arg1 arg2 ...)  ...`
- Comments: `# what a wonderful day`

Here are some new techniques:

- Strict unit mode: `./catt.exe --su myfile.txt`
- Strict unit and associator mode: `./catt.exe --sua myfile.txt`
- Equality checking in pasting context: `assert (pasting) | u = v`
- Definition of arbitrary context: `{x :: *} {y :: *} (a :: src => tgt) (b :: src => tgt) ...`
- Type assertion in any context: `let name (context) : [ source => target ] = word`

Class 3 – Activities

Activity 3.1. This activity investigates the associator $\alpha_{f,g,h}$. (Use `assert`)

(a) In $\textbf{Catt}_{\textbf{su}}$, show the associator equals the identity whenever one argument is an identity.

(b) In $\textbf{Catt}_{\textbf{sua}}$, show that the associator always equals the identity.

Activity 3.2. This activity builds on Activity 1.3. (Use `assert`)

(a) In $\textbf{Catt}_{\textbf{sua}}$, show that the triangle, pentagon, unit and associahedron coherences are equal to the identity.

(b) In $\textbf{Catt}_{\textbf{sua}}$, show that the interchanger coherence is *not* equal to an identity.

Activity 3.3. In $\textbf{Catt}_{\textbf{sua}}$, formalize the Eckmann-Hilton isotopy from slide **??**, and show that it has the type $s \bullet t \to t \bullet s$. (Use `let`)

Activity 3.4. In $\textbf{Catt}_{\textbf{sua}}$, formalize the Third Reidemeister Move from Activity 2.2(d). (Use `let`)