Computational Models of Higher Categories
Lecture 2

Jamie Vicary
University of Cambridge

Midland Graduate School in the Foundations of Computing Science
University of Birmingham
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Interchange law

In Class 1, we will build the interchange law as a 3-cell. In a 2-category, it is an *equation*.

Lemma. In a 2-category, all suitably composable 2-morphisms m, n, p, q satisfy the interchange law:

$$(m \circ n) \bullet (p \circ q) = (m \bullet p) \circ (n \bullet q)$$

Proof. This holds due to properties of the category $\mathbf{C}(A, B) \times \mathbf{C}(B, C)$, and from the fact that $- \circ -\ : \mathbf{C}(A, B) \times \mathbf{C}(B, C) \to \mathbf{C}(A, C)$ is a functor:

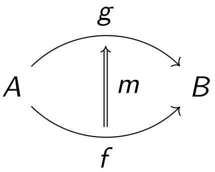
$$\begin{aligned}
(m \bullet n) \circ (p \bullet q) &\equiv \circ(m \bullet n, p \bullet q) \\
&= \circ\big((m, p) \bullet (n, q)\big) && \text{(composition in } \mathbf{C} \times \mathbf{C}\text{)} \\
&= \big(\circ(m, p)\big) \bullet \big(\circ(n, q)\big) && \text{(functoriality of } \circ\text{)} \\
&= (m \circ p) \bullet (n \circ q)
\end{aligned}$$

Remember functoriality: $F(g \circ f) = F(g) \circ F(f)$. This is a good consistency check.

The form of this equation shows the difficulty we may have working with higher categories.

Graphical calculus for 2-categories

In the disk/pasting representation, objects are represented by points, 1-cells by horizontal lines, and 2-cells by regions:



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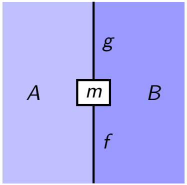
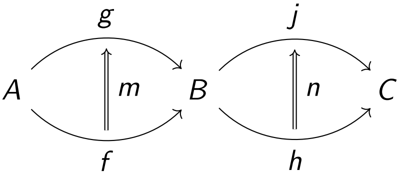


The *graphical calculus* is an alternative notation that *dualizes* this representation.

In this calculus, objects are represented by regions, 1-morphisms by vertical lines, and 2-morphisms by vertices.

Graphical calculus for 2-categories

Horizontal and vertical composition is represented like this:



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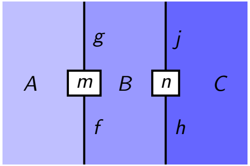
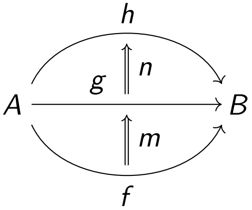


$= \quad m \circ n$



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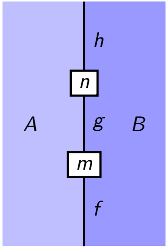


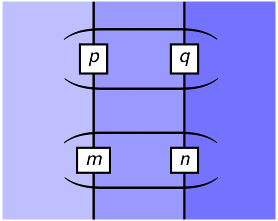
$= \quad m \bullet n$

The units, associator, and left and right unitors are not depicted.
Coherence is essential for this to make sense.

The interchange law, revisited

Let's look again at the interchange law:

$$(m \circ n) \bullet (p \circ q) \qquad = \qquad (m \bullet p) \circ (n \bullet q)$$



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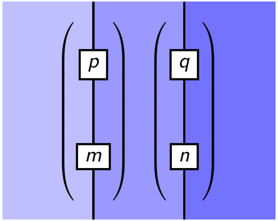


In the graphical calculus, we have a grid of cells in the plane.

The brackets aren't part of the notation. Dropping them, the equation becomes trivial!

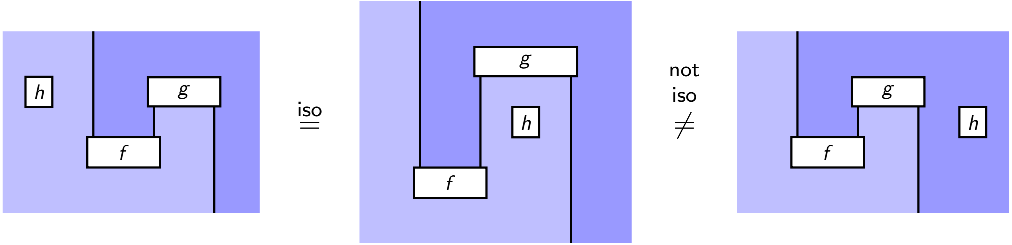
The apparent complexity of the theory of monoidal categories — α, λ, ρ, coherence, interchange — was in fact complexity of the *geometry of the plane*. When we use a geometrical notation, the complexity vanishes.

Planar isotopy

Two diagrams are *planar isotopic* when one can be deformed into the other, such that:

- diagrams remain confined to a rectangular region of the plane;
- input and output wires terminate at the lower and upper boundaries of the rectangle;
- components of the diagram never intersect.

Here are examples of isotopic and non-isotopic diagrams:



We will allow heights of the diagrams to change, and allow input and output wires to slide horizontally along the boundary, although they must never change order.

Correctness of the graphical calculus

We can now state the correctness theorem.

Theorem. A well-formed equation between morphisms in a 2-category follows from the axioms if and only if it holds in the graphical language up to planar isotopy.

Let f and g be composite 2-morphisms such that the equation $f = g$ is well-formed, and consider the following statements:

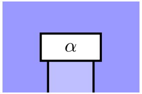
- $P(f, g) =$ 'under the axioms of a 2-category, $f = g$'
- $Q(f, g) =$ 'graphically, f and g are planar isotopic'

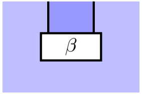
Soundness is the assertion that for all such f and g, $P(f, g) \Rightarrow Q(f, g)$. It is easy to prove: just check each axiom.

Completeness is the reverse assertion, that for all such f and g, $Q(f, g) \Rightarrow P(f, g)$. It is hard to prove; one must show that planar isotopy is generated by a finite set of moves, each being implied by the 2-category axioms.

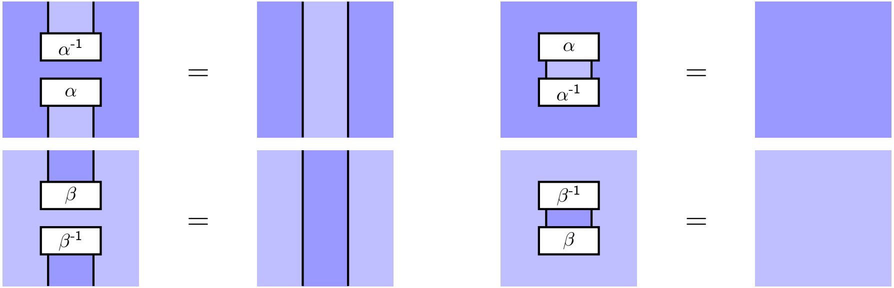
Equivalence

Definition. In a 2-category, an *equivalence* is a pair of 1-morphisms $f : x \to y$ and $g : y \to x$, and invertible 2-morphisms $\alpha : f \circ g \Rightarrow \mathrm{id}(x)$ and $\beta : g \circ f \Rightarrow \mathrm{id}(y)$:





Invertibility has the following graphical form:

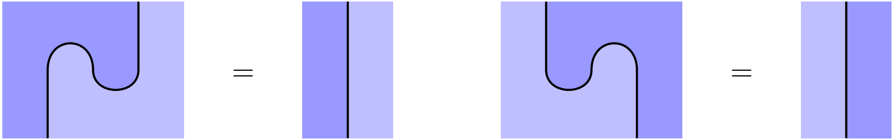


Lemma. An equivalence in **Cat** is exactly an ordinary equivalence of categories.

Duality

Definition. In a 2-category, a 1-morphism $f : x \to y$ has a *right dual* $g : y \to x$ when there are 2-morphisms $\alpha : f \circ g \Rightarrow \mathrm{id}(x)$ and $\beta : g \circ f \Rightarrow \mathrm{id}(B)$

satisfying the snake equations:



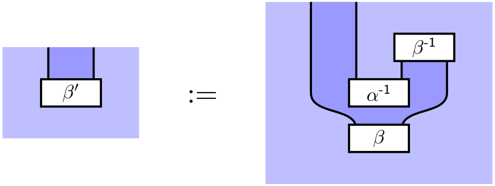
Lemma. In **Cat**, a duality $f \dashv g$ is exactly an adjunction of functors.

Promoting an equivalence

We now prove an interesting theorem relating equivalences and duals.

Theorem. In a 2-category, every equivalence gives rise to a dual equivalence.

Proof. Suppose we have an equivalence in a 2-category, witnessed by invertible 2-morphisms α and β. Then we can build a new equivalence witnessed by α and β', with β' defined like this:



Since β' is composed from invertible 2-morphisms it must itself be invertible.

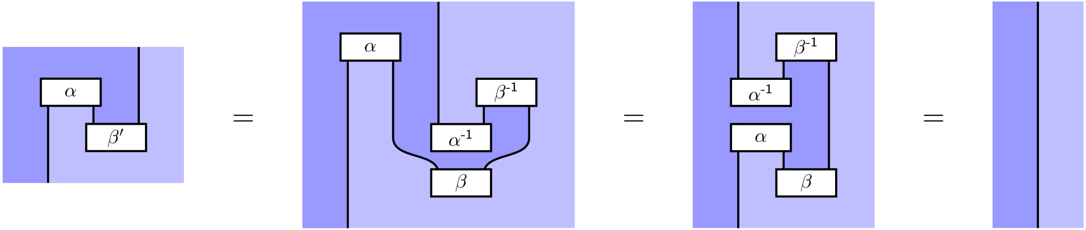
Also, notice that β and β' the same type.

So α and β' together still give the data of an equivalence.

Promoting an equivalence

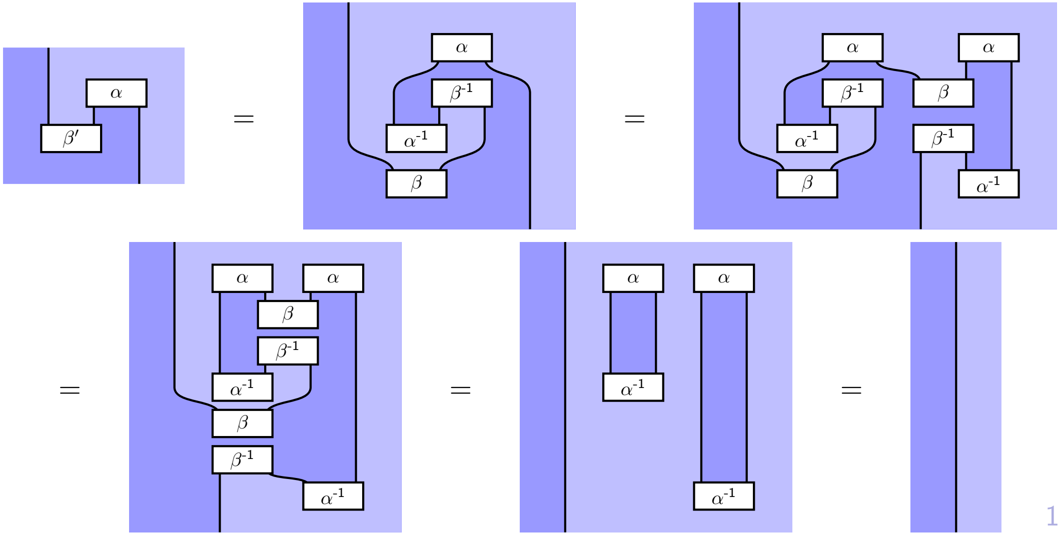
We now demonstrate that the snake equations are satisfied by α, β'.

We prove the first snake equation as follows:



Promoting an equivalence

The second snake equation is demonstrated as follows:



Defining 3-categories

The algebraic definition of 3-category (or *tricategory*) is difficult.

However, the graphical calculus allows us to understand 3-categories quite easily.

The graphical calculus for 2-categories is 2-dimensional:

- objects correspond to planes;
- 1-morphisms correspond to wires;
- 2-morphisms correspond to vertices.

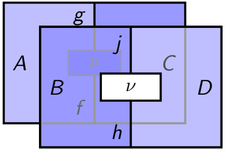
For 3-categories, we extend this as follows:

- objects correspond to volumes
- 1-morphisms correspond to surfaces
- 2-morphisms correspond to wires
- 3-morphisms correspond to vertices

We still need to take care with our notion of isotopy, but the theory of these diagrams is now quite well understood.

Defining 3-categories

We can compose structures horizontally, vertically or "depthwise":



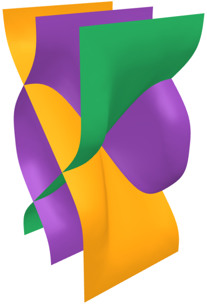
Components can move freely in their separate layers, giving the *interchanger* 3-morphism:

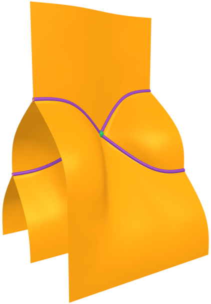
Unit 1-morphisms correspond to "empty surfaces".

The proof assistant homotopy.io

Now we're ready to take a look at the proof assistant. You can follow along on your laptop.

https://beta.homotopy.io





(f, g, h example)

Degenerate higher categories

An n-category is *k-degenerate* if it has one object, one 1-cell, ..., and one $(k-1)$-cell.

A k-degenerate n-category behaves more like an $(n-k)$-category, as k dims are trivial.

However, because it is sitting above the degenerate k dimensions, this $(n-k)$-category will be endowed with extra structure. Let's see what happens with 1-categories:

- A 1-degenerate 1-category is a monoid.
- A 2-degenerate 2-category is a commutative monoid.

After this it stabilises: an n-degenerate n-category is a commutative monoid for $n \geq 2$.

Now let's try the same thing with 2-categories:

- A 1-degenerate 2-category is a *monoidal category*.
- A 2-degenerate 3-category is a *braided monoidal category*.
- A 3-degenerate 4-category is a *symmetric monoidal category*.

Here it again stabilises: an n-degenerate $(n+1)$-category is symmetric monoidal for $n \geq 3$.

The general pattern is now clear. This is called the *periodic table of higher categories*.

Definitions and theorems

Working in a free ∞-category, the concepts of *definition* and *theorem* merge.

Theorems. Let p, q be k-cells, and suppose we conjecture there exists a $(k+1)$-cell $p \to q$.

A proof would be a composite $(k+1)$-cell $c = p \to p' \to \cdots \to q$.

This data can be conveniently encoded as follows:

- A new $(k+1)$-dimensional generator $\mathit{thm} : p \to q$. (This is the theorem statement.)
- A new invertible $(k+2)$-dimensional generator $\mathit{pf} : \mathit{thm} \to c$. (This is the proof.)

Definitions. Consider a complex composite $(k+1)$-cell $c : p \to q$. If we use it frequently, we may want to define a new generator *token* as a shorthand. We achieve that as follows:

- A new $(k+1)$-dimensional generator $\mathit{token} : p \to q$.
- A new invertible $(k+2)$-dimensional generator $\mathit{def} : \mathit{token} \to c$.

These situations are identical!

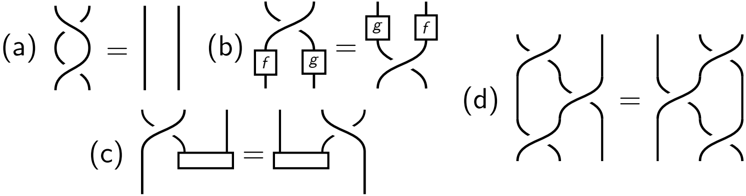
The generators *pf*/*def* let us inline or abstract *thm*/*token* locally within a large proof.

Class 2 – Activities

Use the proof assistant homotopy.io to complete these activities.

Activity 2.1. Build the example planar isotopy from slide 6.

Activity 2.2. Prove these braided monoidal category identities:



Activity 2.3. In a 2-category, a *monad* is a 1-cell $f : x \to x$, along with 2-cells $m : f \circ f \Rightarrow f$ and $u : \mathrm{id}(x) \Rightarrow f$, satisfying associativity and unitality equations.

(a) Use homotopy.io to encode the definition of a monad. View the data in 3d.

(b) Add 4-cells representing the pentagon and triangle laws. View the data in 3d and 4d.

(c) (Hard.) Add a 5-cell representing the associahedron law for the pentagon.
See page 10 of this article: `https://arxiv.org/abs/1301.1053`

Activity 2.4. Formalize the equivalence promotion theorem from slide 10.

Activity 2.5. Prove that for an adjunction $f \dashv g$, we have $g \simeq g'$ if and only if $f \dashv g'$.