Formalization of Mathematics for Fun and Profit

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Summary of talk

- From *Principia* to the computer age
- Formalization in current mathematics
- Recent achievements in formalization
- Reliability of machine-checked proof
- Other uses of formal proofs

From *Principia* to the computer age

100 years since Principia Mathematica

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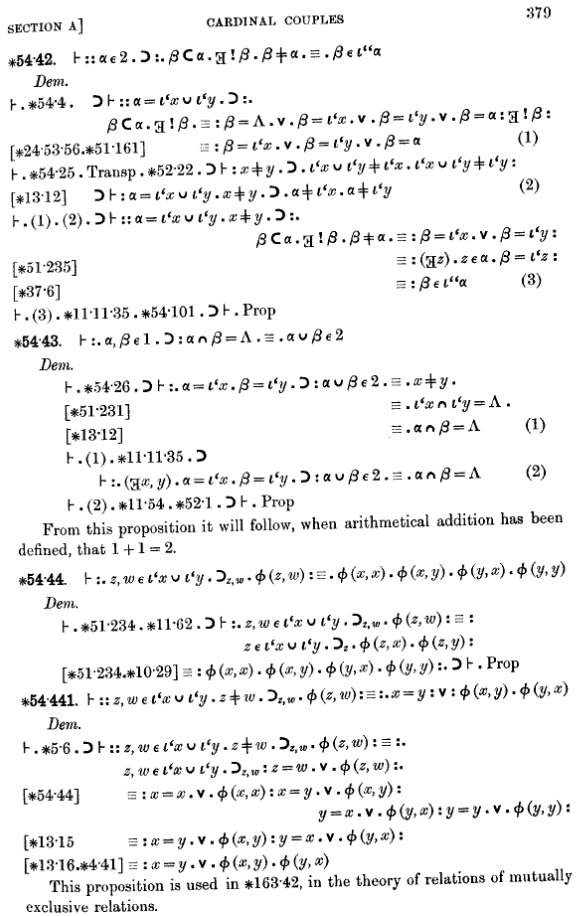
- Subsequently, the idea of actually formalizing proofs has not been taken very seriously, and few mathematicians do it today.

But thanks to the rise of the computer, the actual formalization of mathematics is attracting more interest.

Formal proofs are difficult by hand

*"my intellect never quite recovered from the strain of writing [*Principia Mathematica*]. I have been ever since definitely less capable of dealing with difficult abstractions than I was before." (Russell,* Autobiography*)*

A formal proof from 1910



SECTION A] CARDINAL COUPLES 379

$*54{\cdot}42.\quad \vdash :: \alpha \in 2 . \supset :. \beta \subset \alpha . \exists ! \beta . \beta \neq \alpha . \equiv . \beta \in \iota``\alpha$

Dem.

$\vdash . *54{\cdot}4 . \quad \supset \vdash :: \alpha = \iota`x \cup \iota`y . \supset :.$
$\beta \subset \alpha . \exists ! \beta . \equiv : \beta = \Lambda . \vee . \beta = \iota`x . \vee . \beta = \iota`y . \vee . \beta = \alpha : \exists ! \beta :$
$[*24{\cdot}53{\cdot}56.*51{\cdot}161] \quad \equiv : \beta = \iota`x . \vee . \beta = \iota`y . \vee . \beta = \alpha \qquad (1)$
$\vdash . *54{\cdot}25 . \text{Transp} . *52{\cdot}22 . \supset \vdash : x \neq y . \supset . \iota`x \cup \iota`y \neq \iota`x . \iota`x \cup \iota`y \neq \iota`y :$
$[*13{\cdot}12] \quad \supset \vdash : \alpha = \iota`x \cup \iota`y . x \neq y . \supset . \alpha \neq \iota`x . \alpha \neq \iota`y \qquad (2)$
$\vdash . (1) . (2) . \supset \vdash :: \alpha = \iota`x \cup \iota`y . x \neq y . \supset :.$
$\beta \subset \alpha . \exists ! \beta . \beta \neq \alpha . \equiv : \beta = \iota`x . \vee . \beta = \iota`y :$
$[*51{\cdot}235] \quad \equiv : (\exists z) . z \in \alpha . \beta = \iota`z :$
$[*37{\cdot}6] \quad \equiv : \beta \in \iota``\alpha \qquad (3)$
$\vdash . (3) . *11{\cdot}11{\cdot}35 . *54{\cdot}101 . \supset \vdash . \text{Prop}$

$*54{\cdot}43.\quad \vdash :. \alpha, \beta \in 1 . \supset : \alpha \cap \beta = \Lambda . \equiv . \alpha \cup \beta \in 2$

Dem.

$\vdash . *54{\cdot}26 . \supset \vdash :. \alpha = \iota`x . \beta = \iota`y . \supset : \alpha \cup \beta \in 2 . \equiv . x \neq y .$
$[*51{\cdot}231] \quad \equiv . \iota`x \cap \iota`y = \Lambda .$
$[*13{\cdot}12] \quad \equiv . \alpha \cap \beta = \Lambda \qquad (1)$
$\vdash . (1) . *11{\cdot}11{\cdot}35 . \supset$
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$\vdash . (2) . *11{\cdot}54 . *52{\cdot}1 . \supset \vdash . \text{Prop}$

From this proposition it will follow, when arithmetical addition has been defined, that $1 + 1 = 2$.

$*54{\cdot}44.\quad \vdash :. z, w \in \iota`x \cup \iota`y . \supset_{z,w} . \phi(z,w) : \equiv . \phi(x,x) . \phi(x,y) . \phi(y,x) . \phi(y,y)$

Dem.

$\vdash . *51{\cdot}234 . *11{\cdot}62 . \supset \vdash :. z, w \in \iota`x \cup \iota`y . \supset_{z,w} . \phi(z,w) : \equiv :$
$z \in \iota`x \cup \iota`y . \supset_z . \phi(z,x) . \phi(z,y) :$
$[*51{\cdot}234.*10{\cdot}29] \equiv : \phi(x,x) . \phi(x,y) . \phi(y,x) . \phi(y,y) :. \supset \vdash . \text{Prop}$

$*54{\cdot}441.\quad \vdash :: z, w \in \iota`x \cup \iota`y . z \neq w . \supset_{z,w} . \phi(z,w) : \equiv :. x = y : \vee : \phi(x,y) . \phi(y,x)$

Dem.

$\vdash . *5{\cdot}6 . \supset \vdash :: z, w \in \iota`x \cup \iota`y . z \neq w . \supset_{z,w} . \phi(z,w) : \equiv :.$
$z, w \in \iota`x \cup \iota`y . \supset_{z,w} : z = w . \vee . \phi(z,w) :.$
$[*54{\cdot}44] \quad \equiv : x = x . \vee . \phi(x,x) : x = y . \vee . \phi(x,y) :$
$y = x . \vee . \phi(y,x) : y = y . \vee . \phi(y,y) :$
$[*13{\cdot}15 \quad \equiv : x = y . \vee . \phi(x,y) : y = x . \vee . \phi(y,x) :$
$[*13{\cdot}16.*4{\cdot}41] \equiv : x = y . \vee . \phi(x,y) . \phi(y,x)$

This proposition is used in $*163{\cdot}42$, in the theory of relations of mutually exclusive relations.

This is p379 of Whitehead and Russell's *Principia Mathematica*.

Zooming in . . .

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The importance of computers for formal proof

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Because of these dual connections, interest in formal proofs is strongest among computer scientists, but some 'mainstream' mathematicians are becoming interested too.

A formal proof from 2010

```
let PNT = prove
 (`(((\n. &(CARD {p | prime p /\ p <= n}) / (&n / log(&n)))
    ---> &1) sequentially`,
  REWRITE_TAC[PNT_PARTIAL_SUMMATION] THEN
  REWRITE_TAC[SUM_PARTIAL_PRE] THEN
  REWRITE_TAC[GSYM REAL_OF_NUM_ADD; SUB_REFL; CONJUNCT1 LE] THEN
  SUBGOAL_THEN `{p | prime p /\ p = 0} = {}` SUBST1_TAC THENL
   [REWRITE_TAC[EXTENSION; IN_ELIM_THM; NOT_IN_EMPTY] THEN
    MESON_TAC[PRIME_IMP_NZ];
    ALL_TAC] THEN
  REWRITE_TAC[SUM_CLAUSES; REAL_MUL_RZERO; REAL_SUB_RZERO] THEN
  MATCH_MP_TAC REALLIM_TRANSFORM_EVENTUALLY THEN
  EXISTS_TAC
   `\n. ((&n + &1) / log(&n + &1) *
         sum {p | prime p /\ p <= n} (\p. log(&p) / &p) -
         sum (1..n)
         (\k. sum {p | prime p /\ p <= k} (\p. log(&p) / &p) *
              ((&k + &1) / log(&k + &1) - &k / log(&k)))) / (&n / log(&n))` THEN
  CONJ_TAC THENL
   [REWRITE_TAC[EVENTUALLY_SEQUENTIALLY] THEN EXISTS_TAC `1` THEN SIMP_TAC[];
    ALL_TAC] THEN
  MATCH_MP_TAC REALLIM_TRANSFORM THEN
  EXISTS_TAC
   `\n. ((&n + &1) / log(&n + &1) * log(&n) -
         sum (1..n)
         (\k. log(&k) * ((&k + &1) / log(&k + &1) - &k / log(&k)))) /
        (&n / log(&n))` THEN
  REWRITE_TAC[] THEN CONJ_TAC THENL
   [REWRITE_TAC[REAL_ARITH
     `(a * x - s) / b - (a * x' - s') / b:real =
      ((s' - s) - (x' - x) * a) / b`] THEN
    REWRITE_TAC[GSYM SUM_SUB_NUMSEG; GSYM REAL_SUB_RDISTRIB] THEN
    REWRITE_TAC[REAL_OF_NUM_ADD] THEN
    MATCH_MP_TAC SUM_PARTIAL_LIMIT_ALT THEN
```

Zooming in . . .

At least the theorems are more substantial:

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Though whether formal proofs have become more digestible to the non-expert is perhaps questionable . . .

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Newell and Simon's paper on a more elegant proof of one result in PM was rejected by JSL because it was co-authored by a machine.

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Correctness of a formal proof is an objective question, algorithmically checkable in principle.

Mathematics is reduced to sets

The explication of mathematical concepts in terms of sets is now quite widely accepted (see *Bourbaki*).

- A real number is a set of rational numbers ...
- A Turing machine is a quintuple $(\Sigma, A, \ldots)$

Statements in such terms are generally considered clearer and more objective. (Consider pathological functions from real analysis ...)

Symbolism is important

The use of symbolism in mathematics has been steadily increasing over the centuries:

> *"[Symbols] have invariably been introduced to make things easy. [...] by the aid of symbolism, we can make transitions in reasoning almost mechanically by the eye, which otherwise would call into play the higher faculties of the brain. [...] Civilisation advances by extending the number of important operations which can be performed without thinking about them." (Whitehead,* An Introduction to Mathematics*)*

Formalization is the key to rigour

Formalization now has a important conceptual role in principle:

> *"... the correctness of a mathematical text is verified by comparing it, more or less explicitly, with the rules of a formalized language." (Bourbaki,* Theory of Sets*)*
> *"A Mathematical proof is rigorous when it is (or could be) written out in the first-order predicate language $L(\in)$ as a sequence of inferences from the axioms ZFC, each inference made according to one of the stated rules." (Mac Lane,* Mathematics: Form and Function*)*

What about in practice?

Mathematicians don't use logical symbols

Variables were used in logic long before they appeared in mathematics, but logical symbolism is rare in current mathematics. Logical relationships are usually expressed in natural language, with all its subtlety and ambiguity.
Logical symbols like '$\Rightarrow$' and '$\forall$' are used *ad hoc*, mainly for their abbreviatory effect.

> *"as far as the mathematical community is concerned George Boole has lived in vain" (Dijkstra)*

Mathematicians don't do formal proofs . . .

The idea of actual formalization of mathematical proofs has not been taken very seriously:

> *"this mechanical method of deducing some mathematical theorems has no practical value because it is too complicated in practice." (Rasiowa and Sikorski,* The Mathematics of Metamathematics*)*
>
> *"[. . .] the tiniest proof at the beginning of the Theory of Sets would already require several hundreds of signs for its complete formalization. [. . .] formalized mathematics cannot in practice be written down in full [. . .] We shall therefore very quickly abandon formalized mathematics" (Bourbaki,* Theory of Sets*)*

. . . Poincaré's had a particular aversion . . .

I see in logistic only shackles for the inventor. It is no aid to conciseness — far from it, and if twenty-seven equations were necessary to establish that 1 is a number, how many would be needed to prove a real theorem? If we distinguish, with Whitehead, the individual x, the class of which the only member is x and [...] the class of which the only member is the class of which the only member is x [...], do you think these distinctions, useful as they may be, go far to quicken our pace?

Are proofs in doubt?

Mathematical proofs are subjected to peer review, but errors often escape unnoticed.

> *"Professor Offord and I recently committed ourselves to an odd mistake (Annals of Mathematics (2) 49, 923, 1.5). In formulating a proof a plus sign got omitted, becoming in effect a multiplication sign. The resulting false formula got accepted as a basis for the ensuing fallacious argument. (In defence, the final result was known to be true.)" (Littlewood,* Miscellany*)*

A book by Lecat gave 130 pages of errors made by major mathematicians up to 1900.
A similar book today would no doubt fill many volumes.

Even elegant textbook proofs can be wrong

"The second edition gives us the opportunity to present this new version of our book: It contains three additional chapters, substantial revisions and new proofs in several others, as well as minor amendments and improvements, many of them based on the suggestions we received. It also misses one of the old chapters, about the "problem of the thirteen spheres," whose proof turned out to need details that we couldn't complete in a way that would make it brief and elegant." (Aigner and Ziegler, Proofs from the Book*)*

Most doubtful informal proofs

What are the proofs where we do in practice worry about correctness?

- Those that are just very long and involved. Classification of finite simple groups, Seymour-Robertson graph minor theorem
- Those that involve extensive computer checking that cannot in practice be verified by hand. Four-colour theorem, Hales's proof of the Kepler conjecture
- Those that are about very technical areas where complete rigour is painful. Some branches of proof theory, formal verification of hardware or software

Recent achievements in formalization

Formalized theorems and libraries of mathematics

Interactive provers have been used to check quite non-trivial results, albeit not close to today's research frontiers, e.g.

- Jordan Curve Theorem — Tom Hales (HOL Light), Andrzej Trybulec et al. (Mizar)
- Prime Number Theorem — Jeremy Avigad et al (Isabelle/HOL), John Harrison (HOL Light)
- Dirichlet's Theorem — John Harrison (HOL Light)
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According to the *Formalizing 100 theorems* page, 88% of a list of the 'top 100 mathematical theorems' have been formalized using interactive theorem provers.
In the process, provers are building up ever-larger libraries of pre-proved theorems that can be deployed in future proofs.

The four-colour Theorem

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In 2005, Georges Gonthier formalized the entire proof in Coq, making use of the "SSReflect" proof language and replacing ad-hoc programs by evaluation within the logical kernel.

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- Uses the "SSReflect" proof language for Coq that was used in the four-colour proof.

The Kepler conjecture

The *Kepler conjecture* states that no arrangement of identical balls in ordinary 3-dimensional space has a higher packing density than the obvious 'cannonball' arrangement.
Hales, working with Ferguson, arrived at a proof in 1998:

- 300 pages of mathematics: geometry, measure, graph theory and related combinatorics, ...
- 40,000 lines of supporting computer code: graph enumeration, nonlinear optimization and linear programming.

Hales submitted his proof to *Annals of Mathematics* ...

The response of the reviewers

After a full four years of deliberation, the reviewers returned:

> *"The news from the referees is bad, from my perspective. They have not been able to certify the correctness of the proof, and will not be able to certify it in the future, because they have run out of energy to devote to the problem. This is not what I had hoped for.*
> *Fejes Toth thinks that this situation will occur more and more often in mathematics. He says it is similar to the situation in experimental science — other scientists acting as referees can't certify the correctness of an experiment, they can only subject the paper to consistency checks. He thinks that the mathematical community will have to get used to this state of affairs."*

The birth of Flyspeck

Hales's proof was eventually published, and no significant error has been found in it. Nevertheless, the verdict is disappointingly lacking in clarity and finality.
As a result of this experience, the journal changed its editorial policy on computer proof so that it will no longer even try to check the correctness of computer code.
Dissatisfied with this state of affairs, Hales initiated a project called *Flyspeck* to completely formalize the proof.

Flyspeck

Flyspeck = 'Formal Proof of the Kepler Conjecture'.

> *"In truth, my motivations for the project are far more complex than a simple hope of removing residual doubt from the minds of few referees. Indeed, I see formal methods as fundamental to the long-term growth of mathematics. (Hales,* The Kepler Conjecture*)*

The formalization effort has been running for a few years now with a significant group of people involved, some doing their PhD on Flyspeck-related formalization.
In parallel, Hales has simplified the informal proof using ideas from Marchal, significantly cutting down on the formalization work.

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- A method has been developed by Alexey Solovyev to prove all the nonlinear optimization results, running in many parallel sessions of HOL Light.

Reliability of machine-checked proof

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Why should we believe that these proofs are more reliable than human proofs?
What if the underlying logic is inconsistent or the proof checker is faulty?

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- Even the flakiest theorem provers are far more reliable than most human hand proofs.
- Problems in specification and modelling are more likely.
- Nothing is ever 100% certain, and a foundational death spiral adds little value.

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- It's unsatisfactory that we urge formality on others while developing provers so casually.
- It should be beyond reasonable doubt that we do or don't have a formal proof.
- A quest for perfection is worthy, even if the goal is unattainable.

Prover architecture

The reliability of a theorem prover increases dramatically if its correctness depends only on a small amount of code.

- de Bruijn approach — generate proofs that can be certified by a simple, separate checker.
- LCF approach — reduce all rules to sequences of primitive inferences implemented by a small logical kernel.

The checker or kernel can be much simpler than the prover as a whole.
But it is still non-trivial . . .

HOL Light

HOL Light is an extreme case of the LCF approach. The entire critical core is 430 lines of code:

- 10 rather simple primitive inference rules
- 2 conservative definitional extension principles
- 3 mathematical axioms (infinity, extensionality, choice)

Everything, even arithmetic on numbers, is done by reduction to the primitive basis.

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- Treatment of polymorphic types in definitions

It would still be nice to verify the core . . .

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To eliminate all of these:

> *Formalize the intended set-theoretic semantics of the logic and prove that the code implements inference rules that are sound w.r.t. this semantics.*

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However there are two apparent problems with 'HOL in HOL' . . .

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Actually prove two slightly different statements:

- $\mathrm{HOL} \vdash \mathit{Con}(\mathrm{HOL} - \{\infty\})$
- $\mathrm{HOL} + I \vdash \mathit{Con}(\mathrm{HOL})$.

Other uses of formal proofs

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> *What more do we know if we have proved a theorem by restricted means than if we merely know that it is true?*

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- ACL2 $\rightarrow$ HOL4: Gordon, Hunt, Kaufmann & Reynolds 2006

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More in the spirit of the original QED vision is the Logosphere project, which uses the Twelf logical framework as the common 'metalogic':
`http://www.logosphere.org`

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Finally, mainly thanks to Josef Urban, there has been an explosion of interest in this area.
Paradoxically, learning benefits from the large formal libraries associated with *interactive* theorem provers, even though the more natural 'home' might seem to be *automated* theorem proving.

Machine learning and ATP on Flyspeck

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> *It is shown that 39% of the 14185 theorems could be proved in a push-button mode (without any high-level advice and user interaction) in 30 seconds of real time on a fourteen-CPU workstation.*

HOL(y)Hammer

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It can sometimes find proofs that a human (this one, anyway) missed

> *Theorem* `FACE_OF_POLYHEDRON_POLYHEDRON` *states that a face of a polyhedron [...] is again a polyhedron. The HOL Light proof takes 23 lines [...] but a much simpler proof was found by the AI/ATP automation, based on [...] the* `FACE_OF_STILLCONVEX` *theorem: a face t of any convex set s is equal to the intersection of s with the affine hull of t. To finish the proof, one needs just three 'obvious' facts: Every polyhedron is convex (*`POLYHEDRON_IMP_CONVEX`*), the intersection of two polyhedra is again a polyhedron (*`POLYHEDRON_INTER`*), and affine hull is always a polyhedron (*`POLYHEDRON_AFFINE_HULL`*).*

Thank you!