# Formal verification of floating-point arithmetic at Intel

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6 June 2012

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- The rocket veered off its flight path and exploded, just 40 seconds into the flight sequence.

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- A Scud missile travels $500m$ in that time

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- Intel eventually set aside US $475 million to cover the costs.

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For example:

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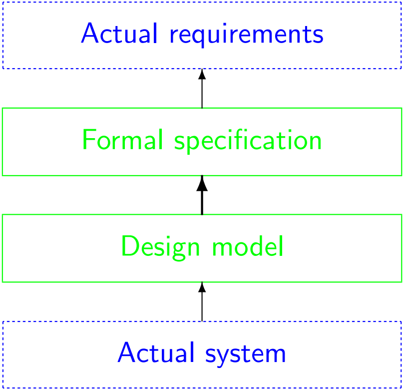
- Vastly higher number of possible states of a complex microarchitecture.

# Formal verification

Formal verification: mathematically prove the correctness of a *design* with respect to a mathematical *formal specification*.



Actual requirements

Formal specification

Design model

Actual system

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Similarly, extensive testing of hardware or software may still miss errors that would be revealed by a formal proof.

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However:

- Difficult and time-consuming.
- Only as reliable as the formal models used.
- How can we be sure the proof is right?

# Formal verification is hard

Writing out a completely formal proof of correctness for real-world hardware and software is difficult.

- Must specify intended behaviour formally
- Need to make many hidden assumptions explicit
- Requires long detailed proofs, difficult to review

The state of the art is quite limited.
Software verification has been around since the 60s, but there have been few major successes.

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However, these are rare and apparently well controlled by existing engineering best practice.

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- Lemmas 1, 2, and 3 were all false.
- The proof of the main induction in the final theorem was wrong.
- The main result, however, was correct!

# Machine-checked proof

A more promising approach is to have the proof checked (or even generated) by a computer program.

- It can reduce the risk of mistakes.
- The computer can automate some parts of the proofs.

There are limits on the power of automation, so detailed human guidance is often necessary.

# A variety of verification methods

There is a diverse world of formal verification methods, trading automation for generality / efficiency, most of which are in active use at Intel.

- Propositional tautology/equivalence checking (FEV)
- Symbolic simulation
- Symbolic trajectory evaluation (STE)
- Temporal logic model checking
- Combined decision procedures (SMT)
- First order automated theorem proving
- Interactive theorem proving

# A spectrum of formal techniques

Traditionally, formal verification has been focused on complete proofs of functional correctness.

But recently there have been notable successes elsewhere for ‘semi-formal’ methods involving abstraction or more limited property checking.

- Airbus A380 avionics
- Microsoft SLAM/SDV

One can also consider applying theorem proving technology to support testing or other traditional validation methods like path coverage.

These are all areas of interest at Intel.

# Our work

We have formally verified correctness of various floating-point algorithms.

- Division and square root (Marstein-style, using fused multiply-add to do Newton-Raphson or power series approximation with delicate final rounding).
- Transcendental functions like *log* and *sin* (table-driven algorithms using range reduction and a core polynomial approximations).

Proofs use the HOL Light prover

- `http://www.cl.cam.ac.uk/users/jrh/hol-light`

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- Verifying errors in polynomial approximations

# Example: tangent algorithm

- The input number $X$ is first reduced to $r$ with approximately $|r| \leq \pi/4$ such that $X = r + N\pi/2$ for some integer $N$. We now need to calculate $\pm tan(r)$ or $\pm cot(r)$ depending on $N$ modulo 4.

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- If the reduced argument $r$ is still not small enough, it is separated into its leading few bits $B$ and the trailing part $x = r - B$, and the overall result computed from $tan(x)$ and pre-stored functions of $B$, e.g.

$$tan(B + x) = tan(B) + \frac{\frac{1}{sin(B)cos(B)} tan(x)}{cot(B) - tan(x)}$$

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- Now a power series approximation is used for $tan(r)$, $cot(r)$ or $tan(x)$ as appropriate.

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Most of these parts are non-trivial. Moreover, some of them require more pure mathematics than might be expected.

# Why mathematics?

Controlling the error in range reduction becomes difficult when the reduced argument $X - N\pi/2$ is small.
To check that the computation is accurate enough, we need to know:

> *How close can a floating point number be to an integer multiple of* $\pi/2$*?*

Even deriving the power series (for $0 < |x| < \pi$):

$$cot(x) = 1/x - \frac{1}{3}x - \frac{1}{45}x^3 - \frac{2}{945}x^5 - \ldots$$

is much harder than you might expect.

# Why HOL Light?

We need a general theorem proving system with:

- High standard of logical rigor and reliability
- Ability to mix interactive and automated proof
- Programmability for domain-specific proof tasks
- A substantial library of pre-proved mathematics

Other theorem provers such as ACL2, Coq and PVS have also been used for verification in this area.

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This experience seems quite common.

# Perspectives and future prospects

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- We need more research on making formal verification more efficient and automatic so it can be applied more widely, and applied by relative non-experts.
- We need computer science curricula at universities to provide more rigorous treatment of mathematical rigor, logic and formal proof so that more programmers and engineers are able to deploy formal techniques.