Automated and Interactive Theorem Proving
1: Background & Propositional Logic

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Marktoberdorf 2007

Thu 2nd August 2007 (08:30–09:15)

What I will talk about

Aim is to cover some of the most important approaches to computer-aided proof in classical logic.

1. Background and propositional logic
2. First-order logic, with and without equality
3. Decidable problems in logic and algebra
4. Combination and certification of decision procedures
5. Interactive theorem proving

What I won't talk about

- Decision procedures for temporal logic, model checking (well covered in other courses)
- Higher-order logic (my own interest but off the main topic; will see some of this in other courses)
- Undecidability and incompleteness (I don't have enough time)
- Methods for constructive logic, modal logic, other nonclassical logics (I don't know much anyway)

A practical slant

Our approach to logic will be highly constructive!

Most of what is described is implemented by explicit code that can be obtained here:

`http://www.cl.cam.ac.uk/users/jrh/atp/`

See also my interactive higher-order logic prover HOL Light:

`http://www.cl.cam.ac.uk/users/jrh/hol-light/`

which incorporates many decision procedures in a certified way.

Propositional Logic

We probably all know what propositional logic is.

English	Standard	Boolean	Other
false	$\bot$	0	F
true	$\top$	1	T
not p	$\neg p$	$\overline{p}$	$-p$, $\sim p$
p and q	$p \wedge q$	pq	$p\&q$, $p \cdot q$
p or q	$p \vee q$	$p + q$	$p \mid q$, p *or* q
p implies q	$p \Rightarrow q$	$p \leq q$	$p \rightarrow q$, $p \supset q$
p iff q	$p \Leftrightarrow q$	$p = q$	$p \equiv q$, $p \sim q$

In the context of circuits, it's often referred to as 'Boolean algebra', and many designers use the Boolean notation.

Is propositional logic boring?

Traditionally, propositional logic has been regarded as fairly boring.

- There are severe limitations to what can be said with propositional logic.
- Propositional logic is trivially decidable in theory
- The usual methods aren't efficient enough for interesting problems.

But . . .

No!

The last decade has seen a remarkable upsurge of interest in propositional logic.

In fact, it’s arguably the hottest topic in automated theorem proving!

Why the resurgence?

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Why the resurgence?

- There *are* many interesting problems that can be expressed in propositional logic
- Efficient algorithms *can* often decide large, interesting problems

A practical counterpart to the theoretical reductions in NP-completeness theory.

Logic and circuits

The correspondence between digital logic circuits and propositional logic has been known for a long time.

Digital design	Propositional Logic
circuit	formula
logic gate	propositional connective
input wire	atom
internal wire	subexpression
voltage level	truth value

Many problems in circuit design and verification can be reduced to propositional tautology or satisfiability checking ('SAT').

For example optimization correctness: $\phi \Leftrightarrow \phi'$ is a tautology.

Combinatorial problems

Many other apparently difficult combinatorial problems can be encoded as Boolean satisfiability (SAT), e.g. scheduling, planning, geometric embeddibility, even factorization.

$$
\begin{aligned}
\neg(\quad & (out_0 \Leftrightarrow x_0 \wedge y_0) \wedge \\
& (out_1 \Leftrightarrow (x_0 \wedge y_1 \Leftrightarrow \neg(x_1 \wedge y_0))) \wedge \\
& (v_2^2 \Leftrightarrow (x_0 \wedge y_1) \wedge x_1 \wedge y_0) \wedge \\
& (u_2^0 \Leftrightarrow ((x_1 \wedge y_1) \Leftrightarrow \neg v_2^2)) \wedge \\
& (u_2^1 \Leftrightarrow (x_1 \wedge y_1) \wedge v_2^2) \wedge \\
& (out_2 \Leftrightarrow u_2^0) \wedge (out_3 \Leftrightarrow u_2^1) \wedge \\
& \neg out_0 \wedge out_1 \wedge out_2 \wedge \neg out_3)
\end{aligned}
$$

Read off the factorization $6 = 2 \times 3$ from a refuting assignment.

Efficient methods

The naive truth table method is quite impractical for formulas with more than a dozen primitive propositions.

Practical use of propositional logic mostly relies on one of the following algorithms for deciding tautology or satisfiability:

- Binary decision diagrams (BDDs)
- The Davis-Putnam method (DP, DPLL)
- Stålmarck's method

We'll sketch the basic ideas behind Davis-Putnam and Stålmarck's method.

DP and DPLL

Actually, the original Davis-Putnam procedure is not much used now.

What is usually called the Davis-Putnam method is actually a later refinement due to Davis, Loveland and Logemann (hence DPLL).

We formulate it as a test for *satisfiability*. It has three main components:

- Transformation to conjunctive normal form (CNF)
- Application of simplification rules
- Splitting

Normal forms

In ordinary algebra we can reach a ‘sum of products’ form of an expression by:

- Eliminating operations other than addition, multiplication and negation, e.g. $x - y \mapsto x + -y$.
- Pushing negations inwards, e.g. $-(-x) \mapsto x$ and $-(x + y) \mapsto -x + -y$.
- Distributing multiplication over addition, e.g. $x(y + z) \mapsto xy + xz$.

In logic we can do exactly the same, e.g. $p \Rightarrow q \mapsto \neg p \vee q$, $\neg(p \wedge q) \mapsto \neg p \vee \neg q$ and $p \wedge (q \vee r) \mapsto (p \wedge q) \vee (p \wedge r)$.

The first two steps give ‘negation normal form’ (NNF).

Following with the last (distribution) step gives ‘disjunctive normal form’ (DNF), analogous to a sum-of-products.

Conjunctive normal form

Conjunctive normal form (CNF) is the dual of DNF, where we reverse the roles of ‘and’ and ‘or’ in the distribution step to reach a ‘product of sums’:

$$\begin{array}{rcl} p \lor (q \land r) & \mapsto & (p \lor q) \land (p \lor r) \\ (p \land q) \lor r & \mapsto & (p \lor r) \land (q \lor r) \end{array}$$

Reaching such a CNF is the first step of the Davis-Putnam procedure.

Unfortunately the naive distribution algorithm can cause the size of the formula to grow exponentially — not a good start. Consider for example:

$$(p_1 \land p_2 \land \cdots \land p_n) \lor (q_1 \land p_2 \land \cdots \land q_n)$$

Definitional CNF

A cleverer approach is to introduce new variables for subformulas. Although this isn't logically equivalent, it does preserve satisfiability.

$$(p \lor (q \land \neg r)) \land s$$

introduce new variables for subformulas:

$$(p_1 \Leftrightarrow q \land \neg r) \land (p_2 \Leftrightarrow p \lor p_1) \land (p_3 \Leftrightarrow p_2 \land s) \land p_3$$

then transform to (3-)CNF in the usual way:

$$(\neg p_1 \lor q) \land (\neg p_1 \lor \neg r) \land (p_1 \lor \neg q \lor r) \land$$
$$(\neg p_2 \lor p \lor p_1) \land (p_2 \lor \neg p) \land (p_2 \lor \neg p_1) \land$$
$$(\neg p_3 \lor p_2) \land (\neg p_3 \lor s) \land (p_3 \lor \neg p_2 \lor \neg s) \land p_3$$

Clausal form

It's convenient to think of the CNF form as a set of sets:

- Each disjunction $p_1 \vee \cdots \vee p_n$ is thought of as the set $\{p_1, \ldots, p_n\}$, called a *clause*.
- The overall formula, a conjunction of clauses $C_1 \wedge \cdots \wedge C_m$ is thought of as a set $\{C_1, \ldots, C_m\}$.

Since 'and' and 'or' are associative, commutative and idempotent, nothing of logical significance is lost in this interpretation.

Special cases: an empty clause means $\bot$ (and is hence unsatisfiable) and an empty set of clauses means $\top$ (and is hence satisfiable).

Simplification rules

At the core of the Davis-Putnam method are two transformations on the set of clauses:

I The 1-literal rule: if a unit clause p appears, remove $\neg p$ from other clauses and remove all clauses including p.

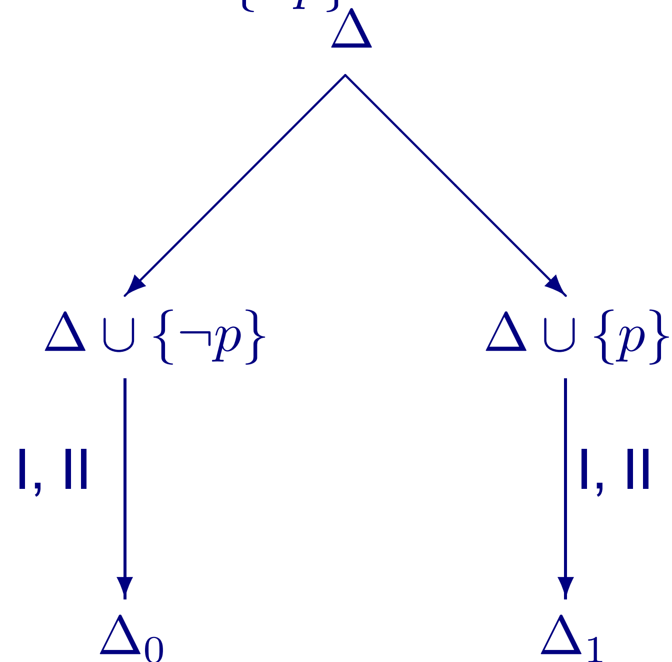
II The affirmative-negative rule: if p occurs *only* negated, or *only* unnegated, delete all clauses involving p.

These both preserve satisfiability of the set of clause sets.

Splitting

In general, the simplification rules will not lead to a conclusion. We need to perform case splits.

Given a clause set Δ, simply choose a variable p, and consider the two new sets $\Delta \cup \{p\}$ and $\Delta \cup \{\neg p\}$.



In general, these case-splits need to be nested.

Industrial strength SAT solvers

For big applications, there are several important modifications to the basic DPLL algorithm:

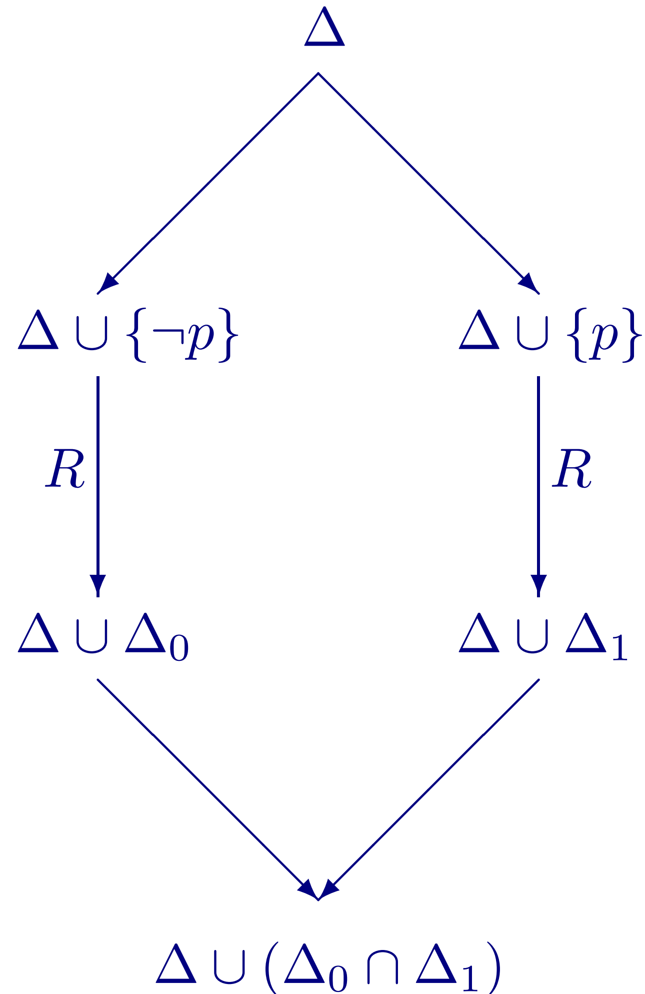
- Highly efficient data structures
- Good heuristics for picking ‘split’ variables
- Intelligent non-chronological backtracking / conflict clauses

Some well-known provers are BerkMin, zChaff and Minisat.

These often shine because of careful attention to low-level details like memory hierarchy, not cool algorithmic ideas.

Stålmarck's algorithm

Stålmarck's 'dilemma' rule attempts to avoid nested case splits by feeding back common information from both branches.



Summary

- Propositional logic is no longer a neglected area of theorem proving
- A wide variety of practical problems can usefully be encoded in SAT
- There is intense interest in efficient algorithms for SAT
- Many of the most successful systems are still based on minor refinements of the ancient Davis-Putnam procedure
- Can we invent a better SAT algorithm?

Automated and Interactive Theorem Proving
2: First-order logic with and without equality

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Summary

- First order logic
- Naive Herbrand procedures
- Unification
- Adding equality
- Knuth-Bendix completion

First-order logic

Start with a set of *terms* built up from variables and constants using function application:

$$x + 2 \cdot y \equiv +(x, \cdot(2(), y))$$

Create atomic formulas by applying relation symbols to a set of terms

$$x > y \equiv > (x, y)$$

Create complex formulas using quantifiers

- $\forall x.\ P[x]$ — for all x, $P[x]$
- $\exists x.\ P[x]$ — there exists an x such that $P[x]$

Quantifier examples

The order of quantifier nesting is important. For example

$\forall x.\, \exists y.\, loves(x,y)$ — everyone loves someone
$\exists x.\, \forall y.\, loves(x,y)$ — somebody loves everyone
$\exists y.\, \forall x.\, loves(x,y)$ — someone is loved by everyone

This says that a function $\mathbb{R} \to \mathbb{R}$ is continuous:

$$\forall \epsilon.\, \epsilon > 0 \Rightarrow \forall x.\, \exists \delta.\, \delta > 0 \wedge \forall x'.\, |x' - x| < \delta \Rightarrow |f(x') - f(x)| < \varepsilon$$

while this one says it is *uniformly* continuous, an important distinction

$$\forall \epsilon.\, \epsilon > 0 \Rightarrow \exists \delta.\, \delta > 0 \wedge \forall x.\, \forall x'.\, |x' - x| < \delta \Rightarrow |f(x') - f(x)| < \varepsilon$$

Skolemization

Skolemization relies on this observation (related to the axiom of choice):

$$(\forall x.\ \exists y.\ P[x, y]) \Leftrightarrow \exists f.\ \forall x.\ P[x, f(x)]$$

For example, a function is surjective (onto) iff it has a right inverse:

$$(\forall x.\ \exists y.\ g(y) = x) \Leftrightarrow (\exists f.\ \forall x.\ g(f(x)) = x$$

Can't quantify over functions in first-order logic.

But we get an *equisatisfiable* formula if we just introduce a new function symbol.

$$\begin{aligned} &\forall x_1, \ldots, x_n.\ \exists y.\ P[x_1, \ldots, x_n, y] \\ &\to \forall x_1, \ldots, x_n.\ P[x_1, \ldots, x_n, f(x_1, \ldots, x_n)] \end{aligned}$$

Now we just need a satisfiability test for universal formulas.

First-order automation

The underlying domains can be arbitrary, so we can't do an exhaustive analysis, but must be slightly subtler.

We can reduce the problem to propositional logic using the so-called *Herbrand theorem* and *compactness theorem*, together implying:

> Let $\forall x_1, \ldots, x_n.\ P[x_1, \ldots, x_n]$ be a first order formula with only the indicated universal quantifiers (i.e. the body $P[x_1, \ldots, x_n]$ is quantifier-free). Then the formula is satisfiable iff all finite sets of 'ground instances' $P[t_1^i, \ldots, t_n^i]$ that arise by replacing the variables by arbitrary variable-free terms made up from functions and constants in the original formula is *propositionally* satisfiable.

Still only gives a *semidecision* procedure, a kind of proof search.

Example

Suppose we want to prove the 'drinker's principle'

$$\exists x.\, \forall y.\, D(x) \Rightarrow D(y)$$

Negate the formula, and prove negation unsatisfiable:

$$\neg(\exists x.\, \forall y.\, D(x) \Rightarrow D(y))$$

Convert to prenex normal form: $\forall x.\, \exists y.\, D(x) \wedge \neg D(y)$

Skolemize: $\forall x.\, D(x) \wedge \neg D(f(x))$

Enumerate set of ground instances, first $D(c) \wedge \neg D(f(c))$ is not unsatisfiable, but the next is:

$$(D(c) \wedge \neg D(f(c))) \wedge (D(f(c)) \wedge \neg D(f(f(c)))$$

Instantiation versus unification

The first automated theorem provers actually used that approach.

It was to test the propositional formulas resulting from the set of ground-instances that the Davis-Putnam method was developed.

However, more efficient than enumerating ground instances is to use *unification* to choose instantiations intelligently.

For example, choose instantiation for x and y so that $D(x)$ and $\neg(D(f(y)))$ are complementary.

Unification

Given a set of pairs of terms

$$S = \{(s_1, t_1), \ldots, (s_n, t_n)\}$$

a *unifier* of S is an instantiation σ such that each

$$\sigma s_i = \sigma t_i$$

If a unifier exists there is a *most general* unifier (MGU), of which any other is an instance.

MGUs can be found by straightforward recursive algorithm.

Unification-based theorem proving

Many theorem-proving algorithms based on unification exist:

- Tableaux
- Resolution
- Model elimination
- Connection method
- ...

Resolution

Propositional resolution is the rule:

$$\frac{p \vee A \quad \neg p \vee B}{A \vee B}$$

and full first-order resolution is the generalization

$$\frac{P \vee A \quad Q \vee B}{\sigma(A \vee B)}$$

where σ is an MGU of literal sets P and Q^-.

Adding equality

We often want to restrict ourselves to validity in *normal* models where ‘equality means equality’.

- Add extra axioms for equality and use non-equality decision procedures
- Use other preprocessing methods such as Brand transformation or STE
- Use special rules for equality such as paramodulation or superposition

Equality axioms

Given a formula p, let the *equality axioms* be equivalence:

$$\forall x.\ x = x$$

$$\forall x\ y.\ x = y \Rightarrow y = x$$

$$\forall x\ y\ z.\ x = y \land y = z \Rightarrow x = z$$

together with *congruence* rules for each function and predicate in p:

$$\forall \overline{xy}.\ x_1 = y_1 \land \cdots \land x_n = y_n \Rightarrow f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n)$$

$$\forall \overline{xy}.\ x_1 = y_1 \land \cdots \land x_n = y_n \Rightarrow R(x_1, \ldots, x_n) \Rightarrow R(y_1, \ldots, y_n)$$

Brand transformation

Adding equality axioms has a bad reputation in the ATP world.

Simple substitutions like $x = y \Rightarrow f(y) + f(f(x)) = f(x) + f(f(y))$ need many applications of the rules.

Brand's transformation uses a different translation to build in equality, involving 'flattening'

$$(x \cdot y) \cdot z = x \cdot (y \cdot z)$$

$$x \cdot y = w_1 \Rightarrow w_1 \cdot z = x \cdot (y \cdot z)$$

$$x \cdot y = w_1 \land y \cdot z = w_2 \Rightarrow w_1 \cdot z = x \cdot w_2$$

Still not conclusively better.

Paramodulation and related methods

Often better to add special rules such as paramodulation:

$$\frac{C \vee s \doteq t \qquad D \vee P[s']}{\sigma\ (C \vee D \vee P[t])}$$

Works best with several restrictions including the use of orderings to orient equations.

Easier to understand for pure equational logic.

Normalization by rewriting

Use a set of equations left-to-right as rewrite rules to simplify or normalize a term:

- Use some kind of ordering (e.g. lexicographic path order) to ensure termination
- Difficulty is ensuring confluence

Failure of confluence

Consider these axioms for groups:

$$(x \cdot y) \cdot z = x \cdot (y \cdot z)$$

$$1 \cdot x = x$$

$$i(x) \cdot x = 1$$

They are not confluent because we can rewrite

$$(i(x) \cdot x) \cdot y \longrightarrow i(x) \cdot (x \cdot y)$$

$$(i(x) \cdot x) \cdot y \longrightarrow 1 \cdot y$$

Knuth-Bendix completion

Key ideas of Knuth-Bendix completion:

- Use unification to identify most general situations where confluence fails (‘critical pairs’)
- Add critical pairs, suitably oriented, as new equations and repeat

This process completes the group axioms, deducing some non-trivial consequences along the way.

Completion of group axioms

$$i(x \cdot y) = i(y) \cdot i(x)$$

$$i(i(x)) = x$$

$$i(1) = 1$$

$$x \cdot i(x) = 1$$

$$x \cdot i(x) \cdot y = y$$

$$x \cdot 1 = x$$

$$i(x) \cdot x \cdot y = y$$

$$1 \cdot x = x$$

$$i(x) \cdot x = 1$$

$$(x \cdot y) \cdot z = x \cdot y \cdot z$$

Summary

- Can't solve first-order logic by naive method, but Herbrand's theorem gives a proof search procedure
- Unification is normally a big improvement on straightforward search through the Herbrand base
- Can incorporate equality either by preprocessing or special rules
- Knuth-Bendix completion is an important approach to equality handling and the same ideas reappear in other first-order methods.

Automated and Interactive Theorem Proving
3: Decidable problems in logic and algebra

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Summary

- Decidable fragments of pure logic
- Quantifier elimination
- Important arithmetical examples
- Algebra and word problems
- Geometric theorem proving

Decidable problems

Although first order validity is undecidable, there are special cases where it is decidable, e.g.

- AE formulas: no function symbols, universal quantifiers before existentials in prenex form
- Monadic formulas: no function symbols, only unary predicates

Decidable problems

Although first order validity is undecidable, there are special cases where it is decidable, e.g.

- AE formulas: no function symbols, universal quantifiers before existentials in prenex form
- Monadic formulas: no function symbols, only unary predicates

All ‘syllogistic’ reasoning can be reduced to the monadic fragment:

If all M are P, and all S are M, then all S are P

can be expressed as the monadic formula:

$$(\forall x.\ M(x) \Rightarrow P(x)) \land (\forall x.\ S(x) \Rightarrow M(x)) \Rightarrow (\forall x.\ S(x) \Rightarrow P(x))$$

Why AE is decidable

The negation of an AE formula is an EA formula to be refuted:

$$\exists x_1, \ldots, x_n. \forall y_1, \ldots, y_m. P[x_1, \ldots, x_n, y_1, \ldots, y_m]$$

and after Skolemization we still have no functions:

$$\forall y_1, \ldots, y_m. P[c_1, \ldots, c_n, y_1, \ldots, y_m]$$

So there are only finitely many ground instances to check for satisfiability.

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and after Skolemization we still have no functions:

$$\forall y_1, \ldots, y_m. P[c_1, \ldots, c_n, y_1, \ldots, y_m]$$

So there are only finitely many ground instances to check for satisfiability.

Since the equality axioms are purely universal formulas, adding those doesn't disturb the AE/EA nature, so we get Ramsey's decidability result.

The finite model property

Another way of understanding decidability results is that fragments like AE and monadic formulas have the finite model property:

> If the formula in the fragment has a model it has a finite model.

Any fragment with the finite model property is decidable: search for a model and a disproof in parallel.

Often we even know the exact size we need consider: e.g. size 2^n for monadic formula with n predicates.

In practice, we quite often find finite countermodels to false formulas.

Failures of the FMP

However many formulas with simple quantifier prefixes *don't* have the FMP:

- $(\forall x.\ \neg R(x,x)) \wedge (\forall x.\ \exists z.\ R(x,z)) \wedge$
  $(\forall x\ y\ z.\ R(x,y) \wedge R(y,z) \Rightarrow R(x,z))$
- $(\forall x.\ \neg R(x,x)) \wedge (\forall x.\ \exists y.\ R(x,y) \wedge \forall z.\ R(y,z) \Rightarrow R(x,z)))$
- $\neg(\ \ (\forall x.\ \neg(F(x,x)) \wedge$
  $(\forall x\ y.\ F(x,y) \Rightarrow F(y,x)) \wedge$
  $(\forall x\ y.\ \neg(x = y) \Rightarrow \exists ! z.\ \ F(x,z) \wedge F(y,z))$
  $\Rightarrow \exists u.\ \forall v.\ \neg(v = u) \Rightarrow F(u,v))$

Failures of the FMP

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  $(\forall x\ y\ z.\ R(x,y) \wedge R(y,z) \Rightarrow R(x,z))$

- $(\forall x.\ \neg R(x,x)) \wedge (\forall x.\ \exists y.\ R(x,y) \wedge \forall z.\ R(y,z) \Rightarrow R(x,z)))$

- $\neg(\ \ (\forall x.\ \neg(F(x,x)) \wedge$
  $(\forall x\ y.\ F(x,y) \Rightarrow F(y,x)) \wedge$
  $(\forall x\ y.\ \neg(x = y) \Rightarrow \exists z.\ \ F(x,z) \wedge F(y,z) \wedge$
  $\forall w.\ F(x,w) \wedge F(y,w) \Rightarrow w = z)$
  $\Rightarrow \exists u.\ \forall v.\ \neg(v = u) \Rightarrow F(u,v))$

The theory of equality

A simple but useful decidable theory is the universal theory of equality with function symbols, e.g.

$$\forall x.\ f(f(f(x)) = x \land f(f(f(f(f(x))))) = x \Rightarrow f(x) = x$$

after negating and Skolemizing we need to test a ground formula for satisfiability:

$$f(f(f(c)) = c \land f(f(f(f(f(c))))) = c \land \neg(f(c) = c)$$

Two well-known algorithms:

- Put the formula in DNF and test each disjunct using one of the classic 'congruence closure' algorithms.
- Reduce to SAT by introducing a propositional variable for each equation between subterms and adding constraints.

Decidable theories

More useful in practical applications are cases not of *pure* validity, but validity in special (classes of) models, or consequence from useful axioms, e.g.

- Does a formula hold over all rings (Boolean rings, non-nilpotent rings, integral domains, fields, algebraically closed fields, . . .)
- Does a formula hold in the natural numbers or the integers?
- Does a formula hold over the real numbers?
- Does a formula hold in all real-closed fields?
- . . .

Because arithmetic comes up in practice all the time, there's particular interest in theories of arithmetic.

Theories

These can all be subsumed under the notion of a *theory*, a set of formulas T closed under logical validity. A theory T is:

- *Consistent* if we never have $p \in T$ and $(\neg p) \in T$.
- *Complete* if for closed p we have $p \in T$ or $(\neg p) \in T$.
- *Decidable* if there's an algorithm to tell us whether a given closed p is in T

Note that a complete theory generated by an r.e. axiom set is also decidable.

Quantifier elimination

Often, a quantified formula is T-equivalent to a quantifier-free one:

- $\mathbb{C} \models (\exists x.\ x^2 + 1 = 0) \Leftrightarrow \top$
- $\mathbb{R} \models (\exists x. ax^2+bx+c = 0) \Leftrightarrow a \neq 0 \wedge b^2 \geq 4ac \vee a = 0 \wedge (b \neq 0 \vee c = 0)$
- $\mathbb{Q} \models (\forall x.\ x < a \Rightarrow x < b) \Leftrightarrow a \leq b$
- $\mathbb{Z} \models (\exists k\ x\ y.\ ax = (5k + 2)y + 1) \Leftrightarrow \neg(a = 0)$

We say a theory T admits *quantifier elimination* if *every* formula has this property.

Assuming we can decide variable-free formulas, quantifier elimination implies completeness.

And then an *algorithm* for quantifier elimination gives a decision method.

Important arithmetical examples

- Presburger arithmetic: arithmetic equations and inequalities with addition but *not multiplication*, interpreted over $\mathbb{Z}$ or $\mathbb{N}$.
- Tarski arithmetic: arithmetic equations and inequalities with addition and multiplication, interpreted over $\mathbb{R}$ (or any real-closed field)
- Complex arithmetic: arithmetic equations with addition and multiplication interpreted over $\mathbb{C}$ (or other algebraically closed field of characteristic 0).

Important arithmetical examples

- Presburger arithmetic: arithmetic equations and inequalities with addition but *not multiplication*, interpreted over $\mathbb{Z}$ or $\mathbb{N}$.
- Tarski arithmetic: arithmetic equations and inequalities with addition and multiplication, interpreted over $\mathbb{R}$ (or any real-closed field)
- Complex arithmetic: arithmetic equations with addition and multiplication interpreted over $\mathbb{C}$ (or other algebraically closed field of characteristic 0).

However, arithmetic with multiplication over $\mathbb{Z}$ is not even semidecidable, by Gödel's theorem.

Nor is arithmetic over $\mathbb{Q}$ (Julia Robinson), nor just solvability of equations over $\mathbb{Z}$ (Matiyasevich). Equations over $\mathbb{Q}$ unknown.

History of real quantifier elimination

- 1930: Tarski discovers quantifier elimination procedure for this theory.
- 1948: Tarski's algorithm published by RAND
- 1954: Seidenberg publishes simpler algorithm
- 1975: Collins develops and *implements* cylindrical algebraic decomposition (CAD) algorithm
- 1983: Hörmander publishes very simple algorithm based on ideas by Cohen.
- 1990: Vorobjov improves complexity bound to doubly exponential in number of quantifier *alternations*.

Current implementations

There are quite a few simple versions of real quantifier elimination, even in computer algebra systems like Mathematica.

Among the more heavyweight implementations are:

- `qepcad` — `http://www.cs.usna.edu/~qepcad/B/QEPCAD.html`
- `REDLOG` — `http://www.fmi.uni-passau.de/~redlog/`

Word problems

Want to decide whether one set of equations implies another in a class of algebraic structures:

$$\forall \overline{x}.\ s_1 = t_1 \land \cdots \land s_n = t_n \Rightarrow s = t$$

For rings, we can assume it's a standard polynomial form

$$\forall \overline{x}.\ p_1(\overline{x}) = 0 \land \cdots \land p_n(\overline{x}) = 0 \Rightarrow q(\overline{x}) = 0$$

Word problem for rings

$$\forall \overline{x}.\ p_1(\overline{x}) = 0 \wedge \cdots \wedge p_n(\overline{x}) = 0 \Rightarrow q(\overline{x}) = 0$$

holds in all rings iff

$$q \in \mathsf{Id}_{\mathbb{Z}} \langle p_1, \ldots, p_n \rangle$$

i.e. there exist 'cofactor' polynomials with integer coefficients such that

$$p_1 \cdot q_1 + \cdots + p_n \cdot q_n = q$$

Special classes of rings

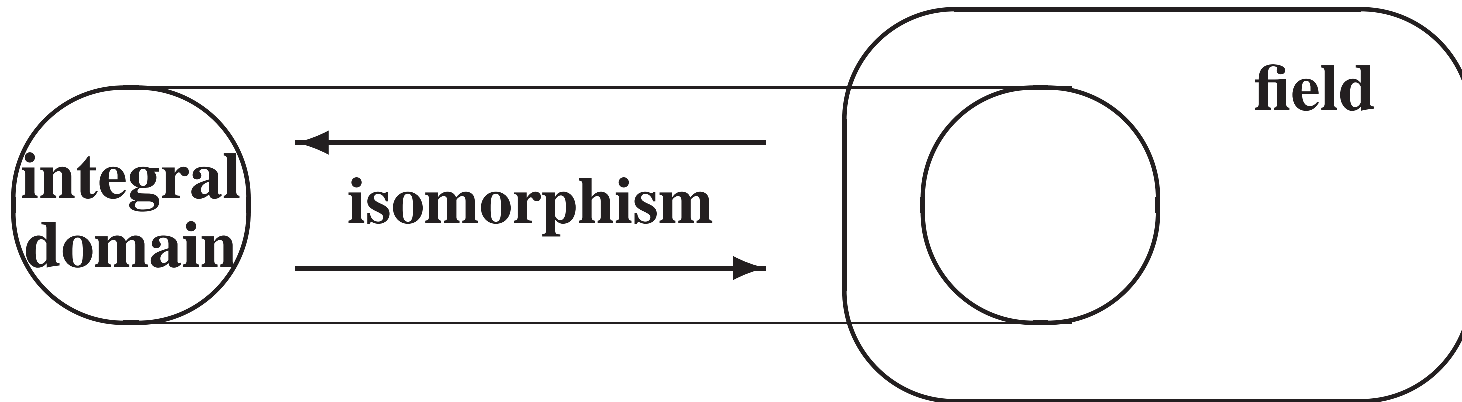
- Torsion-free: $\overbrace{x + \cdots + x}^{n \text{ times}} = 0 \Rightarrow x = 0$ for $n \geq 1$
- Characteristic p: $\overbrace{1 + \cdots + 1}^{n \text{ times}} = 0$ iff $p | n$
- Integral domains: $x \cdot y = 0 \Rightarrow x = 0 \vee y = 0$ (and $1 \neq 0$).

Special word problems

$$\forall \overline{x}.\ p_1(\overline{x}) = 0 \wedge \cdots \wedge p_n(\overline{x}) = 0 \Rightarrow q(\overline{x}) = 0$$

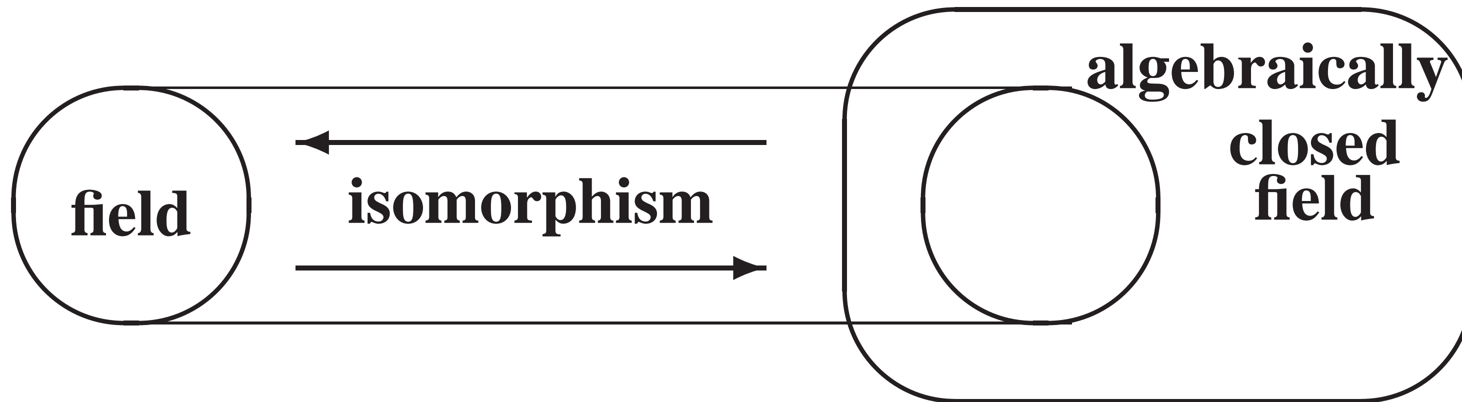
- Holds in all rings iff $q \in \mathsf{Id}_{\mathbb{Z}} \langle p_1, \ldots, p_n \rangle$
- Holds in all torsion-free rings iff $q \in \mathsf{Id}_{\mathbb{Q}} \langle p_1, \ldots, p_n \rangle$
- Holds in all integral domains iff $q^k \in \mathsf{Id}_{\mathbb{Z}} \langle p_1, \ldots, p_n \rangle$ for some $k \geq 0$
- Holds in all integral domains of characteristic 0 iff $q^k \in \mathsf{Id}_{\mathbb{Q}} \langle p_1, \ldots, p_n \rangle$ for some $k \geq 0$

Embedding in field of fractions



Universal formula in the language of rings holds in all integral domains [of characteristic p] iff it holds in all fields [of characteristic p].

Embedding in algebraic closure



Universal formula in the language of rings holds in all fields [of characteristic p] iff it holds in all algebraically closed fields [of characteristic p]

Connection to the Nullstellensatz

Also, algebraically closed fields of the same characteristic are elementarily equivalent.

For a universal formula in the language of rings, all these are equivalent:

- It holds in all integral domains of characteristic 0
- It holds in all fields of characteristic 0
- It holds in all algebraically closed fields of characteristic 0
- It holds in any given algebraically closed field of characteristic 0
- It holds in $\mathbb{C}$

Penultimate case is basically the Hilbert Nullstellensatz.

Gröbner bases

Can solve all these ideal membership goals in various ways.

The most straightforward uses *Gröbner bases*.

Use polynomial $m_1 + m_2 + \cdots + m_p = 0$ as a rewrite rule $m_1 = -m_2 + \cdots + -m_p$ for a 'head' monomial according to ordering.

Perform operation analogous to Knuth-Bendix completion to get expanded set of equations that is confluent, a *Gröbner basis*.

Geometric theorem proving

In principle can solve most geometric problems by using coordinate translation then Tarski's real quantifier elimination.

Example: A, B, C are collinear iff
$(A_x - B_x)(B_y - C_y) = (A_y - B_y)(B_x - C_x)$

In practice, it's much faster to use decision procedures for complex numbers. Remarkably, many geometric theorems remain true in this more general context.

As well as Gröbner bases, Wu pioneered the approach using characteristic sets (Ritt-Wu triangulation).

Summary

- Some fragments of pure first-order logic are decidable
- Quantifier elimination for arithmetical theories is potentially very useful
- Tarski algebra is powerful in principle, limited in practice
- Many word problems have efficient solutions using ideal membership and can be solved using Gröbner bases
- Geometry theorem proving using complex coordinates is surprisingly effective

Automated and Interactive Theorem Proving 4: Combining and certifying decision procedures

John Harrison

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Marktoberdorf 2007

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Summary

- Need to combine multiple decision procedures
- Basics of Nelson-Oppen method
- Proof-producing decision procedures
- Separate certification

Need for combinations

In applications we often need to combine decision methods from different domains.

$x - 1 < n \wedge \neg(x < n) \Rightarrow a[x] = a[n]$

An arithmetic decision procedure could easily prove

$x - 1 < n \wedge \neg(x < n) \Rightarrow x = n$

but could not make the additional final step, even though it looks trivial.

Most combinations are undecidable

Adding almost any additions, especially uninterpreted, to the usual decidable arithmetic theories destroys decidability.

Some exceptions like BAPA ('Boolean algebra + Presburger arithmetic').

This formula over the reals constrains P to define the integers:

$$(\forall n.\ P(n+1) \Leftrightarrow P(n)) \land (\forall n.\ 0 \leq n \land n < 1 \Rightarrow (P(n) \Leftrightarrow n = 0))$$

and this one in Presburger arithmetic defines squaring:

$$(\forall n.\ f(-n) = f(n)) \land (f(0) = 0) \land$$
$$(\forall n.\ 0 \leq n \Rightarrow f(n+1) = f(n) + n + n + 1)$$

and so we can define multiplication.

Quantifier-free theories

However, if we stick to so-called ‘quantifier-free’ theories, i.e. deciding universal formulas, things are better.

Two well-known methods for combining such decision procedures:

- Nelson-Oppen
- Shostak

Nelson-Oppen is more general and conceptually simpler.

Shostak seems more efficient where it does work, and only recently has it really been understood.

Nelson-Oppen basics

Key idea is to combine theories $T_1, \ldots, T_n$ with *disjoint signatures*. For instance

- T_1: numerical constants, arithmetic operations
- T_2: list operations like cons, head and tail.
- T_3: other uninterpreted function symbols.

The only common function or relation symbol is '$=$'.

This means that we only need to share formulas built from equations among the component decision procedure, thanks to the *Craig interpolation theorem*.

The interpolation theorem

Several slightly different forms; we'll use this one (by compactness, generalizes to theories):

> If $\models \phi_1 \land \phi_2 \Rightarrow \bot$ then there is an 'interpolant' ψ, whose only free variables and function and predicate symbols are those occurring in *both* ϕ_1 and ϕ_2, such that $\models \phi_1 \Rightarrow \psi$ and $\models \phi_2 \Rightarrow \neg\psi$.

This is used to assure us that the Nelson-Oppen method is complete, though we don't need to produce general interpolants in the method.

In fact, interpolants can be found quite easily from proofs, including Herbrand-type proofs produced by resolution etc.

Nelson-Oppen I

Proof by example: refute the following formula in a mixture of Presburger arithmetic and uninterpreted functions:

$$f(v-1)-1=v+1 \wedge f(u)+1=u-1 \wedge u+1=v$$

First step is to *homogenize*, i.e. get rid of atomic formulas involving a mix of signatures:

$$u+1=v \wedge v_1+1=u-1 \wedge v_2-1=v+1 \wedge v_2=f(v_3) \wedge v_1=f(u) \wedge v_3=v-1$$

so now we can split the conjuncts according to signature:

$$(u+1=v \wedge v_1+1=u-1 \wedge v_2-1=v+1 \wedge v_3=v-1) \wedge$$
$$(v_2=f(v_3) \wedge v_1=f(u))$$

Nelson-Oppen II

If the entire formula is contradictory, then there's an interpolant ψ such that in Presburger arithmetic:

$$\mathbb{Z} \models u + 1 = v \land v_1 + 1 = u - 1 \land v_2 - 1 = v + 1 \land v_3 = v - 1 \Rightarrow \psi$$

and in pure logic:

$$\models v_2 = f(v_3) \land v_1 = f(u) \land \psi \Rightarrow \bot$$

We can assume it only involves variables and equality, by the interpolant property and disjointness of signatures.

Subject to a technical condition about finite models, the pure equality theory admits quantifier elimination.

So we can assume ψ is a propositional combination of equations between variables.

Nelson-Oppen III

In our running example, $u = v_3 \wedge \neg(v_1 = v_2)$ is one suitable interpolant, so

$$\mathbb{Z} \models u + 1 = v \wedge v_1 + 1 = u - 1 \wedge v_2 - 1 = v + 1 \wedge v_3 = v - 1 \Rightarrow u = v_3 \wedge \neg(v_1 = v_2)$$

in Presburger arithmetic, and in pure logic:

$$\models v_2 = f(v_3) \wedge v_1 = f(u) \Rightarrow u = v_3 \wedge \neg(v_1 = v_2) \Rightarrow \bot$$

The component decision procedures can deal with those, and the result is proved.

Nelson-Oppen IV

Could enumerate all significantly different potential interpolants.

Better: case-split the original problem over all possible equivalence relations between the variables (5 in our example).

$$T_1, \ldots, T_n \models \phi_1 \wedge \cdots \wedge \phi_n \wedge ar(P) \Rightarrow \bot$$

So by interpolation there's a C with

$$T_1 \models \phi_1 \wedge ar(P) \Rightarrow C$$

$$T_2, \ldots, T_n \models \phi_2 \wedge \cdots \wedge \phi_n \wedge ar(P) \Rightarrow \neg C$$

Since $ar(P) \Rightarrow C$ or $ar(P) \Rightarrow \neg C$, we must have one theory with $T_i \models \phi_i \wedge ar(P) \Rightarrow \bot$.

Nelson-Oppen V

Still, there are quite a lot of possible equivalence relations (bell$(5) = 52$), leading to large case-splits.

An alternative formulation is to repeatedly let each theory deduce new disjunctions of equations, and case-split over them.

$$T_i \models \phi_i \Rightarrow x_1 = y_1 \vee \cdots \vee x_n = y_n$$

This allows two important optimizations:

- If theories are *convex*, need only consider pure equations, no disjunctions.
- Component procedures can actually produce equational consequences rather than waiting passively for formulas to test.

Shostak's method

Can be seen as an optimization of Nelson-Oppen method for common special cases. Instead of just a decision method each component theory has a

- Canonizer — puts a term in a T-canonical form
- Solver — solves systems of equations

Shostak's original procedure worked well, but the theory was flawed on many levels. In general his procedure was incomplete and potentially nonterminating.

It's only recently that a full understanding has (apparently) been reached.

See Yices (`http://yices.csl.sri.com`) for one implementation.

Certification of decision procedures

We might want a decision procedure to produce a ‘proof’ or ‘certificate’

- Doubts over the correctness of the core decision method
- Desire to use the proof in other contexts

This arises in at least two real cases:

- Fully expansive (e.g. ‘LCF-style’) theorem proving.
- Proof-carrying code

Certifiable and non-certifiable

The most desirable situation is that a decision procedure should produce a short certificate that can be checked easily.

Factorization and primality is a good example:

- Certificate that a number is not prime: the factors! (Others are also possible.)
- Certificate that a number is prime: Pratt, Pocklington, Pomerance, ...

This means that primality checking is in NP $\cap$ co-NP (we now know it's in P).

Certifying universal formulas over $\mathbb{C}$

Use the (weak) *Hilbert Nullstellensatz*:

The polynomial equations $p_1(x_1, \ldots, x_n) = 0$, ..., $p_k(x_1, \ldots, x_n) = 0$ in an algebraically closed field have *no* common solution iff there are polynomials $q_1(x_1, \ldots, x_n)$, ..., $q_k(x_1, \ldots, x_n)$ such that the following polynomial identity holds:

$$q_1(x_1, \ldots, x_n) \cdot p_1(x_1, \ldots, x_n) + \cdots + q_k(x_1, \ldots, x_n) \cdot p_k(x_1, \ldots, x_n) = 1$$

All we need to certify the result is the cofactors $q_i(x_1, \ldots, x_n)$, which we can find by an instrumented Gröbner basis algorithm.

The checking process involves just algebraic normalization (maybe still not totally trivial. . .)

Certifying universal formulas over $\mathbb{R}$

There is a similar but more complicated Nullstellensatz (and Positivstellensatz) over $\mathbb{R}$.

The general form is similar, but it's more complicated because of all the different orderings.

It inherently involves sums of squares (SOS), and the certificates can be found efficiently using semidefinite programming (Parillo . . .)

Example: easy to check

$$\forall a\ b\ c\ x.\ ax^2 + bx + c = 0 \Rightarrow b^2 - 4ac \geq 0$$

via the following SOS certificate:

$$b^2 - 4ac = (2ax + b)^2 - 4a(ax^2 + bx + c)$$

Less favourable cases

Unfortunately not all decision procedures seem to admit a nice separation of proof from checking.

Then if a proof is required, there seems no significantly easier way than generating proofs along each step of the algorithm.

Example: Cohen-Hörmander algorithm implemented in HOL Light by McLaughlin (CADE 2005).

Works well, useful for small problems, but about $1000\times$ slowdown relative to non-proof-producing implementation.

Should we use reflection, i.e. verify the code itself?

Summary

- There is a need for combinations of decision methods
- For general quantifier prefixes, relatively few useful results.
- Nelson-Oppen and Shostak give useful methods for universal formulas.
- We sometimes also want decision procedures to produce proofs
- Some procedures admit efficient separation of search and checking, others do not.
- Interesting research topic: new ways of compactly certifying decision methods.

Automated and Interactive Theorem Proving
5: Interactive theorem proving

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Marktoberdorf 2007

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Interactive theorem proving (1)

In practice, many interesting problems can’t be automated completely:

- They don’t fall in a practical decidable subset
- Pure first order proof search is not a feasible approach with, e.g. set theory

Interactive theorem proving (1)

In practice, most interesting problems can't be automated completely:

- They don't fall in a practical decidable subset
- Pure first order proof search is not a feasible approach with, e.g. set theory

In practice, we need an interactive arrangement, where the user and machine work together.

The user can delegate simple subtasks to pure first order proof search or one of the decidable subsets.

However, at the high level, the user must guide the prover.

Interactive theorem proving (2)

The idea of a more 'interactive' approach was already anticipated by pioneers, e.g. Wang (1960):

> [...] the writer believes that perhaps machines may more quickly become of practical use in mathematical research, not by proving new theorems, but by formalizing and checking outlines of proofs, say, from textbooks to detailed formalizations more rigorous that *Principia* [Mathematica], from technical papers to textbooks, or from abstracts to technical papers.

However, constructing an effective and programmable combination is not so easy.

SAM

First successful family of interactive provers were the SAM systems:

> Semi-automated mathematics is an approach to theorem-proving which seeks to combine automatic logic routines with ordinary proof procedures in such a manner that the resulting procedure is both efficient and subject to human intervention in the form of control and guidance. Because it makes the mathematician an essential factor in the quest to establish theorems, this approach is a departure from the usual theorem-proving attempts in which the computer *unaided* seeks to establish proofs.

SAM V was used to settle an open problem in lattice theory.

Three influential proof checkers

- AUTOMATH (de Bruijn, . . .) — Implementation of type theory, used to check non-trivial mathematics such as Landau's *Grundlagen*
- Mizar (Trybulec, . . .) — Block-structured natural deduction with 'declarative' justifications, used to formalize large body of mathematics
- LCF (Milner et al) — Programmable proof checker for Scott's Logic of Computable Functions written in new functional language ML.

Ideas from all these systems are used in present-day systems. (Corbineau's declarative proof mode for Coq . . .)

Sound extensibility

Ideally, it should be possible to customize and program the theorem-prover with domain-specific proof procedures.

However, it's difficult to allow this without compromising the soundness of the system.

A very successful way to combine extensibility and reliability was pioneered in LCF.

Now used in Coq, HOL, Isabelle, Nuprl, ProofPower, ...

Key ideas behind LCF

- Implement in a strongly-typed functional programming language (usually a variant of ML)
- Make `thm` ('theorem') an abstract data type with only simple primitive inference rules
- Make the implementation language available for arbitrary extensions.

First-order axioms (1)

$$\vdash \quad p \Rightarrow (q \Rightarrow p)$$

$$\vdash \quad (p \Rightarrow q \Rightarrow r) \Rightarrow (p \Rightarrow q) \Rightarrow (p \Rightarrow r)$$

$$\vdash \quad ((p \Rightarrow \bot) \Rightarrow \bot) \Rightarrow p$$

$$\vdash \quad (\forall x.\, p \Rightarrow q) \Rightarrow (\forall x.\, p) \Rightarrow (\forall x.\, q)$$

$$\vdash \quad p \Rightarrow \forall x.\, p \quad \textbf{[Provided } x \notin \mathsf{FV}(p)\textbf{]}$$

$$\vdash \quad (\exists x.\, x = t) \quad \textbf{[Provided } x \notin \mathsf{FVT}(t)\textbf{]}$$

$$\vdash \quad t = t$$

$$\vdash \quad s_1 = t_1 \Rightarrow ... \Rightarrow s_n = t_n \Rightarrow f(s_1, .., s_n) = f(t_1, .., t_n)$$

$$\vdash \quad s_1 = t_1 \Rightarrow ... \Rightarrow s_n = t_n \Rightarrow P(s_1, .., s_n) \Rightarrow P(t_1, .., t_n)$$

First-order axioms (2)

$$\vdash \quad (p \Leftrightarrow q) \Rightarrow p \Rightarrow q$$

$$\vdash \quad (p \Leftrightarrow q) \Rightarrow q \Rightarrow p$$

$$\vdash \quad (p \Rightarrow q) \Rightarrow (q \Rightarrow p) \Rightarrow (p \Leftrightarrow q)$$

$$\vdash \quad \top \Leftrightarrow (\bot \Rightarrow \bot)$$

$$\vdash \quad \neg p \Leftrightarrow (p \Rightarrow \bot)$$

$$\vdash \quad p \wedge q \Leftrightarrow (p \Rightarrow q \Rightarrow \bot) \Rightarrow \bot$$

$$\vdash \quad p \vee q \Leftrightarrow \neg(\neg p \wedge \neg q)$$

$$\vdash \quad (\exists x.\ p) \Leftrightarrow \neg(\forall x.\ \neg p)$$

First-order rules

Modus Ponens rule:

$$\frac{\vdash p \Rightarrow q \qquad \vdash p}{\vdash q}$$

Generalization rule:

$$\frac{\vdash p}{\vdash \forall x.\ p}$$

LCF kernel for first order logic (1)

Define type of first order formulas:

```
type term = Var of string | Fn of string * term list;;

type formula = False
             | True
             | Atom of string * term list
             | Not of formula
             | And of formula * formula
             | Or of formula * formula
             | Imp of formula * formula
             | Iff of formula * formula
             | Forall of string * formula
             | Exists of string * formula;;
```

LCF kernel for first order logic (2)

Define some useful helper functions:

```
let mk_eq s t = Atom(R("=",[s;t]));;

let rec occurs_in s t =
  s = t or
  match t with
    Var y -> false
  | Fn(f,args) -> exists (occurs_in s) args;;

let rec free_in t fm =
  match fm with
    False | True -> false
  | Atom(R(p,args)) -> exists (occurs_in t) args
  | Not(p) -> free_in t p
  | And(p,q) | Or(p,q) | Imp(p,q) | Iff(p,q) -> free_in t p or free_in t q
  | Forall(y,p) | Exists(y,p) -> not(occurs_in (Var y) t) & free_in t p;;
```

LCF kernel for first order logic (3)

```
module Proven : Proofsystem =
  struct type thm = formula
         let axiom_addimp p q = Imp(p,Imp(q,p))
         let axiom_distribimp p q r = Imp(Imp(p,Imp(q,r)),Imp(Imp(p,q),Imp(p,r)))
         let axiom_doubleneg p = Imp(Imp(Imp(p,False),False),p)
         let axiom_allimp x p q = Imp(Forall(x,Imp(p,q)),Imp(Forall(x,p),Forall(x,q)))
         let axiom_impall x p =
           if not (free_in (Var x) p) then Imp(p,Forall(x,p)) else failwith "axiom_impall"
         let axiom_existseq x t =
           if not (occurs_in (Var x) t) then Exists(x,mk_eq (Var x) t) else failwith "axiom_existseq"
         let axiom_eqrefl t = mk_eq t t
         let axiom_funcong f lefts rights =
            itlist2 (fun s t p -> Imp(mk_eq s t,p)) lefts rights (mk_eq (Fn(f,lefts)) (Fn(f,rights)))
         let axiom_predcong p lefts rights =
            itlist2 (fun s t p -> Imp(mk_eq s t,p)) lefts rights (Imp(Atom(p,lefts),Atom(p,rights)))
         let axiom_iffimp1 p q = Imp(Iff(p,q),Imp(p,q))
         let axiom_iffimp2 p q = Imp(Iff(p,q),Imp(q,p))
         let axiom_impiff p q = Imp(Imp(p,q),Imp(Imp(q,p),Iff(p,q)))
         let axiom_true = Iff(True,Imp(False,False))
         let axiom_not p = Iff(Not p,Imp(p,False))
         let axiom_or p q = Iff(Or(p,q),Not(And(Not(p),Not(q))))
         let axiom_and p q = Iff(And(p,q),Imp(Imp(p,Imp(q,False)),False))
         let axiom_exists x p = Iff(Exists(x,p),Not(Forall(x,Not p)))
         let modusponens pq p =
           match pq with Imp(p',q) when p = p' -> q | _ -> failwith "modusponens"
         let gen x p = Forall(x,p)
         let concl c = c
  end;;
```

Derived rules

The primitive rules are very simple. But using the LCF technique we can build up a set of derived rules. The following derives $p \Rightarrow p$:

```
let imp_refl p = modusponens (modusponens (axiom_distribimp p (Imp(p,p)) p)
                                          (axiom_addimp p (Imp(p,p))))
                             (axiom_addimp p p);;
```

Derived rules

The primitive rules are very simple. But using the LCF technique we can build up a set of derived rules. The following derives $p \Rightarrow p$:

```
let imp_refl p = modusponens (modusponens (axiom_distribimp p (Imp(p,p)) p)
                                          (axiom_addimp p (Imp(p,p))))
                             (axiom_addimp p p);;
```

While this process is tedious at the beginning, we can quickly reach the stage of automatic derived rules that

- Prove propositional tautologies
- Perform Knuth-Bendix completion
- Prove first order formulas by standard proof search and translation

Fully-expansive decision procedures

Real LCF-style theorem provers like HOL have many powerful derived rules.

Mostly just mimic standard algorithms like rewriting but by inference. For cases where this is difficult:

- Separate certification (my previous lecture)
- Reflection (Tobias's lectures)

Proof styles

Directly invoking the primitive or derived rules tends to give proofs that are *procedural*.

A *declarative* style (*what* is to be proved, not *how*) can be nicer:

- Easier to write and understand independent of the prover
- Easier to modify
- Less tied to the details of the prover, hence more portable

Mizar pioneered the declarative style of proof.

Recently, several other declarative proof languages have been developed, as well as declarative shells round existing systems like HOL and Isabelle.

Finding the right style is an interesting research topic.

Procedural proof example

```
let NSQRT_2 = prove
 (`!p q. p * p = 2 * q * q ==> q = 0`,
  MATCH_MP_TAC num_WF THEN REWRITE_TAC[RIGHT_IMP_FORALL_THM] THEN
  REPEAT STRIP_TAC THEN FIRST_ASSUM(MP_TAC o AP_TERM `EVEN`) THEN
  REWRITE_TAC[EVEN_MULT; ARITH] THEN REWRITE_TAC[EVEN_EXISTS] THEN
  DISCH_THEN(X_CHOOSE_THEN `m:num` SUBST_ALL_TAC) THEN
  FIRST_X_ASSUM(MP_TAC o SPECL [`q:num`; `m:num`]) THEN
  ASM_REWRITE_TAC[ARITH_RULE
   `q < 2 * m ==> q * q = 2 * m * m ==> m = 0 <=>
    (2 * m) * 2 * m = 2 * q * q ==> 2 * m <= q`] THEN
  ASM_MESON_TAC[LE_MULT2; MULT_EQ_0; ARITH_RULE `2 * x <= x <=> x = 0`]);;
```

Declarative proof example

```
let NSQRT_2 = prove
 (`!p q. p * p = 2 * q * q ==> q = 0`,
  suffices_to_prove
   `!p. (!m. m < p ==> (!q. m * m = 2 * q * q ==> q = 0))
        ==> (!q. p * p = 2 * q * q ==> q = 0)`
   (wellfounded_induction) THEN
  fix [`p:num`] THEN
  assume("A") `!m. m < p ==> !q. m * m = 2 * q * q ==> q = 0` THEN
  fix [`q:num`] THEN
  assume("B") `p * p = 2 * q * q` THEN
  so have `EVEN(p * p) <=> EVEN(2 * q * q)` (trivial) THEN
  so have `EVEN(p)` (using [ARITH; EVEN_MULT] trivial) THEN
  so consider (`m:num`,"C",`p = 2 * m`) (using [EVEN_EXISTS] trivial) THEN
  cases ("D",`q < p \/ p <= q`) (arithmetic) THENL
   [so have `q * q = 2 * m * m ==> m = 0` (by ["A"] trivial) THEN
    so we're finished (by ["B"; "C"] algebra);
    so have `p * p <= q * q` (using [LE_MULT2] trivial) THEN
    so have `q * q = 0` (by ["B"] arithmetic) THEN
    so we're finished (algebra)]);;
```

Is automation even more declarative?

```
let LEMMA_1 = SOS_RULE
 `p EXP 2 = 2 * q EXP 2
  ==> (q = 0 \/ 2 * q - p < p /\ ~(p - q = 0)) /\
      (2 * q - p) EXP 2 = 2 * (p - q) EXP 2`;;

let NSQRT_2 = prove
 (`!p q. p * p = 2 * q * q ==> q = 0`,
  REWRITE_TAC[GSYM EXP_2] THEN MATCH_MP_TAC num_WF THEN MESON_TAC[LEMMA_1]);;
```

The Seventeen Provers of the World (1)

- ACL2 — Highly automated prover for first-order number theory without explicit quantifiers, able to do induction proofs itself.
- Alfa/Agda — Prover for constructive type theory integrated with dependently typed programming language.
- B prover — Prover for first-order set theory designed to support verification and refinement of programs.
- Coq — LCF-like prover for constructive Calculus of Constructions with reflective programming language.
- HOL (HOL Light, HOL4, ProofPower) — Seminal LCF-style prover for classical simply typed higher-order logic.
- IMPS — Interactive prover for an expressive logic supporting partially defined functions.

The Seventeen Provers of the World (2)

- Isabelle/Isar — Generic prover in LCF style with a newer declarative proof style influenced by Mizar.
- Lego — Well-established framework for proof in constructive type theory, with a similar logic to Coq.
- Metamath — Fast proof checker for an exceptionally simple axiomatization of standard ZF set theory.
- Minlog — Prover for minimal logic supporting practical extraction of programs from proofs.
- Mizar — Pioneering system for formalizing mathematics, originating the declarative style of proof.
- Nuprl/MetaPRL — LCF-style prover with powerful graphical interface for Martin-Löf type theory with new constructs.

The Seventeen Provers of the World (3)

- Omega — Unified combination in modular style of several theorem-proving techniques including proof planning.
- Otter/IVY — Powerful automated theorem prover for pure first-order logic plus a proof checker.
- PVS — Prover designed for applications with an expressive classical type theory and powerful automation.
- PhoX — prover for higher-order logic designed to be relatively simple to use in comparison with Coq, HOL etc.
- Theorema — Ambitious integrated framework for theorem proving and computer algebra built inside Mathematica.

For more, see Freek Wiedijk, *The Seventeen Provers of the World*, Springer Lecture Notes in Computer Science vol. 3600, 2006.

Summary

- In practice, we need a combination of interaction and automation for difficult proofs.
- Interactive provers / proof checkers are the workhorses in verification applications, even if they use automated subsystems.
- LCF gives a good way of realizing a combination of soundness and extensibility.
- Different proof styles may be preferable, and they can be supported on top of an LCF-style core.
- There are many interactive provers out there with very different characteristics!