Formal proof: current progress and outstanding challenges

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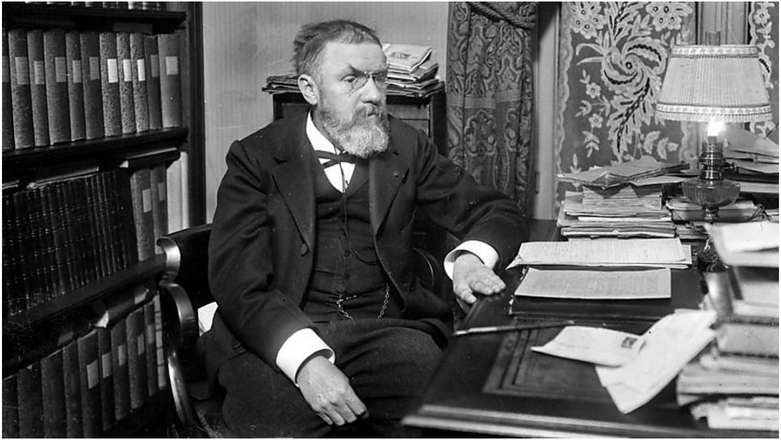
5th May 2014 (11:00–12:00)

Summary of talk

- A century of formal proof
  - Poincaré on formal proof
  - From Principia Mathematica to the computer age
  - Major milestones in formalization
  - Development of mathematical libraries
- Current perspectives
  - The provers of the world
  - Foundations
  - Software architecture
  - Proof languages
  - Automation
  - Libraries
- More about HOL Light
  - Foundations and architecture
  - Decision procedures and automation
  - A tour of the libraries
- The future

A century of formal proof

What would Poincaré have thought?



Poincaré’s had a distinct aversion to formal logic

I see in logistic only shackles for the inventor. It is no aid to conciseness — far from it, and if twenty-seven equations were necessary to establish that 1 is a number, how many would be needed to prove a real theorem?

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I see in logistic only shackles for the inventor. It is no aid to conciseness — far from it, and if twenty-seven equations were necessary to establish that 1 is a number, how many would be needed to prove a real theorem? If we distinguish, with Whitehead, the individual x, the class of which the only member is x and [...] the class of which the only member is the class of which the only member is x [...], do you think these distinctions, useful as they may be, go far to quicken our pace?

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- This was a very productive mistake: the new realization led to a much deeper understanding of dynamical systems and laid the foundations of modern chaos theory.
- However it was embarrassing and expensive for all concerned — Poincaré spent more than the competition prize money paying for the journal issues to be recalled and reprinted.

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- This practical formal mathematics was to forestall objections to Russell and Whitehead's 'logicist' thesis, not a goal in itself.
- The development was difficult and painstaking, and has probably been studied in detail by very few.
- Subsequently, the idea of actually formalizing proofs has not been taken very seriously.

Even Russell did not enjoy doing formal proofs

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However, now we have computers to check and even automatically generate formal proofs.

Our goal is now not so much philosophical, but to achieve a real, practical, useful increase in the precision and accuracy of mathematical proofs.

The importance of computers for formal proof

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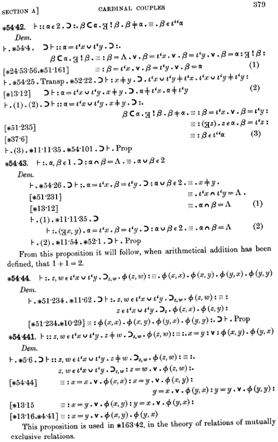
Because of these dual connections, interest in formal proofs is strongest among computer scientists, but some 'mainstream' mathematicians are becoming interested too.

A formal proof from 1910



SECTION A] CARDINAL COUPLES 379

∗54·42. $\vdash :: \alpha \in 2 \,.\supset :.\, \beta \subset \alpha \,.\, \exists ! \beta \,.\, \beta \neq \alpha \,.\equiv .\, \beta \in \iota\text{“}\alpha$

Dem.

$$\vdash .*54\cdot4 .\quad \supset \vdash :: \alpha = \iota\text{‘}x \cup \iota\text{‘}y \,.\supset :.$$
$$\beta \subset \alpha \,.\, \exists ! \beta \,.\equiv : \beta = \Lambda \,.\vee .\, \beta = \iota\text{‘}x \,.\vee .\, \beta = \iota\text{‘}y \,.\vee .\, \beta = \alpha : \exists ! \beta :$$
$$[*24\cdot53\cdot56 . *51\cdot161] \quad \equiv : \beta = \iota\text{‘}x \,.\vee .\, \beta = \iota\text{‘}y \,.\vee .\, \beta = \alpha \qquad (1)$$
$$\vdash .*54\cdot25 .\, \text{Transp} .*52\cdot22 .\supset \vdash : x \neq y \,.\supset .\, \iota\text{‘}x \cup \iota\text{‘}y \neq \iota\text{‘}x \,.\, \iota\text{‘}x \cup \iota\text{‘}y \neq \iota\text{‘}y :$$
$$[*13\cdot12] \quad \supset \vdash : \alpha = \iota\text{‘}x \cup \iota\text{‘}y \,.\, x \neq y \,.\supset .\, \alpha \neq \iota\text{‘}x \,.\, \alpha \neq \iota\text{‘}y \qquad (2)$$
$$\vdash .(1).(2).\supset \vdash :: \alpha = \iota\text{‘}x \cup \iota\text{‘}y \,.\, x \neq y \,.\supset :.$$
$$\beta \subset \alpha \,.\, \exists ! \beta \,.\, \beta \neq \alpha \,.\equiv : \beta = \iota\text{‘}x \,.\vee .\, \beta = \iota\text{‘}y :$$
$$[*51\cdot235] \quad \equiv : (\exists z) \,.\, z \in \alpha \,.\, \beta = \iota\text{‘}z :$$
$$[*37\cdot6] \quad \equiv : \beta \in \iota\text{“}\alpha \qquad (3)$$
$$\vdash .(3). *11\cdot11\cdot35 . *54\cdot101 .\supset \vdash .\, \text{Prop}$$

∗54·43. $\vdash :. \alpha, \beta \in 1 \,.\supset : \alpha \cap \beta = \Lambda \,.\equiv .\, \alpha \cup \beta \in 2$

Dem.

$$\vdash .*54\cdot26 .\supset \vdash :. \alpha = \iota\text{‘}x \,.\, \beta = \iota\text{‘}y \,.\supset : \alpha \cup \beta \in 2 \,.\equiv .\, x \neq y \,.$$
$$[*51\cdot231] \quad \equiv .\, \iota\text{‘}x \cap \iota\text{‘}y = \Lambda \,.$$
$$[*13\cdot12] \quad \equiv .\, \alpha \cap \beta = \Lambda \qquad (1)$$
$$\vdash .(1). *11\cdot11\cdot35 .\supset$$
$$\vdash :. (\exists x, y) \,.\, \alpha = \iota\text{‘}x \,.\, \beta = \iota\text{‘}y \,.\supset : \alpha \cup \beta \in 2 \,.\equiv .\, \alpha \cap \beta = \Lambda \qquad (2)$$
$$\vdash .(2). *11\cdot54 . *52\cdot1 .\supset \vdash .\, \text{Prop}$$

From this proposition it will follow, when arithmetical addition has been defined, that $1+1=2$.

∗54·44. $\vdash :. z, w \in \iota\text{‘}x \cup \iota\text{‘}y \,.\supset_{z,w} .\, \phi(z,w) : \equiv .\, \phi(x,x) \,.\, \phi(x,y) \,.\, \phi(y,x) \,.\, \phi(y,y)$

Dem.

$$\vdash .*51\cdot234 . *11\cdot62 .\supset \vdash :. z, w \in \iota\text{‘}x \cup \iota\text{‘}y \,.\supset_{z,w} .\, \phi(z,w) : \equiv :$$
$$z \in \iota\text{‘}x \cup \iota\text{‘}y \,.\supset_z .\, \phi(z,x) \,.\, \phi(z,y) :$$
$$[*51\cdot234 . *10\cdot29] \equiv : \phi(x,x) \,.\, \phi(x,y) \,.\, \phi(y,x) \,.\, \phi(y,y) :.\supset \vdash .\, \text{Prop}$$

∗54·441. $\vdash :: z, w \in \iota\text{‘}x \cup \iota\text{‘}y \,.\, z \neq w \,.\supset_{z,w} .\, \phi(z,w) : \equiv :.\, x = y : \vee : \phi(x,y) \,.\, \phi(y,x)$

Dem.

$$\vdash .*5\cdot6 .\supset \vdash :: z, w \in \iota\text{‘}x \cup \iota\text{‘}y \,.\, z \neq w \,.\supset_{z,w} .\, \phi(z,w) : \equiv :.$$
$$z, w \in \iota\text{‘}x \cup \iota\text{‘}y \,.\supset_{z,w} : z = w \,.\vee .\, \phi(z,w) :.$$
$$[*54\cdot44] \quad \equiv : x = x \,.\vee .\, \phi(x,x) : x = y \,.\vee .\, \phi(x,y) :$$
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$$[*13\cdot15 \quad \equiv : x = y \,.\vee .\, \phi(x,y) : y = x \,.\vee .\, \phi(y,x) :$$
$$[*13\cdot16 . *4\cdot41] \equiv : x = y \,.\vee .\, \phi(x,y) \,.\, \phi(y,x)$$

This proposition is used in ∗163·42, in the theory of relations of mutually exclusive relations.

This is p379 of Whitehead and Russell's *Principia Mathematica*.

Zooming in . . .

$*$**54·43.** $\vdash :. \alpha, \beta \in 1 . \supset : \alpha \cap \beta = \Lambda . \equiv . \alpha \cup \beta \in 2$

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\begin{aligned}
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A formal proof from 2010

```
let PNT = prove
 (`((\n. &(CARD {p | prime p /\ p <= n}) / (&n / log(&n)))
    ---> &1) sequentially`,
  REWRITE_TAC[PNT_PARTIAL_SUMMATION] THEN
  REWRITE_TAC[SUM_PARTIAL_PRE] THEN
  REWRITE_TAC[GSYM REAL_OF_NUM_ADD; SUB_REFL; CONJUNCT1 LE] THEN
  SUBGOAL_THEN `{p | prime p /\ p = 0} = {}` SUBST1_TAC THENL
   [REWRITE_TAC[EXTENSION; IN_ELIM_THM; NOT_IN_EMPTY] THEN
    MESON_TAC[PRIME_IMP_NZ];
    ALL_TAC] THEN
  REWRITE_TAC[SUM_CLAUSES; REAL_MUL_RZERO; REAL_SUB_RZERO] THEN
  MATCH_MP_TAC REALLIM_TRANSFORM_EVENTUALLY THEN
  EXISTS_TAC
   `\n. ((&n + &1) / log(&n + &1) *
         sum {p | prime p /\ p <= n} (\p. log(&p) / &p) -
         sum (1..n)
         (\k. sum {p | prime p /\ p <= k} (\p. log(&p) / &p) *
              ((&k + &1) / log(&k + &1) - &k / log(&k)))) / (&n / log(&n))` THEN
  CONJ_TAC THENL
   [REWRITE_TAC[EVENTUALLY_SEQUENTIALLY] THEN EXISTS_TAC `1` THEN SIMP_TAC[];
    ALL_TAC] THEN
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   `\n. ((&n + &1) / log(&n + &1) * log(&n) -
         sum (1..n)
         (\k. log(&k) * ((&k + &1) / log(&k + &1) - &k / log(&k)))) /
        (&n / log(&n))` THEN
  REWRITE_TAC[] THEN CONJ_TAC THENL
   [REWRITE_TAC[REAL_ARITH
     `(a * x - s) / b - (a * x' - s') / b:real =
      ((s' - s) - (x' - x) * a) / b`] THEN
    REWRITE_TAC[GSYM SUM_SUB_NUMSEG; GSYM REAL_SUB_RDISTRIB] THEN
    REWRITE_TAC[REAL_OF_NUM_ADD] THEN
    MATCH_MP_TAC SUM_PARTIAL_LIMIT_ALT THEN
```

Zooming in ...

At least the theorems are more substantial:

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Moreover, we can arrange to have more readable proofs — see for example Bill Richter's talk.

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These are demonstrations that the technology can handle long and difficult proofs, and even that some leading mathematicians like Hales are willing to use them.

Formalized theorems and libraries of mathematics

Also important is the progress made on more modest building-blocks for mathematics, still including quite substantial results, e.g.

- Jordan Curve Theorem — Tom Hales (HOL Light), Andrzej Trybulec et al. (Mizar)
- Prime Number Theorem — Jeremy Avigad et al (Isabelle/HOL), John Harrison (HOL Light)
- First and second Cartan Theorems — Marco Maggesi et al (HOL Light)

In the process, provers are building up ever-larger libraries of pre-proved theorems that can be deployed in future proofs.

Current perspectives

A few notable general-purpose theorem provers

There is a diverse (perhaps too diverse?) world of proof assistants, with these being just a few:

- ACL2
- Agda
- Coq
- HOL (HOL Light, HOL4, ProofPower, HOL Zero)
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See Freek Wiedijk's book *The Seventeen Provers of the World* (Springer-Verlag lecture notes in computer science volume 3600) for descriptions of many systems and proofs that $\sqrt{2}$ is irrational.

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- There is now interest in a new foundational approach, homotopy type theory, with experimental implementations.

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There have even recently been papers about versions of Milawa (a simplified ACL2) and HOL Light verified right down to machine code.

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Mizar pioneered the declarative style of proof. Recently, several other declarative proof languages have been developed, as well as declarative shells round existing systems like HOL and Isabelle.

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- Reimplement algorithms to perform proofs as they proceed
- Have suitable 'certificates' produced by an external tool checked in the inference kernel.
- Extend kernel with verified implementation (*reflection*).

Libraries

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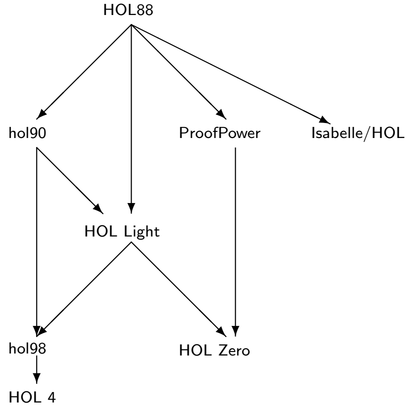
- The earliest large mathematical library, still perhaps the largest is the Mizar Mathematical Library (MML), following the style of mathematical papers with extracted text and references.
- Many theorem provers including Coq, HOL Light and Isabelle/HOL (including the ‘archive of formal proofs’) also have large and every-expanding mathematical libraries.

More about HOL Light

The HOL family DAG

There are many HOL provers, of which HOL Light is just one, all descended from Mike Gordon's original HOL system in the late 1980s.



HOL Light primitive rules (1)

$$\frac{}{\vdash t = t}\ \texttt{REFL}$$

$$\frac{\Gamma \vdash s = t \quad \Delta \vdash t = u}{\Gamma \cup \Delta \vdash s = u}\ \texttt{TRANS}$$

$$\frac{\Gamma \vdash s = t \quad \Delta \vdash u = v}{\Gamma \cup \Delta \vdash s(u) = t(v)}\ \texttt{MK\_COMB}$$

$$\frac{\Gamma \vdash s = t}{\Gamma \vdash (\lambda x.\ s) = (\lambda x.\ t)}\ \texttt{ABS}$$

$$\frac{}{\vdash (\lambda x.\ t)x = t}\ \texttt{BETA}$$

HOL Light primitive rules (2)

$$\frac{}{\{p\} \vdash p} \; \texttt{ASSUME}$$

$$\frac{\Gamma \vdash p = q \quad \Delta \vdash p}{\Gamma \cup \Delta \vdash q} \; \texttt{EQ\_MP}$$

$$\frac{\Gamma \vdash p \quad \Delta \vdash q}{(\Gamma - \{q\}) \cup (\Delta - \{p\}) \vdash p = q} \; \texttt{DEDUCT\_ANTISYM\_RULE}$$

$$\frac{\Gamma[x_1, \ldots, x_n] \vdash p[x_1, \ldots, x_n]}{\Gamma[t_1, \ldots, t_n] \vdash p[t_1, \ldots, t_n]} \; \texttt{INST}$$

$$\frac{\Gamma[\alpha_1, \ldots, \alpha_n] \vdash p[\alpha_1, \ldots, \alpha_n]}{\Gamma[\gamma_1, \ldots, \gamma_n] \vdash p[\gamma_1, \ldots, \gamma_n]} \; \texttt{INST\_TYPE}$$

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HOL Light may represent the most "extreme" application of this philosophy.

- HOL Light's primitive rules are very simple, and the trusted core is just a few hundred lines of code.
- There is an extensive suite of automated tools built on top that all reduce to this foundation.

Some of HOL Light's basic automation

- Simplifier for (conditional, contextual) rewriting.
- Tactic mechanism for mixed forward and backward proofs.
- Tautology checker.
- Automated theorem provers for pure logic, based on tableaux and model elimination.
- Linear arithmetic decision procedures over $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}$.
- Differentiator for real functions.
- Generic normalizers for rings and fields
- General quantifier elimination over $\mathbb{C}$
- Gröbner basis algorithm over fields

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- Decision procedures for general 'triangle law' reasoning in normed spaces and general decision procedure for Hilbert spaces, using decidability results developed in work with Solovay and Arthan.
- 'Without loss of generality' tactics for simplifying goals in geometry by use of special coordinate systems, which can greatly simplify some Flyspeck goals.

A tour of the libraries (1)

Partly as a result of Flyspeck, HOL Light is particularly strong in the area of topology, analysis and geometry in Euclidean space $\mathbb{R}^n$.

File	Lines	Contents
misc.ml	562	Background stuff
vectors.ml	8627	Basic vectors, linear algebra
determinants.ml	3141	Determinant and trace
topology.ml	20235	Basic topological notions
convex.ml	11827	Convex sets and functions
paths.ml	17066	Paths, simple connectedness etc.
polytope.ml	5855	Faces, polytopes, polyhedra etc.
dimension.ml	6794	Dimensional theorems
derivatives.ml	2732	Derivatives
clifford.ml	979	Geometric (Clifford) algebra
integration.ml	17407	Integration
measure.ml	10252	Lebesgue measure

A tour of the libraries (2)

From this foundation complex analysis is developed and used to derive convenient theorems for $\mathbb{R}$ as well as more topological results.

File	Lines	Contents
complexes.ml	2036	Complex numbers
canal.ml	3760	Complex analysis
transcendentals.ml	6981	Real & complex transcendentals
realanalysis.ml	15845	Some analytical stuff on R
moretop.ml	7349	Further topological results
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It would be desirable to generalize much of the material to general topological spaces, metric spaces, measure spaces etc. Some work already by Bill Richter on general topology.

Some examples from topology

The Brouwer fixed point theorem:

```
  |- !f:real^N->real^N s.
        compact s /\ convex s /\ ~(s = {}) /\
        f continuous_on s /\ IMAGE f s SUBSET s
        ==> ?x. x IN s /\ f x = x
```

The Borsuk homotopy extension theorem:

```
|- !f:real^M->real^N g s t u.
        closed_in (subtopology euclidean t) s /\
        (ANR s /\ ANR t \/ ANR u) /\
        f continuous_on t /\ IMAGE f t SUBSET u /\
        homotopic_with (\x. T) (s,u) f g
        ==> ?g'. homotopic_with (\x. T) (t,u) f g' /\
                 g' continuous_on t /\
                 IMAGE g' t SUBSET u /\
                 !x. x IN s ==> g'(x) = g(x)
```

Some examples from convexity

The Krein-Milman (Minkowski) theorem

```
|- !s:real^N->bool.
        convex s /\ compact s
        ==> s = convex hull {x | x extreme_point_of s}
```

Approximation of convex sets by polytopes w.r.t. Hausdorff distance:

```
|- !s:real^N->bool e.
        bounded s /\ convex s /\ &0 < e
        ==> ?p. polytope p /\ s SUBSET p /\
                hausdist(p,s) < e
```

Some examples from measure theory

Steinhaus's theorem:

```
|- !s:real^N->bool.
        lebesgue_measurable s /\ ~negligible s
        ==> ?d. &0 < d /\
                ball(vec 0,d) SUBSET
                {x - y | x IN s /\ y IN s}
```

Luzin's theorem:

```
|- !f:real^M->real^N s e.
        measurable s /\ f measurable_on s /\ &0 < e
        ==> ?k. compact k /\ k SUBSET s /\
                measure(s DIFF k) < e /\
                f continuous_on k
```

Some examples from complex analysis

The Little Picard theorem:

```
|- !f a b.
      f holomorphic_on (:complex) /\
      ~(a = b) /\ IMAGE f (:complex) INTER {a,b} = {}
      ==> ?c. f = \x. c
```

The Riemann mapping theorem:

```
|- !s. open s /\ simply_connected s <=>
       s = {} \/ s = (:complex) \/
       ?f g. f holomorphic_on s /\
             g holomorphic_on ball(Cx(&0),&1) /\
             (!z. z IN s
                  ==> f z IN ball(Cx(&0),&1) /\
                      g(f z) = z) /\
             (!z. z IN ball(Cx(&0),&1)
                  ==> g z IN s /\ f(g z) = z)
```

The future

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- New research in foundations may result in fundamentally better approaches to formalization and even have increasing influence back on mathematics itself.
- Given the diversity of theorem proving systems, it seems there will be still more research into sharing and importing and exporting proofs between them.
- We can further increase the soundness guarantees by rigorous verification down to the lowest levels as well as proof checking and proof auditing.