Mathematical Modeling to Formally Prove Correctness

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The human cost of bugs

Computers are often used in safety-critical systems where a failure could cause loss of life.

- Heart pacemakers
- Aircraft
- Nuclear reactor controllers
- Car engine management systems
- Radiation therapy machines
- Telephone exchanges (!)
- ...

Financial cost of bugs

Even when not a matter of life and death, the consequences of bugs can be quite dramatic.

- In 1996, the Ariane 5 rocket made its first flight
- It was automatically destroyed shortly after takeoff
- The cause was an uncaught exception on floating-point to integer conversion.

Another floating-point bug

Intel has also had at least one major floating-point issue:

- Error in the floating-point division (FDIV) instruction on some early Intel®Pentium® processors
- Very rarely encountered, but was hit by a mathematician doing research in number theory.
- Intel eventually set aside US $475 million to cover the costs.

Things are not getting easier

The environment is becoming even less benign:

- The overall market is much larger, so the potential cost of recall/replacement is far higher.
- New products are ramped faster and reach high unit sales very quickly.
- Competitive pressures are leading to more design complexity.

Some complexity metrics

Recent Intel processor generations (Pentium, P6 and Pentium 4) indicate:

- A 4-fold increase in overall complexity (lines of RTL . . .) per generation
- A 4-fold increase in design bugs per generation.
- Approximately 8000 bugs introduced during design of the Pentium 4.

Fortunately, pre-silicon detection rates are now very close to 100%.

Just enough to keep our heads above water. . .

Limits of testing

Bugs are usually detected by extensive testing, including pre-silicon simulation.

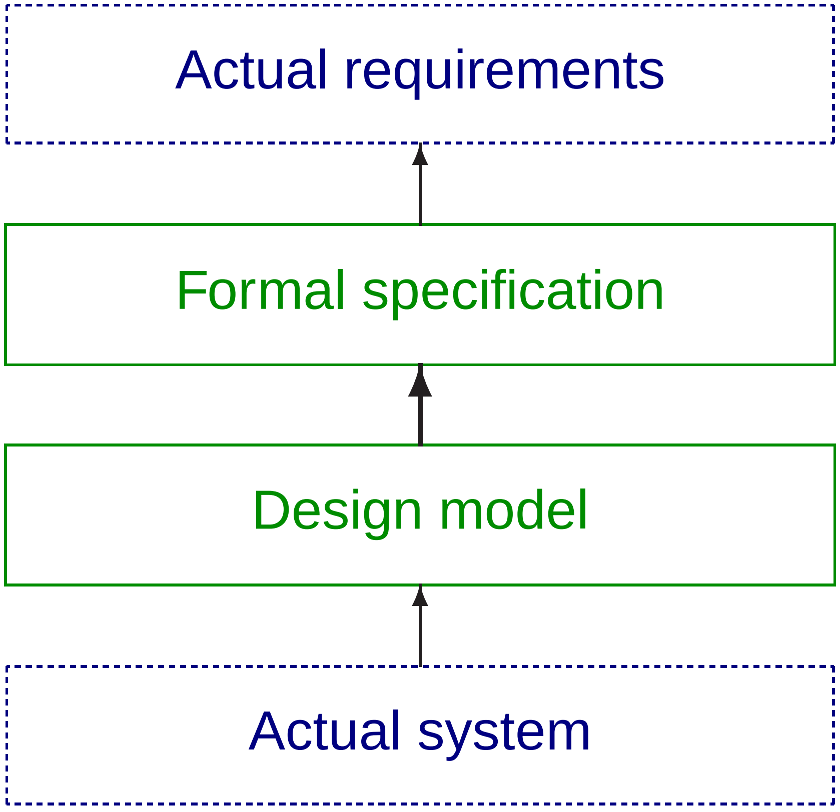
- Slow — especially pre-silicon
- Too many possibilities to test them all

For example:

- 2^{160} possible pairs of floating point numbers (possible inputs to an adder).
- Vastly higher number of possible states of a complex microarchitecture.

Formal verification

Formal verification: mathematically prove the correctness of a *design* with respect to a mathematical *formal specification*.



Actual requirements

Formal specification

Design model

Actual system

Verification vs. testing

Verification has some advantages over testing:

- Exhaustive.
- Improves our intellectual grasp of the system.

However:

- Difficult and time-consuming.
- Only as reliable as the formal models used.
- How can we be sure the proof is right?

Analogy with mathematics

Sometimes even a huge weight of empirical evidence can be misleading.

- $\pi(n) =$ number of primes $\leq n$
- $li(n) = \int_0^n du/ln(u)$

Littlewood proved in 1914 that $\pi(n) - li(n)$ changes sign infinitely often.

No change of sign at all had ever been found despite testing up to $n = 10^{10}$ (in the days before computers).

Similarly, extensive testing of hardware or software may still miss errors that would be revealed by a formal proof.

Formal verification is hard

Writing out a completely formal proof of correctness for real-world hardware and software is difficult.

- Must specify intended behaviour formally
- Need to make many hidden assumptions explicit
- Requires long detailed proofs, difficult to review

The state of the art is quite limited.

Software verification has been around since the 60s, but there have been few major successes.

Machine-checked proof

A more promising approach is to have the proof checked (or even generated) by a computer program.

- It can reduce the risk of mistakes.
- The computer can automate some parts of the proofs.

There are limits on the power of automation, so detailed human guidance is often necessary.

A spectrum of formal techniques

There are various possible levels of rigor in correctness proofs:

- Programming language typechecking
- Lint-like static checks (uninitialized variables . . .)
- Checking of loop invariants and other annotations
- Complete functional verification

FV in the software industry

Some recent success with partial verification in the software world:

- Analysis of Microsoft Windows device drivers using SLAM
- Non-overflow proof for Airbus A380 flight control software

Much less use of full functional verification. Very rare except in highly safety-critical or security-critical niches.

FV in the hardware industry

In the hardware industry, full functional correctness proofs are increasingly becoming common practice.

- Hardware is designed in a more modular way than most software.
- There is more scope for complete automation
- The potential consequences of a hardware error are greater

Formal verification methods

Many different methods are used in formal verification, mostly trading efficiency and automation against generality.

- Propositional tautology checking
- Symbolic simulation
- Symbolic trajectory evaluation
- Temporal logic model checking
- Decidable subsets of first order logic
- First order automated theorem proving
- Interactive theorem proving

Our work

We will focus on our own formal verification activities:

- Formal verification of floating-point operations
- Targeted at the Intel® Itanium® processor family.
- Conducted using the interactive theorem prover HOL Light.

Why floating-point?

There are obvious reasons for focusing on floating-point:

- Known to be difficult to get right, with several issues in the past. We don’t want another FDIV!
- Quite clear specification of how most operations *should* behave. We have the IEEE Standard 754.

However, Intel is also applying FV in many other areas, e.g. control logic, cache coherence, bus protocols . . .

Why interactive theorem proving?

Limited scope for highly automated finite-state techniques like model checking.

It’s difficult even to specify the intended behaviour of complex mathematical functions in bit-level terms.

We need a general framework to reason about mathematics in general while checking against errors.

HOL Light overview

HOL Light is a member of the HOL family of provers, descended from Mike Gordon's original HOL system developed in the 80s.

An LCF-style proof checker for classical higher-order logic built on top of (polymorphic) simply-typed λ-calculus.

HOL Light is designed to have a simple and clean logical foundation.

Current version written in Objective CAML ("OCaml").

What does LCF mean?

The name is a historical accident:

> The original Stanford and Edinburgh LCF systems were for Scott's Logic of Computable Functions.

The main features of the LCF approach to theorem proving are:

- Reduce all proofs to a small number of relatively simple primitive rules
- Use the programmability of the implementation/interaction language to make this practical

No free lunch

There is no practical way of automatically proving highly sophisticated mathematics.

Some isolated successes such as the solution of the Robbins conjecture . . .

Mostly, we content ourselves with automating “routine” parts of the proof.

Automating the routine

We can automate linear inequality reasoning:

$$
\begin{aligned}
& a \leq x \wedge b \leq y \wedge |x-y| < |x-a| \wedge |x-y| < |x-b| \wedge \\
& (b \leq x \Rightarrow |x-a| < |x-b|) \wedge (a \leq y \Rightarrow |y-b| < |x-a|) \\
& \Rightarrow a = b
\end{aligned}
$$

and basic algebraic rearrangement:

$$
\begin{aligned}
& (w_1^2 + x_1^2 + y_1^2 + z_1^2) \cdot (w_2^2 + x_2^2 + y_2^2 + z_2^2) = \\
& (w_1 \cdot w_2 - x_1 \cdot x_2 - y_1 \cdot y_2 - z_1 \cdot z_2)^2 + \\
& (w_1 \cdot x_2 + x_1 \cdot w_2 + y_1 \cdot z_2 - z_1 \cdot y_2)^2 + \\
& (w_1 \cdot y_2 - x_1 \cdot z_2 + y_1 \cdot w_2 + z_1 \cdot x_2)^2 + \\
& (w_1 \cdot z_2 + x_1 \cdot y_2 - y_1 \cdot x_2 + z_1 \cdot w_2)^2
\end{aligned}
$$

The obviousness mismatch

Can also automate some purely logical reasoning such as this:

$$
\begin{aligned}
&(\forall x\ y\ z.\ P(x,y) \land P(y,z) \Rightarrow P(x,z)) \land \\
&(\forall x\ y\ z.\ Q(x,y) \land Q(y,z) \Rightarrow Q(x,z)) \land \\
&(\forall x\ y.\ Q(x,y) \Rightarrow Q(y,x)) \land \\
&(\forall x\ y.\ P(x,y) \lor Q(x,y)) \\
&\Rightarrow (\forall x\ y.\ P(x,y)) \lor (\forall x\ y.\ Q(x,y))
\end{aligned}
$$

As Łoś points out, this is not obvious for most people.

Floating point verification

We’ve used HOL Light to verify the accuracy of floating point algorithms (used in hardware and software) for:

- Division and square root
- Transcendental function such as sin, exp, $atan$.

This involves background work in formalizing:

- Real analysis
- Basic floating point arithmetic

Existing real analysis theory

- Definitional construction of real numbers
- Basic topology
- General limit operations
- Sequences and series
- Limits of real functions
- Differentiation
- Power series and Taylor expansions
- Transcendental functions
- Gauge integration

Examples of useful theorems

```
|- sin(x + y) = sin(x) * cos(y) + cos(x) * sin(y)
|- tan(&n * pi) = &0

|- &0 < x ∧ &0 < y ⇒ (ln(x / y) = ln(x) - ln(y))

|- f contl x ∧ g contl (f x) ⇒ (g o f) contl x

|- (∀x. a <= x ∧ x <= b ⇒ (f diffl (f’ x)) x) ∧
   f(a) <= K ∧ f(b) <= K ∧
   (∀x. a <= x ∧ x <= b ∧ (f’(x) = &0) ⇒ f(x) <= K)
   ⇒ ∀x. a <= x ∧ x <= b ⇒ f(x) <= K
```

HOL floating point theory (1)

A floating point format is identified by a triple of natural numbers `fmt`.

The corresponding set of real numbers is `format(fmt)`, or ignoring the upper limit on the exponent, `iformat(fmt)`.

Floating point rounding returns a floating point approximation to a real number, ignoring upper exponent limits. More precisely

```
round fmt rc x
```

returns the appropriate member of `iformat(fmt)` for an exact value `x`, depending on the rounding mode `rc`, which may be one of `Nearest`, `Down`, `Up` and `Zero`.

HOL floating point theory (2)

For example, the definition of rounding down is:

```
|- (round fmt Down x = closest
        {a | a IN iformat fmt ∧ a <= x} x)
```

We prove a large number of results about rounding, e.g.

```
|- ¬(precision fmt = 0) ∧ x IN iformat fmt
   ⇒ (round fmt rc x = x)
```

that rounding is monotonic:

```
|- ¬(precision fmt = 0) ∧ x <= y
   ⇒ round fmt rc x <= round fmt rc y
```

and that subtraction of nearby floating point numbers is exact:

```
|- a IN iformat fmt ∧ b IN iformat fmt ∧
   a / &2 <= b ∧ b <= &2 * a ⇒ (b - a) IN iformat fmt
```

The $(1 + \epsilon)$ property

Designers often rely on clever “cancellation” tricks to avoid or compensate for rounding errors.

But many routine parts of the proof can be dealt with by a simple conservative bound on rounding error:

```
|- normalizes fmt x ∧
   ¬(precision fmt = 0)
   ⇒ ∃e. abs(e) <= mu rc / &2 pow (precision fmt - 1) ∧
           (round fmt rc x = x * (&1 + e))
```

Derived rules apply this result to computations in a floating point algorithm automatically, discharging the conditions as they go.

Example: tangent algorithm

- The input number X is first reduced to r with approximately $|r| \leq \pi/4$ such that $X = r + N\pi/2$ for some integer N. We now need to calculate $\pm tan(r)$ or $\pm cot(r)$ depending on N modulo 4.
- If the reduced argument r is still not small enough, it is separated into its leading few bits B and the trailing part $x = r - B$, and the overall result computed from $tan(x)$ and pre-stored functions of B, e.g.

$$tan(B + x) = tan(B) + \frac{\frac{1}{sin(B)cos(B)} tan(x)}{cot(B) - tan(x)}$$

- Now a power series approximation is used for $tan(r)$, $cot(r)$ or $tan(x)$ as appropriate.

Overview of the verification

To verify this algorithm, we need to prove:

- The range reduction to obtain r is done accurately.
- The mathematical facts used to reconstruct the result from components are applicable.
- Stored constants such as $tan(B)$ are sufficiently accurate.
- The power series approximation does not introduce too much error in approximation.
- The rounding errors involved in computing with floating point arithmetic are within bounds.

Most of these parts are non-trivial. Moreover, some of them require more pure mathematics than might be expected.

Why mathematics?

Controlling the error in range reduction becomes difficult when the reduced argument $X - N\pi/2$ is small.

To check that the computation is accurate enough, we need to know:

> How close can a floating point number be to an integer multiple of $\pi/2$?

Even deriving the power series (for $0 < |x| < \pi$):

$$cot(x) = 1/x - \frac{1}{3}x - \frac{1}{45}x^3 - \frac{2}{945}x^5 - \dots$$

is much harder than you might expect.

Polynomial approximation errors

Many transcendental functions are ultimately approximated by polynomials in this way.

This usually follows some initial reduction step to ensure that the argument is in a small range, say $x \in [a, b]$.

The *minimax* polynomials used have coefficients found numerically to minimize the maximum error over the interval.

In the formal proof, we need to prove that this is indeed the maximum error, say $\forall x \in [a, b].\ |sin(x) - p(x)| \leq 10^{-62}|x|$.

By using a Taylor series with much higher degree, we can reduce the problem to bounding a pure polynomial with rational coefficients over an interval.

Bounding functions

If a function f differentiable for $a \leq x \leq b$ has the property that $f(x) \leq K$ at all points of zero derivative, as well as at $x = a$ and $x = b$, then $f(x) \leq K$ everywhere.

```
|- (∀x. a <= x ∧ x <= b ⇒ (f diffl (f′ x)) x) ∧
   f(a) <= K ∧ f(b) <= K ∧
   (∀x. a <= x ∧ x <= b ∧ (f′(x) = &0)
        ⇒ f(x) <= K)
   ⇒ (∀x. a <= x ∧ x <= b ⇒ f(x) <= K)
```

Hence we want to be able to isolate zeros of the derivative (which is just another polynomial).

Isolating derivatives

For any differentiable function f, $f(x)$ can be zero only at one point between zeros of the derivative $f'(x)$.

More precisely, if $f'(x) \neq 0$ for $a < x < b$ then if $f(a)f(b) \geq 0$ there are no points of $a < x < b$ with $f(x) = 0$:

```
|- (∀x. a <= x ∧ x <= b ⇒ (f diffl f’(x))(x)) ∧
   (∀x. a < x ∧ x < b ⇒ ¬(f’(x) = &0)) ∧
   f(a) * f(b) >= &0
   ⇒ ∀x. a < x ∧ x < b ⇒ ¬(f(x) = &0)
```

Bounding and root isolation

This gives rise to a recursive procedure for bounding a polynomial and isolating its zeros, by successive differentiation.

```
|- (∀x. a <= x ∧ x <= b ⇒ (f diffl (f’ x)) x) ∧
   (∀x. a <= x ∧ x <= b ⇒ (f’ diffl (f’’ x)) x) ∧
   (∀x. a <= x ∧ x <= b ⇒ abs(f’’(x)) <= K) ∧
   a <= c ∧ c <= x ∧ x <= d ∧ d <= b ∧ (f’(x) = &0)
   ⇒ abs(f(x)) <= abs(f(d)) + (K / &2) * (d - c) pow 2
```

At each stage we actually produce HOL theorems asserting bounds and the enclosure properties of the isolating intervals.

Conclusions

- Formal verification is industrially important, and can be attacked with current theorem proving technology.
- A large part of our work involves building up general theories about both pure mathematics and special properties of floating point numbers.
- It is easy to underestimate the amount of pure mathematics needed for obtaining very practical results.
- The mathematics required is often the sort that is not found in current textbooks: very concrete results but with a proof!
- Using HOL Light, we can confidently integrate all the different aspects of the proof, using programmability to automate tedious parts.