Floating-point reasoning at the bit level

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Overview

- Why am I here?
- Why floating-point matters
- Theorem proving technology
- Floating-point arithmetic
- Square root and reciprocal verification examples
- Bit-level details and decimal arithmetic

Why am I here?

Roughly similar relationship:

- Floating-point numbers $\sim \mathbb{R}$
- Machine words $\sim \mathbb{Z}$ or $\mathbb{N}$

Why am I here?

Roughly similar relationship:

- Floating-point numbers $\sim \mathbb{R}$
- Machine words $\sim \mathbb{Z}$

In some ways the FP-to-reals relationship is more complicated:

- Limited range *and* limited accuracy, not just 'modulo 2^n'.
- Some additional non-numeric data / error indications

Patriot missile failure

During the first Gulf War in 1991, 28 soldiers were killed when a Scud missile struck an army barracks.

- Patriot missile failed to intercept the Scud
- Underlying cause was a computer arithmetic error in computing time since boot
- Internal clock was multiplied by $\frac{1}{10}$ to produce time in seconds
- Actually performed by multiplying 24-bit approximation of $\frac{1}{10}$
- Net error after 100 hours about 0.34 seconds.
- A Scud missile travels $500m$ in that time

Ariane rocket failure

In 1996, the Ariane 5 rocket on its maiden flight was destroyed; the rocket and its cargo were estimated to be worth $\$500M$.

- Cause was an uncaught floating-point exception
- A 64-bit floating-point number representing horizontal velocity was converted to a 16-bit integer
- The number was larger than 2^{15}.
- As a result, the conversion failed.
- The rocket veered off its flight path and exploded, just 40 seconds into the flight sequence.

Vancouver stock exchange

In 1982 the Vancouver stock exchange index was established at a level of 1000.

A couple of years later the index was hitting lows of around 520.

The cause was repeated truncation of the index to 3 decimal digits on each recalculation, several thousand times a day.

On correction, the stock index leapt immediately from 574.081 to 1098.882.

A floating-point bug closer to home

Intel has also had at least one major floating-point issue:

- Error in the floating-point division (FDIV) instruction on some early Intel®Pentium® processors
- Very rarely encountered, but was hit by a mathematician doing research in number theory.
- Intel eventually set aside US $\$475$ million to cover the costs.

Remember the HP-35?

The Hewlett-Packard HP-35 calculator (1972) also had floating-point bugs:

- Exponential function, e.g. $e^{ln(2.02)} = 2.00$
- sin of some small angles completely wrong

At this time HP had already sold 25,000 units, but they advised users of the problem and offered a replacement:

> "We're going to tell everyone, and offer them a replacement. It would be better to never make a dime of profit than to have a product out there with a problem." (Dave Packard.)

Why floating-point?

There are obvious reasons for focusing on floating-point:

- Known to be difficult to get right, with several issues in the past.
  We don't want another FDIV!

- Quite clear specification of how most operations *should* behave.
  We have the IEEE Standard 754.

However, Intel is also applying FV in many other areas.

Why interactive theorem proving?

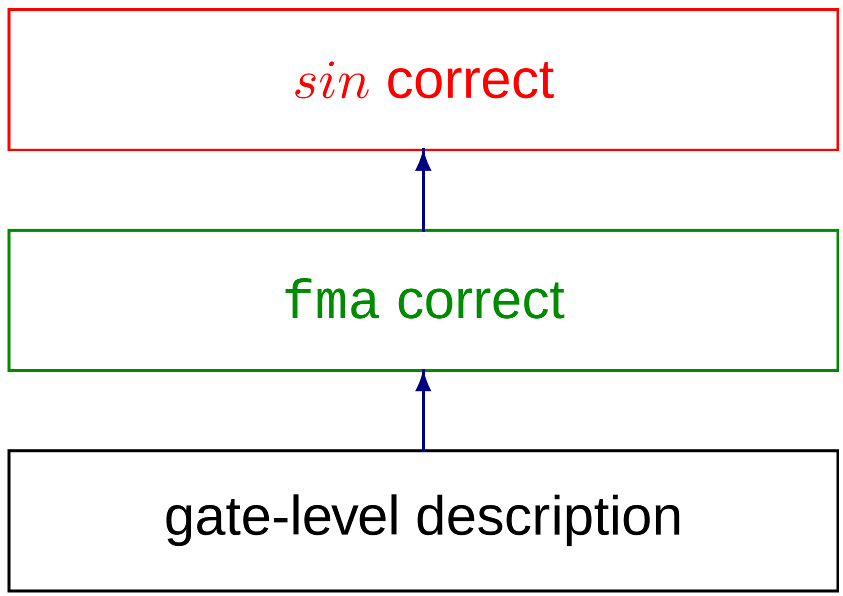
Limited scope for highly automated finite-state techniques like model checking.

It's difficult even to specify the intended behaviour of complex mathematical functions in bit-level terms.

We need a general framework to reason about mathematics in general while checking against errors.

Levels of verification

High-level algorithms assume correct behavior of some hardware primitives.



Proving my assumptions is someone else's job . . .

Characteristics of this work

The verification we're concerned with is somewhat atypical:

- Rather simple according to typical programming metrics, e.g. 5-150 lines of code, often no loops.
- Relies on non-trivial mathematics including number theory, analysis and special properties of floating-point rounding.

Tools that are often effective in other verification tasks, e.g. temporal logic model checkers, are of almost no use.

What do we need?

We need a general theorem proving system with:

- Ability to mix interactive and automated proof
- Programmability for domain-specific proof tasks
- A substantial library of pre-proved mathematics

Theorem provers for floating-point

There are several oof-the-shelf theorem provers that have been used for floating-point verification, some of it in industry:

- ACL2 (used at AMD)
- Coq
- HOL Light (used at Intel)
- PVS

All these are powerful systems with somewhat different strengths and weaknesses.

Interactive versus automatic

From interactive proof checkers to fully automatic theorem provers.

AUTOMATH (de Bruijn)

Mizar (Trybulec)

. . .

PVS (Owre, Rushby, Shankar)

. . .

ACL2 (Boyer, Kaufmann, Moore)

Vampire (Voronkov)

Mathematical versus industrial

Some provers are intended to formalize pure mathematics, others to tackle industrial-scale verification

AUTOMATH (de Bruijn)

Mizar (Trybulec)

...

...

PVS (Owre, Rushby, Shankar)

ACL2 (Boyer, Kaufmann, Moore)

No free lunch

There is no practical way of automatically proving highly sophisticated mathematics.

Some isolated successes such as the solution of the Robbins conjecture . . .

Mostly, we content ourselves with automating “routine” parts of the proof.

HOL Light overview

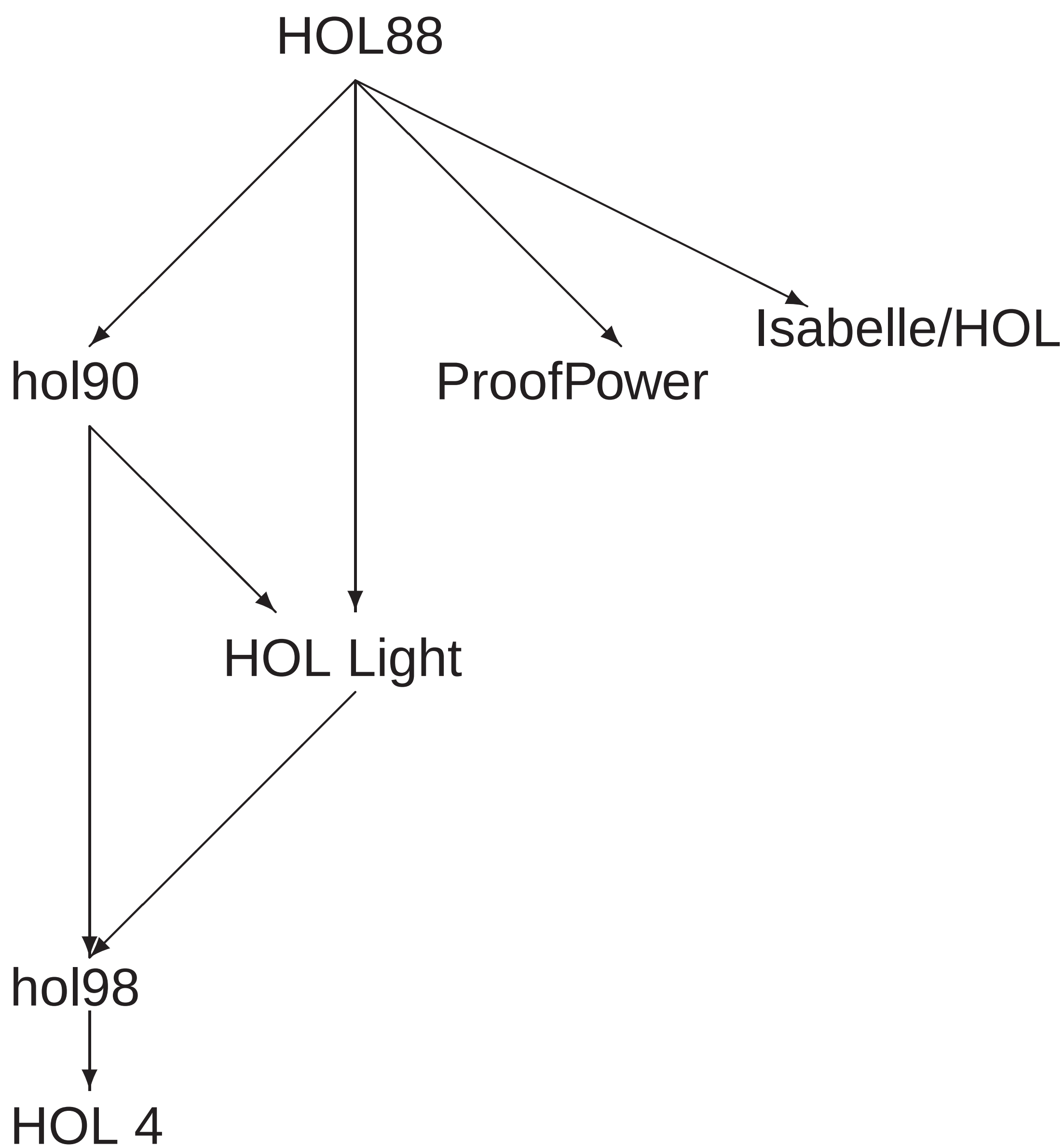
HOL Light is a member of the HOL family of provers, descended from Mike Gordon's original HOL system developed in the 80s.

An LCF-style proof checker for classical higher-order logic built on top of (polymorphic) simply-typed λ-calculus.

HOL Light is designed to have a simple and clean logical foundation.

Written in Objective CAML (OCaml).

The HOL family DAG



HOL Light primitive rules (1)

$$\frac{}{\vdash t = t}\ \texttt{REFL}$$

$$\frac{\Gamma \vdash s = t \quad \Delta \vdash t = u}{\Gamma \cup \Delta \vdash s = u}\ \texttt{TRANS}$$

$$\frac{\Gamma \vdash s = t \quad \Delta \vdash u = v}{\Gamma \cup \Delta \vdash s(u) = t(v)}\ \texttt{MK\_COMB}$$

$$\frac{\Gamma \vdash s = t}{\Gamma \vdash (\lambda x.\ s) = (\lambda x.\ t)}\ \texttt{ABS}$$

$$\frac{}{\vdash (\lambda x.\ t)x = t}\ \texttt{BETA}$$

HOL Light primitive rules (2)

$$\frac{}{\{p\} \vdash p} \; \texttt{ASSUME}$$

$$\frac{\Gamma \vdash p = q \quad \Delta \vdash p}{\Gamma \cup \Delta \vdash q} \; \texttt{EQ\_MP}$$

$$\frac{\Gamma \vdash p \quad \Delta \vdash q}{(\Gamma - \{q\}) \cup (\Delta - \{p\}) \vdash p = q} \; \texttt{DEDUCT\_ANTISYM\_RULE}$$

$$\frac{\Gamma[x_1, \ldots, x_n] \vdash p[x_1, \ldots, x_n]}{\Gamma[t_1, \ldots, t_n] \vdash p[t_1, \ldots, t_n]} \; \texttt{INST}$$

$$\frac{\Gamma[\alpha_1, \ldots, \alpha_n] \vdash p[\alpha_1, \ldots, \alpha_n]}{\Gamma[\gamma_1, \ldots, \gamma_n] \vdash p[\gamma_1, \ldots, \gamma_n]} \; \texttt{INST\_TYPE}$$

Pushing the LCF approach to its limits

The main features of the LCF approach to theorem proving are:

- Reduce all proofs to a small number of relatively simple primitive rules
- Use the programmability of the implementation/interaction language to make this practical

Our work may represent the most “extreme” application of this philosophy.

- HOL Light’s primitive rules are very simple.
- Some of the proofs expand to about 100 million primitive inferences and can take many hours to check.

Some of HOL Light's derived rules

- Simplifier for (conditional, contextual) rewriting.
- Tactic mechanism for mixed forward and backward proofs.
- Tautology checker.
- Automated theorem provers for pure logic, based on tableaux and model elimination.
- Linear arithmetic decision procedures over $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}$.
- Differentiator for real functions.
- Generic normalizers for rings and fields
- General quantifier elimination over $\mathbb{C}$
- Gröbner basis algorithm over fields

Automating the routine

We can automate linear inequality reasoning:

$$a \le x \land b \le y \land |x - y| < |x - a| \land |x - y| < |x - b| \land$$
$$(b \le x \Rightarrow |x - a| < |x - b|) \land (a \le y \Rightarrow |y - b| < |x - a|)$$
$$\Rightarrow a = b$$

and basic algebraic rearrangement:

$$(w_1^2 + x_1^2 + y_1^2 + z_1^2) \cdot (w_2^2 + x_2^2 + y_2^2 + z_2^2) =$$
$$(w_1 \cdot w_2 - x_1 \cdot x_2 - y_1 \cdot y_2 - z_1 \cdot z_2)^2 +$$
$$(w_1 \cdot x_2 + x_1 \cdot w_2 + y_1 \cdot z_2 - z_1 \cdot y_2)^2 +$$
$$(w_1 \cdot y_2 - x_1 \cdot z_2 + y_1 \cdot w_2 + z_1 \cdot x_2)^2 +$$
$$(w_1 \cdot z_2 + x_1 \cdot y_2 - y_1 \cdot x_2 + z_1 \cdot w_2)^2$$

The obviousness mismatch

Can also automate some purely logical reasoning such as this:

$$
\begin{aligned}
&(\forall x\ y\ z.\ P(x,y) \land P(y,z) \Rightarrow P(x,z)) \land \\
&(\forall x\ y\ z.\ Q(x,y) \land Q(y,z) \Rightarrow Q(x,z)) \land \\
&(\forall x\ y.\ Q(x,y) \Rightarrow Q(y,x)) \land \\
&(\forall x\ y.\ P(x,y) \lor Q(x,y)) \\
&\Rightarrow (\forall x\ y.\ P(x,y)) \lor (\forall x\ y.\ Q(x,y))
\end{aligned}
$$

As Łoś points out, this is not obvious for most people.

Real analysis details

Real analysis is especially important in our applications

- Definitional construction of real numbers
- Basic topology
- General limit operations
- Sequences and series
- Limits of real functions
- Differentiation
- Power series and Taylor expansions
- Transcendental functions
- Gauge integration

Floating point numbers

Usually, the floating point numbers are those representable in some number n of significant binary digits, within a certain exponent range:

$$(-1)^s \times d_0.d_1d_2 \cdots d_n \times 2^e$$

where

- $s \in \{0, 1\}$ is the *sign*
- $d_0.d_1d_2 \cdots d_n$ is the *significand* and $d_1d_2 \cdots d_n$ is the *fraction* (aka mantissa).
- e is the exponent.

We often refer to $p = n + 1$ as the *precision*.

HOL floating point formats

We have formalized a generic floating point theory in HOL, which can be applied to all hardware formats, and others supported in software.

A floating point format is identified by a triple of natural numbers fmt.

The corresponding set of real numbers is format(fmt), or ignoring the upper limit on the exponent, iformat(fmt).

```
 |- iformat (E,p,N) =
     { x | ∃s e k. s < 2 ∧ k < 2 EXP p ∧
                   x = --(&1) pow s * &2 pow e *
                       &k / &2 pow N}
```

We distinguish carefully between actual floating point numbers (as bitstrings) and the corresponding real numbers.

Supported formats

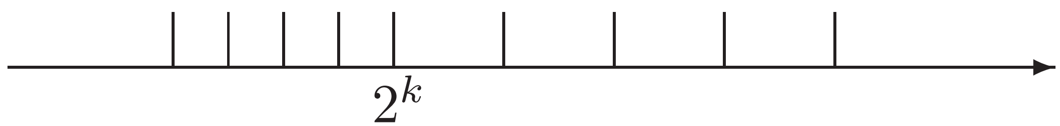
Some of the formats we are concerned with:

Format name	p	E_{min}	E_{max}
Single	24	-126	127
Double	53	-1022	1023
Double-extended	64	-16382	16383
Register	64	-65534	65535

Units in the last place

It’s customary to give a bound on the error in transcendental functions in terms of ‘units in the last place’ (ulps).

While ulps are a standard way of measuring error, there’s a remarkable lack of unanimity in published definitions of the term. One of the merits of a formal treatment is to clear up such ambiguities.



Roughly, a unit in the last place is the gap between adjacent floating point numbers. But at the boundary 2^k between ‘binades’, this distance changes.

Two definitions

Goldberg considers the binade containing the computed result:

> In general, if the floating-point number $d.d \cdots d \times \beta^e$ is used to represent z, it is in error by $|d.d \cdots d - (z/\beta^e)|\beta^{p-1}e$ units in the last place.

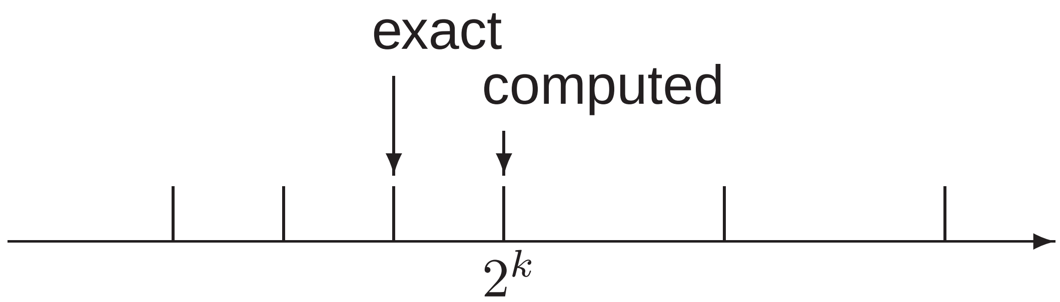
Muller considers the binade containing the exact result:

> The term *ulp(x) (*for *unit in the last place*) denotes the distance between the two floating point numbers that are closest to x.

However these both have counterintuitive properties.

Problems with these definitions (1)

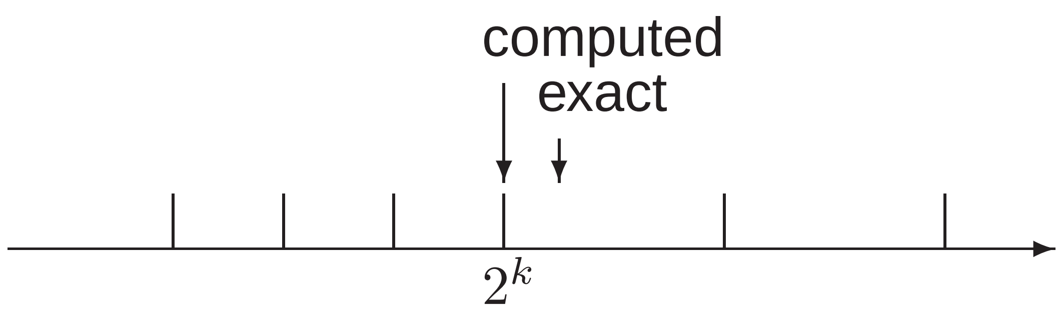
An error of $0.5ulp$ according to Goldberg, but intuitively $1ulp$.



Problems with these definitions (2)

An error of $0.4ulp$ according to Muller, but intuitively 0.2 ulp.

Rounding up is worse...



Our definition: $ulp(x)$ is the distance between the closest pair of floating point numbers a and b with $a \leq x \leq b$. Note that we are counting the exact result 2^k as belonging to the binade *below*.

IEEE-correct operations

The IEEE Standard 754 specifies for all the usual algebraic operations, including square root but *not* transcendentals:

> ...each of these operations shall be performed as if it first produced an intermediate result correct to infinite precision and unbounded range and then coerced this intermediate result to fit in the destination's format [using the specified rounding operation]

The revised version of the IEEE standard extends the set of operations with a fused multiply-add (FMA), which was already implemented on several architectures.

Ternary operation $x \cdot y + z$ (and $x \cdot y - z$ and $z - x \cdot y$).

Rounding (1)

Rounding is controlled by a rounding mode, which is defined in HOL as an enumerated type:

```
roundmode = Nearest | Down | Up | Zero
```

We define notions of ‘closest approximation’ as follows:

```
|- is_closest s x a =
      a IN s ∧ ∀b. b IN s ⇒ abs(b - x) >= abs(a - x)

|- closest s x = εa. is_closest s x a

|- closest_such s p x =
     εa. is_closest s x a ∧
            (∀b. is_closest s x b ∧ p b ⇒ p a)
```

Rounding (2)

Hence the actual definition of rounding:

```
|- (round fmt Nearest x =
        closest_such (iformat fmt)
                     (EVEN o decode_fraction fmt) x) ∧
   (round fmt Down x =
        closest {a | a IN iformat fmt ∧ a <= x} x) ∧
   (round fmt Up x =
        closest {a | a IN iformat fmt ∧ a >= x} x) ∧
   (round fmt Zero x =
        closest {a | a IN iformat fmt ∧ abs a <= abs x} x)
```

Note that this is almost a direct transcription of the standard; no need to talk about ulps etc.

But it is also completely non-constructive!

Theorems about rounding

We prove some basic properties of rounding, e.g. that an already-representable number rounds to itself and conversely:

```
|- a IN iformat fmt ⇒ (round fmt rc a = a)

|- ¬(precision fmt = 0)
   ⇒ ((round fmt rc x = x) = x IN iformat fmt)
```

and that rounding is monotonic in all rounding modes:

```
|- ¬(precision fmt = 0) ∧ x <= y
   ⇒ round fmt rc x <= round fmt rc y
```

There are various other simple properties, e.g. symmetries and skew-symmetries like:

```
|- ¬(precision fmt = 0)
   ⇒ (round fmt Down (--x) = --(round fmt Up x))
```

The $(1 + \epsilon)$ property

Designers often rely on clever "cancellation" tricks to avoid or compensate for rounding errors.

But many routine parts of the proof can be dealt with by a simple conservative bound on rounding error:

```
|- normalizes fmt x ∧
   ¬(precision fmt = 0)
   ⇒ ∃e. abs(e) <= mu rc / &2 pow (precision fmt - 1) ∧
          round fmt rc x = x * (&1 + e)
```

Derived rules apply this result to computations in a floating point algorithm automatically, discharging the conditions as they go.

Exact calculation

A famous theorem about exact calculation:

```
|- a IN iformat fmt ∧ b IN iformat fmt ∧
   a / &2 <= b ∧ b <= &2 * a
   ⇒ (b - a) IN iformat fmt
```

The following shows how we can retrieve the rounding error in multiplication using a fused multiply-add.

```
|- a IN iformat fmt ∧ b IN iformat fmt ∧
   &2 pow (2 * precision fmt - 1) / &2 pow (ulpscale fmt) <= abs(a * b)
   ⇒ (a * b - round fmt Nearest (a * b)) IN iformat fmt
```

Here's a similar one for addition and subtraction:

```
|- x IN iformat fmt ∧ y IN iformat fmt ∧ abs(x) <= abs(y)
   ⇒ (round fmt Nearest (x + y) - y) IN iformat fmt ∧
      (round fmt Nearest (x + y) - (x + y)) IN iformat fmt
```

Proof tools and execution

Several definitions are highly non-constructive, notably rounding.

However, we can prove equivalence to a constructive definition and hence prove particular numerical results:

```
#ROUND_CONV ‘round (10,11,12) Nearest (&22 / &7)‘;;
|- round (10,11,12) Nearest (&22 / &7) = &1609 / &512
```

Internally, HOL derives this using theorems about sufficient conditions for correct rounding.

In ACL2, we would be forced to adopt a non-standard constructive definition, but would then have such proving procedures without further work and highly efficient.

Division and square root

There are several different algorithms for division and square root, and which one is better is a fine choice.

- Digit-by-digit: analogous to pencil-and-paper algorithms but usually with quotient estimation and redundant digits (SRT, Ercegovac-Lang etc.)
- Multiplicative: get faster (e.g. quadratic) convergence by using multiplication, e.g. Newton-Raphson, Goldschmidt, power series.

The Intel® Itanium® architecture uses some interesting multiplicative algorithms relying *purely* on conventional floating-point operations.

Basic ideas due to Peter Markstein (see IBM J. Res. Dev, 1990).

Correctness issues

Easy to get within a bit or so of the right answer, but meeting the IEEE spec is significantly more challenging.

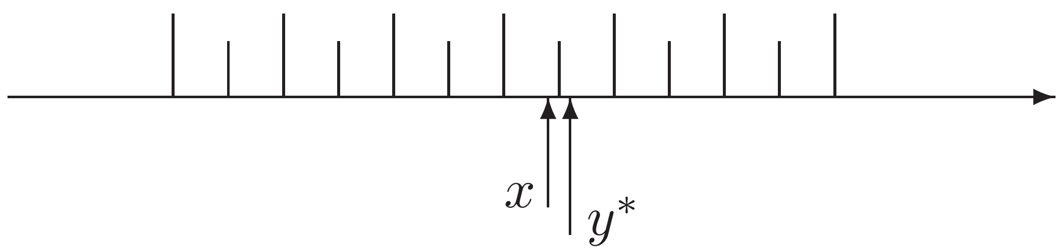
Whatever the overall structure of the algorithm, we can consider its last operation as yielding a result y by rounding an exact value y^*. What is the required property for perfect rounding?

We will concentrate on round-to-nearest mode.

Condition for perfect rounding

Sufficient condition for perfect rounding: the closest floating point number to the exact answer x is also the closest to y^*, the approximate result before the last rounding. That is, the two real numbers x and y^* never fall on opposite sides of a midpoint between two floating point numbers.

In the following diagram this is not true; x would round to the number below it, but y^* to the number above it.



How can we prove this?

A square root algorithm

Single-precision square root, optimized latency, using only single-precision operations (main path).

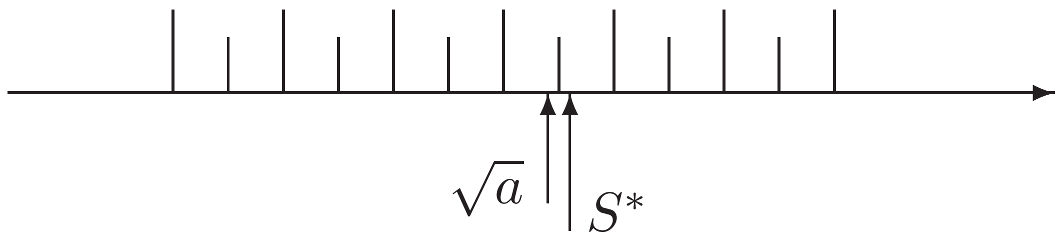
Based on Goldschmidt iteration, refining an initial approximation.

1.	$y_0 = \frac{1}{\sqrt{a}}(1+\epsilon)$	$b = \frac{1}{2}a$	
2.	$z_0 = y_0^2$	$S_0 = ay_0$	
3.	$d = \frac{1}{2} - bz_0$	$k = ay_0 - S_0$	$H_0 = \frac{1}{2}y_0$
4.	$e = 1 + \frac{3}{2}d$	$T_0 = dS_0 + k$	
5.	$S_1 = S_0 + eT_0$	$c = 1 + de$	
6.	$d_1 = a - S_1S_1$	$H_1 = cH_0$	
7.	$S = S_1 + d_1H_1$		

All but the last operation in round-to-nearest; last in native mode.

Condition for perfect rounding

Recall the general condition for perfect rounding. We want to ensure that the two real numbers $\sqrt{a}$ and S^* never fall on opposite sides of a midpoint between two floating point numbers, as here:



Rather than analyzing the rounding of the final approximation explicitly, we can just appeal to general properties of the square root function.

Exclusion zones

It would suffice if we knew for any midpoint m that:

$$|\sqrt{a} - S^*| < |\sqrt{a} - m|$$

In that case $\sqrt{a}$ and S^* cannot lie on opposite sides of m. Here is the formal theorem in HOL:

```
|- ¬(precision fmt = 0) ∧
   (∀m. m IN midpoints fmt ⇒ abs(x - y) < abs(x - m))
   ⇒ round fmt Nearest x = round fmt Nearest y
```

Square root exclusion zones

This is possible to prove, because in fact every midpoint m is surrounded by an 'exclusion zone' of width $\delta_m > 0$ within which the square root of a floating point number cannot occur.

However, this δ can be quite small, considered as a relative error. If the floating point format has precision p, then we can have $\delta_m \approx |m|/2^{2p+2}$.

Example: square root of significand that's all 1s.

Difficult cases

So to ensure the equal rounding property, we need to make the final approximation before the last rounding accurate to *more than twice* the final accuracy.

The fused multiply-add can help us to achieve *just under twice* the accuracy, but to do better is slow and complicated. How can we bridge the gap?

We can use a technique due to Marius Cornea.

Mixed analytic-combinatorial proofs

Only a fairly small number of possible inputs a can come closer than say $2^{-(2p-1)}$.

For all the other inputs, a straightforward relative error calculation (which in HOL we have largely automated) yields the result.

We can then use number-theoretic reasoning to isolate the additional cases we need to consider, then simply *try them and see*!

More than likely we will be lucky, since all the error bounds are worst cases and even if the error is exceeded, it might be in the right direction to ensure perfect rounding anyway.

Isolating difficult cases

Straightforward to show that the difficult cases have mantissas m, considered as p-bit integers, such that one of the following diophantine equations has a solution k for d a small integer.

$$2^{p+2}m = k^2 + d$$

or

$$2^{p+1}m = k^2 + d$$

We consider the equations separately for each chosen d. For example, we might be interested in whether:

$$2^{p+1}m = k^2 - 7$$

has a solution. If so, the possible value(s) of m are added to the set of difficult cases.

Solving the equations

It's quite easy to program HOL to enumerate all the solutions of such diophantine equations, returning a disjunctive theorem of the form:

$$(2^{p+1}m = k^2 + d) \Rightarrow m = n_1 \vee \ldots \vee m = n_i$$

The procedure simply uses even-odd reasoning and recursion on the power of two (effectively so-called 'Hensel lifting'). For example, if

$$2^{25}m = k^2 - 7$$

then we know k must be odd; we can write $k = 2k' + 1$ and get the derived equation:

$$2^{24}m = 2k'^2 + 2k' - 3$$

By more even/odd reasoning, this has no solutions. Always recurse down to an equation that is unsatisfiable or immediately solvable.

General analytical-combinatorial setup

By finding a suitable set of ‘diffifult cases’, one can produce a proof by a mixture of analytical reasoning and explicit checking.

- Find the set of difficult cases S
- Prove the algorithm analytically for all $x \notin S$
- Prove the algorithm by explicit case analysis for $x \in S$

Quite similar to some standard proofs in mathematics, e.g. Bertrand’s conjecture.

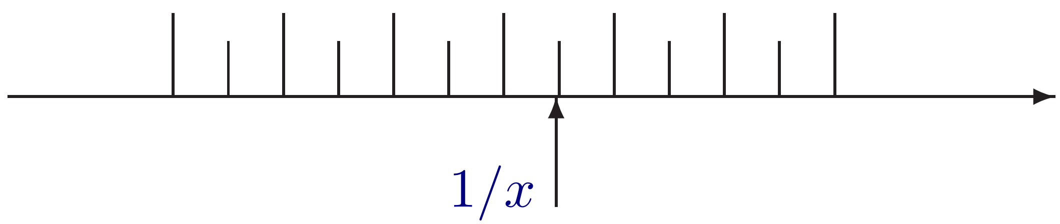
This is particularly useful given that error bounds derived from the $1 + \epsilon$ property are highly conservative.

Another example: Difficult cases for reciprocals

Some algorithms for floating-point division, a/b, can be optimized for the special case of reciprocals ($a = 1$).

A direct analytic proof of the optimized algorithm is sometimes too hard because of the intricacies of rounding.

However, an analytic proof works for all but the 'difficult cases'.



These are floating-point numbers whose reciprocal is very close to another one, or a midpoint, making them trickier to round correctly.

Finding difficult cases with factorization

After scaling to eliminate the exponents, finding difficult cases reduces to a straightforward number-theoretic problem.

A key component is producing the prime factorization of an integer *and* proving that the factors are indeed prime.

In typical applications, the numbers can be 49–227 bits long, so naive approaches based on testing all potential factors are infeasible.

The primality prover is embedded in a HOL derived rule `PRIME_CONV` that maps a numeral to a theorem asserting its primality or compositeness.

Certifying primality

We generate a ‘certificate of primality’ based on Pocklington’s theorem:

```
|- 2 ≤ n ∧
   (n - 1 = q * r) ∧
   n ≤ q EXP 2 ∧
   (a EXP (n - 1) == 1) (mod n) ∧
   (∀p. prime(p) ∧ p divides q
        ⇒ coprime(a EXP ((n - 1) DIV p) - 1,n))
   ⇒ prime(n)
```

The certificate is generated ‘extra-logically’, using the factorizations produced by PARI/GP.

The certificate is then checked by formal proof, using the above theorem.

Typical results

```
0xFFFFFFFFFFFFFFFF 0xFFFFFFFFFFFFFFFD 0xFE421D63446A3B34 0xFBFC17DFE0BEFF04 0xFB940B119826E598
0xFB0089D7241D10FC 0xFA0BF7D05FBE82FC 0xF912590F016D6D04 0xF774DD7F912E1F54 0xF7444DFBF7B20EAC
0xF39EB657E24734AC 0xF36EE790DE069D54 0xF286AD7943D79434 0xEDF09CCC53942014 0xEC4B058D0F7155BC
0xEC1CA6DB6D7BD444 0xE775FF856986AE74 0xE5CB972E5CB972E4 0xE58469F0234F72C4 0xE511C4648E2332C4
0xE3FC771FE3B8FF1C 0xE318DE3C8E6370E4 0xE23B9711DCB88EE4 0xE159BE4A8763011C 0xDF738B7CF7F482E4
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```

Bit-level details

So far we've focused on verification at the level of reals, but we also need to worry about:

- Special floating-point values
- Flag-setting and exception raising

Correct flags/exceptions mostly automatic from correct rounding analysis.

We'll focus on bit-level details.

Going beyond the reals

The IEEE floating-point formats contain additional values beyond just the reals:

- Signed zeros $+0$ and -0.
- Infinities $+\infty$ and $-\infty$
- Not-a-numbers (NaN)

The IEEE Standard specifies the behavior of operations on all of these combinations, except for some underspecification related to NaNs.

Examples of special values

- $1/+0 = +\infty$
- $1/-0 = -\infty$
- $\infty - 1 = \infty$
- $0/0 =$ NaN
- NaN $+ 7 =$ NaN
- NaN $=$ NaN is false, even for identical NaNs(!)

Less obvious ones

While most of the rules for signed zeros are obvious, some seem a little more arbitrary:

- $+0 + +0 = +0$
- $+0 + -0 = +0$
- $+0 - +0 = +0$
- $-0 - +0 = -0$

It's hard to specify this in an elegant way: we just need case analysis in the specification and proof.

Signed zeros in the FMA

We might like to consider addition and multiplication as degenerate cases:

- $x + y = 1 \cdot x + y$
- $x \cdot y = x \cdot y + 0$

Unfortunately this is tricky if we want signed zeros to work:

- $+0 \cdot -0 + +0 = -0 + +0 = +0$

We can also have nonzero exact results for FMA where the FP result underflows to zero.

Decimal arithmetic

The revised version of the IEEE Standard 754 includes *decimal* arithmetic formats.

These are roughly analogous to the binary formats, but also have a notion of *scale*, reflected in a redundant representation:

- $4.00 = 400 \times 10^{-2}$
- $4.0 = 40 \times 10^{-1}$

Even though the numerical values are the same, the representations are distinct.

Operations have specific rules for propagation of scale.

Summary

- Although floats are to reals as bitvectors are to integers, the mathematical relationship is complicated.
- Correct formalization of the floating-point Standard is non-trivial, but quite feasible.
- We have had some success with verification of high-level algorithms based on interactive provers.
- Special values and the new complications of decimal need to be linked with the ‘real’ core verifications.