Formal proofs of hypergeometric sums
Dedicated to the memory of Andrzej Trybulec

John Harrison

Intel Corporation

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Overview

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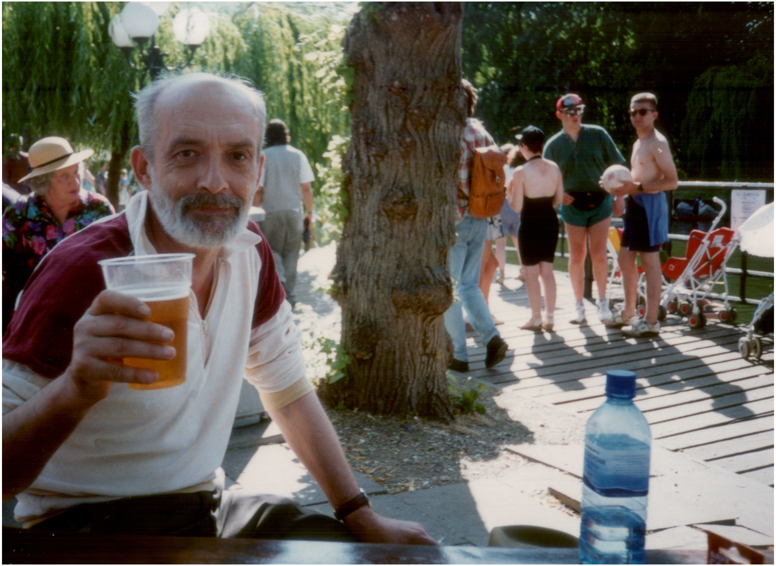
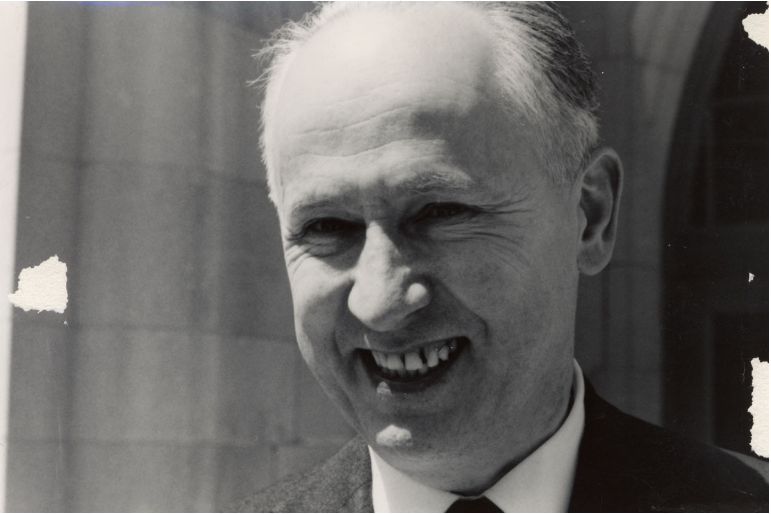
- ☺ Trouble-free formalization: some topological theorems due to Borsuk.
- ☹ Problematic formalization: Wilf-Zeilberger method for hypergeometric summation.
    - Hypergeometric sequences
    - Gosper's algorithm and the WZ method
    - WZ examples, and their difficulties
    - Generic proof of Sylvester's identity, limit formulation of WZ
    - Formalizing the gamma function
    - Avoiding a countable family of algebraic varieties
    - The method at work
    - Automation and conclusions

Memories of Andrzej



Memories of Andrzej

Back to Borsuk . . .



The Borsuk homotopy extension theorem

Fundamental in relating homotopy to extension properties:

```
BORSUK_HOMOTOPY_EXTENSION_HOMOTOPIC =
 |- !f:real^M->real^N g s t u.
       closed_in (subtopology euclidean t) s /\
       (ANR s /\ ANR t \/ ANR u) /\
       f continuous_on t /\
       IMAGE f t SUBSET u /\
       homotopic_with (\x. T) (s,u) f g
       ==> ?g'. homotopic_with (\x. T) (t,u) f g' /\
                g' continuous_on t /\
                IMAGE g' t SUBSET u /\
                !x. x IN s ==> g'(x) = g(x)
```

Bosuk's separation theorem

Characterize separation properties in purely homotopic terms

```
BORSUK_SEPARATION_THEOREM_GEN =
 |- !s:real^N->bool.
     compact s
     ==> ((!c. c IN components((:real^N) DIFF s)
               ==> ~bounded c) <=>
          (!f. f continuous_on s /\
               IMAGE f s SUBSET sphere(vec 0,&1)
             ==> ?c. homotopic_with (\x. T)
                     (s,sphere(vec 0,&1)) f (\x. c)))
```

Note that the $N = 1$ case is a bit different, but this statement works uniformly there too.

Separating space is a homotopy invariant

For compact sets, whether they separate space or not respects homotopy equivalence

```
HOMOTOPY_EQUIVALENT_SEPARATION =
 |- !s t. compact s /\ compact t /\
          s homotopy_equivalent t
          ==> (connected((:real^N) DIFF s) <=>
               connected((:real^N) DIFF t))
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```

This yields in particular a major part of the Jordan Curve Theorem in a more general context

```
JORDAN_BROUWER_SEPARATION =
 |- !s a:real^N r.
        &0 < r /\
        s homeomorphic sphere(a,r)
        ==> ~connected((:real^N) DIFF s)
```

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Reference: '$A = B$' by Marko Petkovšek, Herbert S. Wilf and Doron Zeilberger.

Hypergeometric sequences

A *hypergeometric sequence* (or *term* or *series*) is one where the ratio of successive terms is a rational function of n.

$$a_{n+1}/a_n = r(n) = p(n)/q(n)$$

For example, factorials where $(n+1)!/n! = n+1$, the 'power of 2' function with $2^{n+1}/2^n = 2$.

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For example, factorials where $(n+1)!/n! = n+1$, the 'power of 2' function with $2^{n+1}/2^n = 2$.
We call a function of several variables hypergeometric if it's hypergeometric in each argument separately, e.g. binomial coefficients

$$\binom{n+1}{k} = \frac{n+1}{n-k+1}\binom{n}{k}$$

$$\binom{n}{k+1} = \frac{n-k}{k+1}\binom{n}{k}$$

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However some hypergeometric terms have no hypergeometric antidifference.

Gosper example

Consider the term $t_k = \frac{k \cdot k!}{n^k}\binom{n}{k}$. We'll use the implementation of Gosper's algorithm in Maxima due to Fabrizio Caruso:

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That is $s_k = -k!n^{1-k}\binom{n}{k}$. This lets us easily verify the following *definite* sum, which was problem E 3088 in the "American Mathematical Monthly".

$$\sum_{k=1}^{n} \frac{k \cdot k!}{n^k}\binom{n}{k} = s_{n+1} - s_1 = n$$

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but it turns out $\binom{n}{k}$ has no hypergeometric *antidifference*. The idea of the WZ algorithm is to apply Gosper not to $F(n,k)$ itself, but rather to $F(n+1,k) - F(n,k)$ (or in general a more complicated combination, but we'll ignore that here).

The basic WZ idea

We say that $G(n,k)$ is the WZ-mate of $F(n,k)$, and that F and G form a 'WZ-pair', when

$$F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k)$$

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If $G(n,k)$ has finite support, summing over all (or enough) integers shows $\sum_k F(n+1,k) - \sum_k F(n,k) = 0$, i.e. the sum is *independent of* n.

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3. Conclude that $\sum_k F(n,k)$ is independent of n and so we just need to check the following, which we expect to be easy $\sum_k F(0,k) = 1$

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$$R(n,k) = \frac{-k^2(3n-2k+3)}{2(n-k+1)^2(2n+1)}$$

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3. So $\sum_k F(n,k)$ is independent of n, so we can evaluate the case $n = 0$, which is easy to simplify to 1

A routine formalization?

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> *"Well, at this point we have arrived at a situation that will be referred to throughout this book as a "routinely verifiable" identity. That phrase means roughly that your pet chimpanzee could check out the equation. More precisely it means this. First cancel out all factors that look like c^n or c^k [...] that can be cancelled. Then replace every binomial coefficient in sight by the quotient of factorials that it represents. Finally, cancel out all of the factorials by suitable divisions, leaving only a polynomial identity that involves n and k." (from '$A = B$')*

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- The rational function certificates often have poles, so on the face of it we seem to be simplifying terms of the form $0/0$.

It seems very hard to avoid these issues in a nice and automatable way if we use a straightforward interpretation.

Generic proof of Sylvester's identity

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To handle the general case we use a *limit* argument, that every matrix can be approached arbitrarily closely by an invertible one.

This effectively lets us choose a 'generic' matrix in the main argument.

We want to do the same sort of thing with WZ.

The gamma function

In order to use limits, we need to generalize things from the integers to the reals, defining $\Gamma(z)$ such that $\Gamma(n+1) = n!$

Formalizing the gamma function

We define complex gamma functions via the following limit, though we derive other equivalent forms of the definition

$$\Gamma(z) = \lim_{n\to\infty} \frac{n^z n!}{\prod_{m=0}^{n}(z+m)}$$

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In HOL Light:

```
|- cgamma(z) =
   lim sequentially
    (\n. (Cx(&n) cpow z * Cx(&(FACT n))) /
         cproduct(0..n) (\m. z + Cx(&m)))
```

We derive many useful properties and specialize to the *real* gamma function `gamma`, which is what we use here.

Generalizing to the reals

We establish some definitions to generalize factorials and binomial coefficients to the reals:

```
|- rfact x = gamma(x + &1)

|- rbinom(n,k) = rfact n / (rfact k * rfact (n - k))
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In general, factorials are still not well-defined at negative integers, and similarly not all binomial coefficients make sense.
But they behave very well as limits

```
|- !net nn kk n k.
        (nn ---> &n) net /\ (kk ---> &k) net
        ==> ((\a. rbinom(nn a,kk a))
                ---> &(binom(n,k))) net
```

This lets us justify all the 'naive' manipulations in this context without any case analysis.

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```
|- ratpolyfun p <=>
        ?s. FINITE s /\
            s SUBSET (:num#num) CROSS rational /\
            p = \(x,y). sum s
                  (\((i,j),c). c * x pow i * y pow j)

|- ratty t <=>
     ?p. ratpolyfun p /\ p t = &0 /\ ~(!w. p w = &0)
```

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|- !f c. real_polynomial_function f /\
         ~(!x. f x = c)
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A countable union of nowhere dense sets has empty interior (this is a Baire-type result):

```
|- !g:(real^N->bool)->bool.
       COUNTABLE g /\
       (!s. s IN g ==> closed s /\ interior s = {})
       ==> interior(UNIONS g) = {}
```

The WZ limit theorem

Hence we can obtain a WZ-type theorem that allows one free rein to manipulate terms ‘naively’:

```
|- (!n. FINITE {k | ~(f n k = &0)}) /\
   (!n k. (FF ---> f n k) (at(complex(&n,&k)))) /\
   (!n k. integer k /\ k < &0
          ==> (FF ---> &0) (at(complex(&n,k)))) /\
   (!n B. ?i j. integer i /\ integer j /\
                i < &0 /\ &B <= j /\
                (GG ---> &0) (at(complex(&n,i))) /\
                (GG ---> &0) (at(complex(&n,j)))) /\
   (!n k. ~ratty(n,k)
          ==> FF(complex(n + &1,k)) -
              FF(complex(n,k)) =
              GG(complex(n,k + &1)) -
              GG(complex(n,k))) /\
   sum (:num) (f 0) = 1
   ==> !n. sum (:num) (f n) = 1
```

An example

We define appropriate $F(n, k)$ and $G(n, k)$ for the example $\sum_{k=0}^{n} \binom{n}{k} = 2^n$, as functions $\mathbb{C} \to \mathbb{R}$:

```
|- FF z = rbinom(z$1,z$2) / &2 rpow z$1

|- RR z = z$2 / (&2 * (z$2 - z$1 - &1))
```

An example

We define appropriate $F(n,k)$ and $G(n,k)$ for the example $\sum_{k=0}^{n} \binom{n}{k} = 2^n$, as functions $\mathbb{C} \to \mathbb{R}$:

```
|- FF z = rbinom(z$1,z$2) / &2 rpow z$1

|- RR z = z$2 / (&2 * (z$2 - z$1 - &1))
```

We can justify the WZ-pair property for *all reals* except for a few special cases, all 'ratty'

```
|- ~(n + &1 = &0) /\ ~(n + &1 = k) /\
   ~(k + &1 = &0) /\ ~(n = k)
   ==> FF(complex(n + &1,k)) - FF(complex(n,k)) =
       GG(complex(n,k + &1)) - GG(complex(n,k))
```

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- However, this could be automated too by arguing that the denominator of the certificate, for fixed n, is a polynomial in k and hence is nonzero for large enough $|k|$.
- We believe this is a satisfying, if somewhat involved, interpretation, and that it justifies the WZ method more clearly.