Challenges and Opportunities for Automated Reasoning

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Summary of talk

- Motivation: the need for dependable proof
  - LCF-style theorem proving
  - Intel verification work
  - The Flyspeck project
- Combining tools and certifying results
  - Why is this important?
  - Focus on nonlinear arithmetic
- Beyond standard geometric decision procedures:
  - Without loss of generality
  - Decision procedures for vector spaces

0: Motivation

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How?

- LCF approach à la Milner

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A very powerful motivation for performing rigorous proofs of numerical algorithms!

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- Hales submitted his proof to *Annals of Mathematics* . . .

The response of the reviewers

After a full four years of deliberation, the reviewers returned:

> *"The news from the referees is bad, from my perspective. They have not been able to certify the correctness of the proof, and will not be able to certify it in the future, because they have run out of energy to devote to the problem. This is not what I had hoped for.*
> *Fejes Toth thinks that this situation will occur more and more often in mathematics. He says it is similar to the situation in experimental science — other scientists acting as referees can't certify the correctness of an experiment, they can only subject the paper to consistency checks. He thinks that the mathematical community will have to get used to this state of affairs."*

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- Dissatisfied with this state of affairs, Hales initiated a project called *Flyspeck* to completely formalize the proof.
- "Flyspeck" = "Formal proof of the Kepler Conjecture"

1: Combining tools and certifying results

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- Many powerful facilities in computer algebra systems that we'd like to exploit
- May want to combine work done in different theorem provers, e.g. ACL2, Coq, HOL, Isabelle.

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If the Intel® Software and Services Group (SSG) were split off as a separate company, it would be in the top 10 software companies worldwide.

A diversity of verification problems

This gives rise to a corresponding diversity of verification problems, and of verification solutions.

- Propositional tautology/equivalence checking (FEV)
- Symbolic simulation
- Symbolic trajectory evaluation (STE)
- Temporal logic model checking
- Combined decision procedures (SMT)
- First order automated theorem proving
- Interactive theorem proving

Integrating all these is a challenge!

Flyspeck: a diversity of methods

The Flyspeck proof combines large amounts of pure mathematics, optimization programs and special-purpose programs:

- Standard mathematics including Euclidean geometry and measure theory
- More specialized theoretical results on *hypermaps*, *fans* and packing.
- Enumeration procedure for ‘tame’ graphs
- Many linear programming problems.
- Many nonlinear programming problems.

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- The key is Farkas's Lemma, which implies that for any unsatisfiable set of inequalities, there's a linear combination of them that's 'obviously false' like $1 < 0$.
- Alexey Solovyev's highly optimized implementation of this is essential for Flyspeck.

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- The polynomial equations $p_1(\overline{x}) = 0, \ldots, p_k(\overline{x}) = 0$ in a real closed closed field have *no* common solution iff there are polynomials $q_1(\overline{x}), \ldots, q_k(\overline{x}), s_1(\overline{x}), \ldots, s_m(\overline{x})$ such that

$$q_1(\overline{x}) \cdot p_1(\overline{x}) + \cdots + q_k(\overline{x}) \cdot p_k(\overline{x}) + s_1(\overline{x})^2 + \cdots + s_m(\overline{x})^2 = -1$$

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- The similar but more intricate Positivstellensatz generalizes this to inequalities of all kinds.

Certificates for universal theory of reals (2)

The appropriate certificates can be found in practice via semidefinite programming (SDP). For example $23x^2 + 6xy + 3y^2 - 20x + 5 = 5 \cdot (2x - 1)^2 + 3 \cdot (x + y)^2 \geq 0$ or

$$\forall a\ b\ c\ x.\ ax^2 + bx + c = 0 \Rightarrow b^2 - 4ac \geq 0$$

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However, most standard nonlinear solvers do not return such certificates, and this approach does not obviously generalize to formulas with richer quantifier structure.

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- Core problem becomes bounding polynomial $t(x) - p(x)$ on an interval.
- SOS works very easily in this univariate case: can generate accurate certificates using a more direct method.

Results on Flyspeck

Some simple Flyspeck inequalities, after being expressed componentwise, can be proved efficiently by SOS certification, e.g. this one in HOL Light syntax:

```
!u v w:real^3.dist(u,v) >= &2 /\
     dist(u,w) >= &2 /\
     dist(v,w) >= &2 /\
     norm(u - v) < sqrt(&8)
     ==> norm(w - &1 / &2 % (u + v))
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However, some of the more complex ones seem to be out of reach of current SOS implementations.

Alternative approaches

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Solovyev's highly optimized implementation in HOL Light is already able to prove many difficult inequalities, but efficiency challenges remain.

2: Beyond standard geometric decision procedures

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Many geometric problems can be solved efficiently using coordinate reduction and automated algorithms, e.g.

- Wu's algorithm or Gröbner bases for problems over algebraically closed fields.
- Nonlinear real decision procedures for real-specific cases, e.g. involving inequalities.

However, these are not always efficient when applied in a straightforward manner, especially with the extra problem of generating a complete formal proof.

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- Claims that proving the result in a more special case is nevertheless sufficient to justify the theorem in full generality.
- Often justified by some sort of symmetry or invariance in the problem, particularly in geometry:
  - Choose a convenient origin based on invariance under translation
  - Choose convenient coordinate axes based on rotation invariance

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Often allows the final coordinatewise proof to be much easier and more natural.

Avoiding coordinate reduction

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- Attractive to consider other algorithms (e.g. the area method, bracket algebra, . . .)
- In collaboration with Solovay and Arthan, we considered general decision procedures for various theories of vector spaces
- Many interesting results, both positive and negative, and some practically useful outcomes.

Vector space axioms

$$\forall \mathbf{u}\ \mathbf{v}\ \mathbf{w}.\ \mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$$
$$\forall \mathbf{v}\ \mathbf{w}.\ \mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$$
$$\forall \mathbf{v}.\ \mathbf{0} + \mathbf{v} = \mathbf{v}$$
$$\forall \mathbf{v}.\ -\mathbf{v} + \mathbf{v} = \mathbf{0}$$
$$\forall a\ \mathbf{v}\ \mathbf{w}.\ a(\mathbf{v} + \mathbf{w}) = a\mathbf{v} + a\mathbf{w}$$
$$\forall a\ b\ \mathbf{v}.\ (a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$$
$$\forall \mathbf{v}.\ 1\mathbf{v} = \mathbf{v}$$
$$\forall a\ b\ \mathbf{v}.\ (ab)\mathbf{v} = a(b\mathbf{v})$$

The theory of real inner product spaces

The language of vector spaces plus an inner product operation $\mathcal{V} \times \mathcal{V} \to \mathcal{S}$ written $\langle -, - \rangle$ and satisfying:

$$\forall \mathbf{v}\ \mathbf{w}.\ \langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{w}, \mathbf{v} \rangle$$
$$\forall \mathbf{u}\ \mathbf{v}\ \mathbf{w}.\ \langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$$
$$\forall a\ \mathbf{v}, \mathbf{w}.\ \langle a\mathbf{v}, \mathbf{w} \rangle = a\langle \mathbf{v}, \mathbf{w} \rangle$$
$$\forall \mathbf{v}.\ \langle \mathbf{v}, \mathbf{v} \rangle \geq 0$$
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Decidability of inner product spaces

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Decidability of inner product spaces

- (Solovay): theory of real inner product spaces is decidable, and admits quantifier elimination in a language expanded with inequalities on dimension.
- Since inner product spaces are a conservative extension of vector spaces, the theory of vector spaces is also decidable
- (Arthan) a formula with k vector variables holds in all inner product spaces iff it holds in each $\mathbb{R}^n$ for $0 \leq n \leq k$.

The theory of real normed spaces

The language of vector spaces plus a norm operation $\mathcal{V} \to \mathcal{S}$ written $\| - \|$ and satisfying:

$$\forall \mathbf{v}.\ \|\mathbf{v}\| = 0 \Rightarrow \mathbf{v} = \mathbf{0}$$
$$\forall a\ \mathbf{v}.\ \|a\mathbf{v}\| = |a|\|\mathbf{v}\|$$
$$\forall \mathbf{v}\ \mathbf{w}.\ \|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$$

Normed spaces: better or worse?

- (Solovay) The full theory of real normed spaces is strongly undecidable (same many-one degree as the true Π^2_1 sentences in third-order arithmetic).

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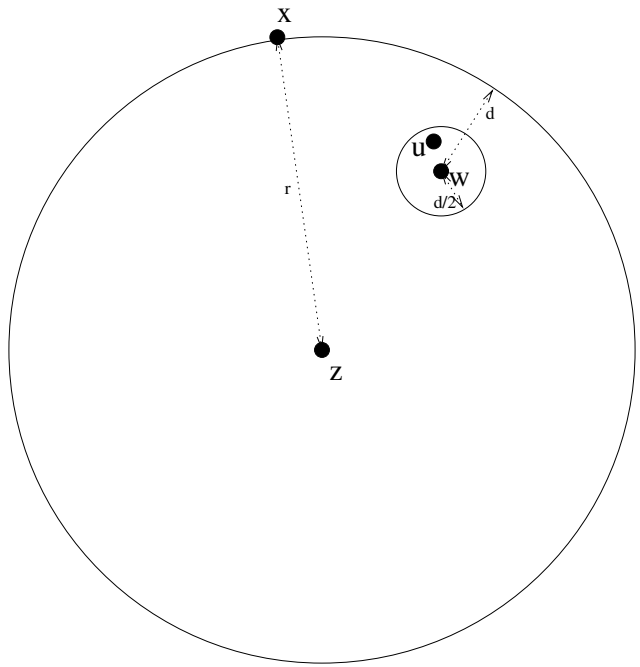
- (Arthan) Even the purely *additive* theory of 2-dimensional normed spaces is strongly undecidable.
- (Harrison) However the $\forall$ (purely universal) fragment of the theory is decidable. In the additive case, can be decided by a generalization of parametrized linear programming.
- (Arthan) This decidability result is quite sharp: both the $\forall\exists$ and $\exists\forall$ fragments, and even the $(\forall) \Rightarrow (\forall)$ fragments are undecidable.

Real application in formalizing complex analysis

An example where our linear normed space procedure is much more efficient than coordinate reduction:

```
|- abs(norm(w - z) - r) = d /\
   norm(u - w) < d / &2 /\
   norm(x - z) = r
   ==> d / &2 <= norm(x - u)
```



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- A useful tool in soundly integrating different proof tools, which has value in verification and in mathematics
- Nonlinear arithmetic is a particularly challenging example for such certification, and has many potential applications.
- There are strong motivations for looking for higher-level (more efficient or conceptual) approaches to such problems.