Symmetry in Petri nets

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Abstract

An algebraic treatment of symmetry in Petri nets is proposed. The standard definition of Petri net is that it has precisely one initial marking. Motivated by work on defining symmetry across models for concurrency, we extend the definitions of forms of net to allow them to have multiple initial markings. Existing coreflections between event structures and occurrence nets and between occurrence nets and P/T nets are generalized, and from them coreflections between categories of nets with symmetry are obtained.

1 Introduction

Petri nets are a widely used model for concurrency. They play a fundamental role analogous to that of transition systems, but, by capturing the effect of events on *local* components of state, it becomes possible to describe how events might occur concurrently, how they might conflict with each other and how they might causally depend on each other. Here an algebraic treatment of symmetry on Petri nets is proposed.

Without doubt symmetry is important and plays a role, at least informally, in many models, and often in the analysis of processes. It is, for instance, present in security protocols due to the repetition of essentially similar sessions [2, 5, 1], can be exploited to increase efficiency in model checking [17], and is present whenever abstract names are involved [6].

Of course, there are undoubtedly several ways to adjoin symmetry to Petri nets. The method we use has some history. It was motivated through the need to extend the expressive power of event structures and the maps between them [23, 24]. One important reason to extend the treatment of symmetry beyond event structures to Petri nets is the potentially more compact and algorithmically-amenable representation nets afford. Another reason for extending the method of adjoining symmetry is to obtain a characterisation of the unfolding of general nets *up to symmetry* [7]: There is an implicit symmetry in a Petri net where a place can be marked more than once. That symmetry is inherited by its unfolding, and, if not made explicit there, will spoil the uniqueness required by its universal characterisation. Once symmetry is added, a coreflection *up to symmetry* between occurrence nets and general nets is obtained.

Roughly, a symmetry in a Petri net is described as a relation between its runs as causal nets, the relation specifying when one run is similar to another up to symmetry; of course, if runs are to be similar, they should have similar futures, as well as pasts. More technically and generally, a relation of symmetry is expressed as a span of open maps which form a pseudo equivalence — it is said to form an equivalence when the span of maps is jointly monic. One motivation for the work in [7] is to apply this general algebraic method to adjoin symmetry to a model, to the instance of Petri nets, and obtain a universal charactersiation of the unfolding of nets. But another motivation is that Petri nets provide a good testing ground for the method of adjoining symmetries.

In our work it became apparent that all but one of the general issues we encountered in considering general nets also arose in considering just safe or P/T nets. The usual categories of Petri nets attach to each net an initial marking to represent the state in which process represented by the net initially lies. The initial marking is essential to understanding the behaviour of the net. Singly-marked nets are, however, unable to express very natural symmetries on nets. Applying the scheme in [23, 24] to obtain a way of defining symmetry on nets, nets with a single initial marking do not even allow the symmetry of the two places in the net  to be expressed. This phenomenon, which appears even for occurrence nets, is frustrating since it only occurs at the initial marking; such symmetry arising, for example, in the postconditions of events can be expressed. While even for safe nets the introduction of symmetry on nets leads us to drop the requirement that a symmetry be a joint monic relation — joint monicity was imposed in [23, 24], but if we were to insist on joint monicity we simply could not express some reasonable symmetries — see the Conclusion.

The work tests a method of adjoining symmetries and provides a rationale for a certain, probably rather innocent, extension of Petri nets. It argues for the enlargement of nets to include multiple initial markings. This extension does however force us to review the existing adjunctions between nets, and in particular the unfoldings of safe and P/T nets into occurrence nets and event structures. In summary, in this paper, we generalize the definition of nets to allow them to have a *set* of initial markings. We extend the existing coreflections* between categories of event structures and occurrence nets and between categories of P/T nets and occurrence nets to this new setting. The coreflection between P/T nets and occurrence nets is shown to extend to categories with symmetry adjoined.

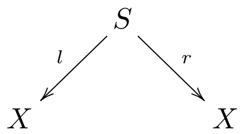
*A coreflection is an adjunction of which the left adjoint is full and faithful. Equivalently [10], a coreflection is an adjunction of which the unit is a natural isomorphism.

Notation

In what follows, it will be necessary to use a little notation when dealing with multisets and sets (a full, formal treatment of the use of multisets in the setting of Petri nets can be found in the appendix of [22]). We write $R \cdot X$ for the result of applying the (multi)relation R to the (multi)set X, $+$ for the union of multisets and $-$ for the partial operation of subtraction of multisets. A partial function f from X to Y will be written $f : X \to_* Y$. We write $f(x) = *$ if f is undefined at x. The image under f of the (multi)set Z comprising elements of X is denoted fZ. We write R^+ for the transitive closure of a relation R and R^* for the reflexive, transitive closure of R.

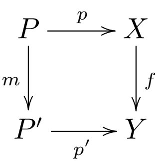
2 Symmetry in concurrency

In [23, 24], a symmetry in model X, an object in a category of models $\mathcal{C}$, is a span

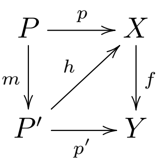


which we write $(X; l, r : S \to X)$. The span should represent a pseudo equivalence, so it is required to satisfy the standard axioms of reflexivity, symmetry and transitivity presented in Appendix A. For simplicity, we assume that the category $\mathcal{C}$ has pullbacks. The morphisms l and r of the symmetry are required to be *open* morphisms [9].

A morphism of $\mathcal{C}$ is open with respect to a path category $\mathbb{P}$ that is a subcategory of $\mathcal{C}$. That is, for a path category $\mathbb{P} \hookrightarrow \mathcal{C}$, a morphism $f : X \to Y$ is said to be $\mathbb{P}$-open if, for any morphism $m : P \to P'$ in $\mathbb{P}$ and morphisms $p : P \to X$ and $p' : P' \to Y$ in $\mathcal{C}$, whenever the diagram



commutes, *i.e.* $fp = p'm$, there is a morphism $h : P' \to X$ such that the two triangles in the following diagram commute



i.e. $hm = p$ and $fh = p'$. Open maps, with respect to suitable path categories, can give rise to well-known forms of bisimulation. For instance, with respect to a category of labelled sequences, spans of open maps of the category of labelled transition systems give rise to Milner and Park's definition of bisimilarity [9]. Spans of open maps of (labelled) event structures and nets, taking paths to be (labelled) partially ordered multisets of events, exhibit a strengthened version [9, 13] of history preserving bisimulation as introduced in [14, 18].

Given a category $\mathcal{C}$ with pullbacks, we will form a category with symmetry $\mathcal{S}_{\mathbb{P}}\mathcal{C}$. The objects of $\mathcal{S}_{\mathbb{P}}\mathcal{C}$ are constructed as above, being tuples $(X; l, r)$ where $l, r : S \to X$ are $\mathbb{P}$-open morphisms in $\mathcal{C}$ that satisfy the axioms of reflexivity, symmetry and transitivity presented in Appendix A. The morphisms of the category $\mathcal{S}_{\mathbb{P}}\mathcal{C}$ are morphisms of $\mathcal{C}$ that preserve symmetry in the sense that a morphism $f : (X; l, r) \to (X'; l', r')$ in $\mathcal{S}_{\mathbb{P}}\mathcal{C}$ is a morphism $f : X \to X'$ in $\mathcal{C}$ for which there exists a morphism $h : S \to S'$ in $\mathcal{C}$ making the two squares in the following diagram commute, where S is the domain of l and r and S' is the domain of l' and r':

$$\begin{array}{ccccc} X & \xleftarrow{\;l\;} & S & \xrightarrow{\;r\;} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f} \\ X' & \xleftarrow[\;l'\;]{} & S' & \xrightarrow[\;r'\;]{} & X' \end{array}$$

The definition of symmetry presented here differs from that in [23] in that it is a *pseudo* equivalence rather than an equivalence. In particular, there an object with symmetry $(X; l, r)$ requires the morphisms l and r to be jointly monic. This relaxation has turned out to be necessary in other situations [19, 7] and is discussed in the conclusion.

2.1 Symmetry, functors and adjunctions

Let $\mathcal{C}$ and $\mathcal{D}$ be categories upon which symmetry can be placed, *i.e.* with pullbacks and subcategories $\mathbb{P}$ and $\mathbb{Q}$, respectively, of paths from which open maps can be drawn. We obtain the categories with symmetry $\mathcal{S}_{\mathbb{P}}\mathcal{C}$ and $\mathcal{S}_{\mathbb{Q}}\mathcal{D}$.

Say that a functor $F : \mathcal{C} \to \mathcal{D}$ preserves open maps if, for any $\mathbb{P}$-open map $f : X \to X'$ of $\mathcal{C}$, the morphism $F(f) : F(X) \to F(X')$ is $\mathbb{Q}$-open in $\mathcal{D}$. Say that F preserves pullbacks of $\mathbb{P}$-open morphisms if, for any two $\mathbb{P}$-open morphisms $f : X \to Y$ and $f' : X' \to Y$ that have a pullback P with pullback morphisms $p : P \to X$ and $p' : P \to X'$ in $\mathcal{C}$, then the object $F(P)$ with pullback morphisms $F(p)$ and $F(p')$ is a pullback of $F(f)$ and $F(f')$ in $\mathcal{D}$.

Proposition 1. *A functor $F : \mathcal{C} \to \mathcal{D}$ between categories described above yields a functor $\mathcal{S}F : \mathcal{S}_{\mathbb{P}}\mathcal{C} \to \mathcal{S}_{\mathbb{Q}}\mathcal{D}$ defined on objects $(X; l, r)$ of $\mathcal{S}_{\mathbb{P}}\mathcal{C}$ as $\mathcal{S}F(X; l, r) = (FX; Fl, Fr)$ and on morphisms $f : (X; l, r) \to (X'; l', r')$ as $\mathcal{S}F(f) = F(f)$ if F preserves open maps and preserves pullbacks of $\mathbb{P}$-open maps.*

Proof. It is easy to see, given that F preserves pullbacks, that $(FX; Fl, Fr)$ satisfies the requirements to be an element of $\mathcal{S}_{\mathbb{Q}}\mathcal{D}$. It is also easy to show that $\mathcal{S}F(f)$ is a map preserving symmetry as a consequence of f being a map preserving symmetry. ∎

Any adjunction

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D} \quad (\bot)$$

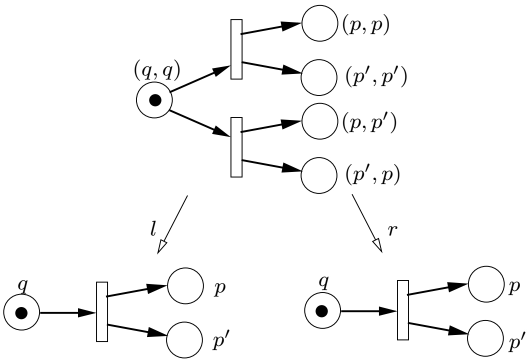
in which the functors F and G satisfy the constraints above of preserving open maps and preserving pullbacks of open maps (noting that the functor G automatically preserves all pullbacks as a consequence of it being a right adjoint) gives rise to an adjunction between the categories enriched with symmetry.

Proposition 2. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories with pullbacks equipped with subcategories $\mathbb{P}$ and $\mathbb{Q}$, respectively, of path objects with respect to which open maps are defined. Suppose that the functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ both preserve open maps, that F preserves pullbacks of $\mathbb{P}$-open morphisms, that G preserves pullbacks of $\mathbb{Q}$-open morphisms, and furthermore that $F \dashv G$, i.e. F is left adjoint to G. The functor $\mathcal{S}F : \mathcal{S}_{\mathbb{P}}\mathcal{C} \to \mathcal{S}_{\mathbb{Q}}\mathcal{D}$ defined in Proposition 1 is left adjoint to the functor $\mathcal{S}G : \mathcal{S}_{\mathbb{Q}}\mathcal{D} \to \mathcal{S}_{\mathbb{P}}\mathcal{C}$, i.e.*

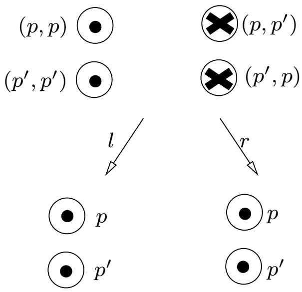
$$\mathcal{S}_{\mathbb{P}}\mathcal{C} \underset{\mathcal{S}G}{\overset{\mathcal{S}F}{\rightleftarrows}} \mathcal{S}_{\mathbb{Q}}\mathcal{D} \quad (\bot) .$$

3 Petri nets

Petri nets were introduced by Petri in 1962 and are an important model of concurrent computation. We now proceed to define the variants of net that we shall consider, referring the reader to [15, 25] for a fuller introduction to net theory. As discussed in the introduction, we generalize nets so that they might possess more than one initial marking. As we do so, a guiding intuition in forming the definitions, particularly in the extended definition of occurrence net, shall be that each initial marking can be thought of as being given



(a) Symmetry of postconditions



(b) Symmetry on initial marking

For each span, the morphism l on conditions projects to the first element of the pair and the morphism r projects to the second.

Figure 1: Net symmetry as spans

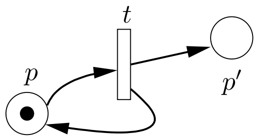


Figure 2: Example P/T net N

rise to by some special, hidden event that is in conflict with all the other events giving rise to the other initial markings.

The key reason in the present setting for extending the definition of nets in this way is that nets with just a single initial marking do not allow obvious symmetries on nets to be expressed. The reader unfamiliar with categories of Petri nets may wish to return to the following account later, skipping directly to Section 3.1.

Consider the span of open morphisms in Figure 1(a) representing the symmetry of the two conditions p and p': the symmetry relates the condition p to p' through the condition (p, p') and through (p', p), so it is symmetric as required. Such a symmetry might arise from unfolding a general net with an event that places two tokens in a single condition [7]. If we were to remove the event from the net, we would still wish to be able to represent the symmetry of p and p'. However, a span satisfying the requirements for being a pseudo equivalence can only be obtained by allowing the span to be from a net with more than one initial marking, as seen in Figure 1(b). We are therefore obliged to consider categories of nets with multiple initial markings.

3.1 P/T nets and safe nets

A P/T net comprises sets of *conditions* (or *places*) and *events* (or *transitions*). Conditions are normally depicted as circles and events as rectangles. There are arcs from conditions to events and arcs from events to conditions, yielding a *flow relation* on the net. In Figure 2, the P/T net N has one event, t, and two conditions, p and p'. The flow relation indicates that p flows into t and t flows into both p and p'. A relatively standard requirement on nets, that simplifies the presentation of Section 5, is to require every event to have at least one condition flowing into it. It is also standard to require there to be no isolated conditions, where a condition is said to be isolated if neither does it occur as a pre- or a postcondition of some event nor does it occur in some initial marking. Other than these, we shall make no further assumptions about P/T nets.

The state of a P/T net is represented by the number of *tokens*, drawn as dots, that each condition contains. This can be considered as a multiset of conditions called the *marking* of the net. In the net N, the condition p holds one token. As discussed, a net is defined with a *set* of *initial markings* representing the set of initial states in which the net could be. Any initial marking is required to contain at most one token in any condition, so each initial marking is itself appropriately described as a set. Formally:

Definition 3. *A P/T net is a 4-tuple*

$$(P, T, F, \mathbb{M})$$

where

- *P is the set of conditions (or places),*
- *T is the set of events (or transitions), disjoint from P,*
- *$F \subseteq (P \times T) \cup (T \times P)$ is the flow relation, and*
- *$\mathbb{M} \subseteq \mathcal{P}ow(P)$ is the set of initial markings, each of which is a set.*

The net must contain no isolated places and, for each event $t \in T$, there must exist $p \in P$ such that $p\,F\,t$.

We shall call a net *singly-marked* if it follows the standard definition, having only one initial marking. A singly-marked net is a tuple (P, T, F, M) of which M is the single initial marking. We shall sometimes, when necessary to explicitly disambiguate the the old singly-marked nets from the nets with sets of initial markings introduced here, call the new nets *multiply-marked*. Be aware, however, that the set of markings of a multiply-marked net could be a singleton set or even empty.

The marking of a P/T net changes according to the occurrence of events: an event t can occur if every condition p that flows into t holds at least one token, *i.e.* each condition that flows into t occurs at least once in the marking. In this situation, the event is said to have *concession*. The resulting marking is obtained by taking a single token from each condition that flows into t and adding a single token to each condition that flows from t. This is called the *token game* for P/T nets. For instance, the token game for the net N in Figure 2 allows the event t to occur in the initial marking drawn, removing a token from condition p and placing tokens in conditions p and p' to yield a marking in which both conditions contain exactly one token.

For any event t and condition p, we adopt the notations:

$$\begin{array}{rclcrcl} {}^\bullet t &=& \{p \mid p\,F\,t\} & \quad & {}^\bullet p &=& \{t \mid t\,F\,p\} \\ t^\bullet &=& \{p \mid t\,F\,p\} & \quad & p^\bullet &=& \{t \mid p\,F\,t\} \end{array}$$

We call the set ${}^\bullet t$ the *preconditions* of t and $t^\bullet$ the *postconditions* of

t. The token game gives rise to a transition relation on markings,

$$M \xrightarrow{t} M' \overset{\triangle}{\iff} {}^\bullet t \leq M \ \&\ M' = M - {}^\bullet t + t^\bullet.$$

Note that we apply the operations of multiset union $+$ and multiset subtraction $-$ to the sets ${}^\bullet t$ and $t^\bullet$, regarding these sets as multisets. A marking that can be obtained through a sequence of events from some initial marking is said to be *reachable* from that initial marking. Observe that although each initial marking of a P/T net is a set, it need not in general be the case that every reachable marking is itself a set. For instance, the net in Figure 2 has the following sequence of event occurrences:

$$\left\{ \begin{array}{c} p\mapsto 1 \\ p'\mapsto 0 \end{array} \right\} \xrightarrow{t} \left\{ \begin{array}{c} p\mapsto 1 \\ p'\mapsto 1 \end{array} \right\} \xrightarrow{t} \left\{ \begin{array}{c} p\mapsto 1 \\ p'\mapsto 2 \end{array} \right\}$$

After two occurrences of the event t, there are two tokens in the condition p'.

We say that a net is *safe* if all its reachable markings are sets.

3.2 Occurrence nets

Occurrence nets were introduced in [12] as a class of net suited to giving the semantics of more general kinds of net in a way that directly represents the causal dependencies of elements of the net, for example that a particular event must have occurred at some earlier stage for a particular condition to become marked, and how the occurrence of elements of the net might conflict with each other. Technically, they can be thought-of as safe nets with acyclic flow relations such that every condition occurs as a postcondition of at most one event, for every condition there is a reachable marking containing that condition, and for every event there is a reachable marking in which the event can occur. We extend their original definition to account for the generalization to having a set of initial markings.

Definition 4. *An* occurrence net $O = (B, E, F, \mathbb{M})$ *is a safe net satisfying the following restrictions:*

1. $\forall M \in \mathbb{M} : \forall b \in M : ({}^\bullet b = \emptyset)$
2. $\forall b' \in B : \exists M \in \mathbb{M} : \exists b \in M : (b\ F^*\ b')$
3. $\forall b \in B : (|{}^\bullet b| \leq 1)$
4. F^+ *is irreflexive and, for all* $e \in E$, *the set* $\{e' \mid e'\ F^*\ e\}$ *is finite*

5. # is irreflexive, where

$$e \#_{\mathrm{m}} e' \stackrel{\triangle}{\iff} e \in E \;\&\; e' \in E \;\&\; e \neq e' \;\&\; {}^{\bullet}e \cap {}^{\bullet}e' \neq \emptyset$$

$$b \#_{\mathrm{m}} b' \stackrel{\triangle}{\iff} \exists M, M' \in \mathbb{M} : (M \neq M' \;\&\; b \in M \;\&\; b' \in M')$$

$$x \# x' \stackrel{\triangle}{\iff} \exists y, y' \in E \cup B : y \#_{\mathrm{m}} y' \;\&\; y \; F^{*} \; x \;\&\; y' \; F^{*} \; x'$$

The flow relation F of an occurrence net O indicates how occurrences of events and conditions causally depend on each other and the relation # indicates how they conflict with each other, with $\#_{\mathrm{m}}$ representing immediate conflict. Two events are in immediate conflict if they share a common precondition, so that the occurrence of one would mean that the other could not occur in any subsequent marking. Two conditions are in immediate conflict if they occur in different initial markings, so if one occurs in a reachable marking there is no subsequent reachable marking in which the other occurs. This corresponds to the intuition at the beginning of this section, that the hidden events giving rise to each initial marking should be in conflict with each other.

The *concurrency* relation $\mathrm{co}_O \subseteq (B \cup E) \times (B \cup E)$ of an occurrence net O may be defined as follows:

$$x \; \mathrm{co}_O \; y \stackrel{\triangle}{\iff} \neg(x \# y \text{ or } x \; F^{+} \; y \text{ or } y \; F^{+} \; x)$$

We often drop the subscript O when we write co_O if the net O is obvious from the context. The concurrency relation is extended to sets of conditions A in the following manner:

$$\mathrm{co}\; A \stackrel{\triangle}{\iff} (\forall b, b' \in A : \; b \; \mathrm{co} \; b') \text{ and } \{e \in E \mid \exists b \in A. e \; F^{*} \; b\} \text{ is finite}$$

Proposition 5. *Let $O = (B, E, F, \mathbb{M})$ be an occurrence net. Any subset $A \subseteq B$ satisfies* co A *iff there exists a reachable marking M of O such that $A \subseteq M$.*

The events and conditions of an occurrence net $(B, E, F, \mathbb{M})$ can only occur from a unique initial marking. For $x \in B \cup E$ and $M \in \mathbb{M}$, write $M \; F^{*} \; x$ if there exists $b \in M$ such that $b \; F^{*} \; x$. It is easy to see that M is unique: for any $M, M' \in \mathbb{M}$, if $M \; F^{*} \; x$ and $M' \; F^{*} \; x$ then $M = M'$.

Proposition 6. *Let $O = (B, E, F, \mathbb{M})$ be an occurrence net. For any $b \in B$ and $M \in \mathbb{M}$, if $M \; F^{*} \; b$ then there exists M' such that $b \in M'$ and M' is reachable from M. For any $e \in E$ and $M \in \mathbb{M}$, if $M \; F^{*} \; e$ then there exists M' such that e has concession in M' and M' is reachable from M.*

An occurrence net O gives rise to a set of singly-marked occurrence nets obtained by splitting the net O at each marking.

Definition 7. *Let $O = (B, E, F, \mathbb{M})$ be an occurrence net. The* marking decomposition *of O is a family of singly-marked occurrence nets $(O_M)_{M \in \mathbb{M}}$ in which the net O_M has conditions B_M and events E_M defined as*

$$B_M = \{b \in B \mid M\ F^*\ b\} \qquad E_M = \{e \in E \mid M\ F^*\ e\},$$

each net O_M inherits the flow relation of O and has initial marking M.

Morphisms of nets are introduced in the following section. It will then be possible to say that an occurrence net can be recovered, up to isomorphism, by placing the elements of its marking decompositions side-by-side, each element with its own initial marking. This will amount to taking the coproduct of the nets in its marking decomposition. Consequently, a multiply marked occurrence net can be partitioned into a family of singly-marked occurrence nets in such a way that the flow relation does not cross the partitions.

3.3 Morphisms and categorical constructions

Morphisms on Petri nets, apart from the slight generalization to multiple initial markings introduced here, were first presented in [20]. They embed the structure of one net into another in a way that preserves the behaviour (the token game) of the original net.

Definition 8. *Let $N = (B, E, F, \mathbb{M})$ and $N' = (B', E', F', \mathbb{M}')$ be P/T nets. A morphism $(\eta, \beta) : N \to N'$ comprises a partial function $\eta : E \to_* E'$ and a relation $\beta \subseteq B \times B'$ satisfying the following criteria:*

- $\forall M \in \mathbb{M}\ \exists M' \in \mathbb{M}' :\ \beta M \subseteq M' \quad \& \quad \forall b' \in M'\ \exists! b \in M :\ \beta(b, b')$,
- $\forall e \in E : \beta {}^\bullet e \subseteq {}^\bullet\eta(e) \quad \& \quad \forall b' \in {}^\bullet\eta(e)\ \exists! b \in {}^\bullet e :\ \beta(b, b')$, *and*
- $\forall e \in E : \beta e^\bullet \subseteq \eta(e)^\bullet \quad \& \quad \forall b' \in \eta(e)^\bullet\ \exists! b \in e^\bullet :\ \beta(b, b')$.

Note that we regard ${}^\bullet\eta(e) = \eta(e)^\bullet = \emptyset$ if $\eta(e) = *$. Regarding these sets as multisets, using multiset notation we might equivalently have written:

$$\forall M \in \mathbb{M} :\ \beta \cdot M \in \mathbb{M}'$$
$$\forall e \in E : \beta \cdot {}^\bullet e = {}^\bullet\eta(e)$$
$$\forall e \in E : \beta \cdot e^\bullet = \eta(e)^\bullet$$

We write $\mathbf{PT}^\sharp$ for the category of P/T nets, $\mathbf{Safe}^\sharp$ for the category of safe nets and $\mathbf{Occ}^\sharp$ for the category of occurrence nets. In each case, we omit the superscript $\sharp$ to indicate the old categories of singly-marked nets, for example writing **Occ** for the category of singly-marked occurrence nets.

We shall say that a net morphism (η, β) is *synchronous* if η is a total function on events. We add the subscript $_{\mathrm{s}}$ to denote categories with only synchronous morphisms, for example writing $\mathbf{Occ}^\sharp_{\mathrm{s}}$ for the category of multiply-marked occurrence nets with synchronous morphisms between them. A morphism (η, β) is a *folding* morphism if it is synchronous and the relation β is also a (total) function. We add the subscript $_{\mathrm{f}}$ to denote categories with only folding morphisms, for example writing $\mathbf{PT}^\sharp_{\mathrm{f}}$ for the category of multiply-marked P/T nets with folding morphisms between them.

A morphism $(\eta, \beta) : N \to N'$ respects the token game for nets in the sense that

$$\text{if } M \xrightarrow{e} M' \text{ in } N \text{ then } \beta \cdot M \xrightarrow{e} \beta \cdot M' \text{ in } N'.$$

It will be useful later to point out that, if the nets N and N' are occurrence nets, the morphism preserves markings giving rise to elements of the occurrence net, it reflects conflict and it reflects the F relation in the following sense:

Proposition 9. *Let $O_1 = (B_1, E_1, F_1, \mathbb{M}_1)$ and $O_2 = (B_2, E_2, F_2, \mathbb{M}_2)$ be occurrence nets. For events $e_1, e'_1 \in E_1$, write $e_1 \asymp_1 e'_1$ iff either $e_1 = e'_1$ or $e_1 \# e'_1$. Define $\asymp_2$ similarly for events in E_2. For any morphism $(\eta, \beta) : O_1 \to O_2$ in $\mathbf{Occ}^\sharp$:*

- *for any $b_1 \in B_1$ and $M \in \mathbb{M}_1$, if $M \ {F_1}^* \ b_1$ and $\beta(b_1, b_2)$ then $\beta \cdot M \ {F_2}^* \ b_2$*
- *for any $e_1 \in E_1$, if $\eta(e_1)$ defined and $M \ {F_1}^* \ e_1$ then $\beta \cdot M \ {F_2}^* \ \eta(e_1)$*
- *for any $e_1, e'_1 \in E_1$ and $e_2, e'_2 \in E_2$:*

$$\eta(e_1) = e_2 \ \& \ \eta(e'_1) = e'_2 \ \& \ e_2 \asymp_2 e'_2 \implies e_1 \asymp_1 e'_1$$

- *for any $b_1, b'_1 \in B_1$ and $b_2, b'_2 \in B_2$:*

$$\beta(b_1, b_2) \ \& \ \beta(b'_1, b'_2) \ \& \ b_2 \asymp_2 b'_2 \implies b_1 \asymp_1 b'_1$$

- *for any $e_2 \in E_2$, $b_1 \in B_1$ and $b_2 \in B_2$:*

$$e_2 \ F_2 \ b_2 \ \& \ \beta(b_1, b_2) \implies \exists! e_1 \in E_1 : \ e_1 \ F_1 \ b_1 \ \& \ \eta(e_1) = e_2$$

- *for any $e_1 \in E_1$, $e_2 \in E_2$ and $b_2 \in B_2$:*

$$\eta(e_1) = e_2 \ \& \ b_2 \ F_2 \ e_2 \implies \exists! b_1 \in B_1 : \ b_1 \ F_1 \ e_1 \ \& \ \beta(b_1, b_2)$$

It follows that morphisms in the category $\mathbf{Occ}^\sharp$ also preserve the concurrency relation on both events and conditions.

Coproducts in the categories of singly-marked safe nets and singly-marked occurrence nets were studied in [20]. There, the construction of $N_1 + N_2$ essentially involves 'gluing' the nets N_1 and N_2

together at their initial markings. The generalization to allow multiple initial markings allows a somewhat simpler construction in the categories $\mathbf{Occ}^\sharp$ and $\mathbf{PT}^\sharp$, where the nets are forced to operate on disjoint sets of conditions.

Proposition 10. *Let $(N_i)_{i\in I}$ be a family of P/T nets where $N_i = (P_i, T_i, F_i, \mathbb{M}_i)$ for each $i \in I$. The net $\sum_{i\in I} N_i = (P, T, F, \mathbb{M})$ defined as*

$$\begin{array}{rcl} P & = & \{\mathsf{in}_i p \mid i \in I \ \&\ p \in P_i\} \\ T & = & \{\mathsf{in}_i t \mid i \in I \ \&\ t \in T_i\} \\ (\mathsf{in}_i x)\ F\ (\mathsf{in}_j y) & \iff & i = j \ \&\ x\ F_i\ y \\ \mathbb{M} & = & \{\{\mathsf{in}_i p \mid p \in M\} \mid i \in I \ \&\ M \in \mathbb{M}_i\} \end{array}$$

is a coproduct in the category $\mathbf{PT}^\sharp$ with coproduct injections $\mathsf{in}_i : N_i \to \sum_{j\in I} N_j$.

Furthermore, the construction gives coproducts in the categories $\mathbf{Occ}^\sharp$ and $\mathbf{Safe}^\sharp$ and the categories with synchronous morphisms.

As mentioned at the end of the previous section, a multiply-marked occurrence net can be recovered (up to isomorphism) by taking the coproduct of the nets arising from each initial marking.

Proposition 11. *Let O be an occurrence net and $(O_M)_{M\in\mathbb{M}}$ be its marking decomposition. Then $O \cong \sum_{M\in\mathbb{M}} O_M$ through an isomorphism natural in O.*

We conclude this section by noting that the category $\mathbf{PT}^\sharp$, in addition to having coproducts, also has pullbacks. This result, a mild generalization of [4] where it was shown that the category of safe nets has pullbacks, is important in being able to enrich P/T nets and occurrence nets with symmetry.

Proposition 12. *The category $\mathbf{PT}^\sharp$ has pullbacks.*

A consequence of the coreflection between $\mathbf{Occ}^\sharp$ and $\mathbf{PT}^\sharp$ will be that the category $\mathbf{Occ}^\sharp$ also has pullbacks.

4 Event structures

Event structures [12, 21] represent a computational process as a set of event occurrences, recording how these event occurrences *causally depend* on each other. An event structure also records how the occurrence of an event indicates that the process has taken a particular branch. For the variant of event structure that we shall consider, called *prime* event structures, this amounts to recording how event occurrences *conflict* with each other.

Definition 13. *A (prime) event structure is a 3-tuple*

$$ES = (E, \leq \#),$$

where

- *E is the set of* events *(more precisely, event occurrences),*
- *$\leq \subseteq E \times E$ is the partial order of* causal dependency, *and*
- *$\# \subseteq E \times E$ is the irreflexive, symmetric binary relation of* conflict*.*

An event structure must satisfy the following axioms:

1. *each event causally depends on only finitely many other events, i.e. $\{e' \mid e' \leq e\}$ is finite for all $e \in E$, and*
2. *if $e_1 \# e_2$ and $e_1 \leq e_1'$ then $e_1' \# e_2$.*

The intuition is that if we have $e \leq e'$ for two events e and e', then the event e must have occurred prior to any occurrence of e'. If we have $e\#e'$, then the occurrence of e precludes the occurrence of event e' at any later stage. An event structure is said to be *elementary* if the conflict relation is empty. The first axiom ensures that an event structure only consists of event occurrences that can eventually take place, not relying on an infinite number of prior event occurrences. The second axiom asserts that if the occurrence of an event e_2 precludes the occurrence of an event e_1 upon which the event e_1' causally depends, then the event e_2 precludes the occurrence of the event e_1'. We say that two events e_1 and e_2 are *concurrent*, written e_1 co e_2, if there is no causal dependency between them and they do not conflict, *i.e.* $e_1 \text{ co } e_2 \iff \neg(e_1 \# e_2 \text{ or } e_1 \leq e_2 \text{ or } e_2 \leq e_1)$. We write $e < e'$ if $e \leq e'$ but $e \neq e'$.

The computational states of an event structure, called its *configurations*, are represented by the sets of events that have occurred. Every configuration must be consistent with the relations of conflict and causal dependency. Formally, $x \subseteq E$ is a configuration of an event structure $(E, \leq, \#)$ if it satisfies the following two properties:

- Conflict-freedom: $\forall e, e' \in x : \neg(e\#e')$
- Downwards-closure: $\forall e, e' \in E : e \leq e' \;\&\; e' \in x \implies e \in x$.

We write $\mathcal{D}(ES)$ for the set of configurations of ES and write $\mathcal{D}^0(ES)$ for the set of finite configurations of ES. We write $\lceil e \rceil$ for $\{e' \mid e' \leq e\}$, the least configuration containing the event e.

4.1 Morphisms

We now introduce morphisms of event structures. A morphism $\eta : ES \to ES'$ is a function from the events of ES to the events of

ES' that expresses how the behaviour of ES embeds into ES' in the sense that the function preserves the the configurations of the event structure and also preserves the atomicity of events.

Definition 14. *Let $ES = (E, \leq, \#)$ and $ES' = (E', \leq', \#')$ be event structures. A morphism $\eta : ES \to ES'$ consists of a partial function $\eta : E \to_* E'$ such that for all $x \in \mathcal{D}(ES)$:*

$$\begin{array}{ll} & \eta x \in \mathcal{D}(ES) \\ \& & \forall e, e' \in x : \; \eta(e), \eta(e') \text{ defined } \; \& \; \eta(e) = \eta(e') \implies e = e' \end{array}$$

A morphism is said to be synchronous *if it is a total function on events.*

In fact, it is only necessary to consider finite configurations $x \in \mathcal{D}^0(ES)$ in the requirement on morphisms above. It is easy to see that if $x \xrightarrow{e} x'$ then $\eta x \xrightarrow{\eta(e)} \eta x'$.

We obtain a category **ES** of (prime) event structures with event structures as objects and morphisms as described above. The identity morphism on an event structure is the identity function on its underlying set of events, and composition of morphisms occurs as composition of functions. We also obtain a category $\mathbf{ES}_\mathrm{s}$ of event structures with synchronous morphisms between then. We write **Elem** for the category of elementary event structures (event structures with no conflict) and $\mathbf{Elem}_\mathrm{s}$ for the category of elementary event structures with synchronous morphisms between them. Elementary event structures can be thought of as paths of event structures and nets, and these categories will later be used to define open maps of event structures and nets.

Before moving on to consider their relationship with Petri nets, we note that the categories **ES** and $\mathbf{ES}_\mathrm{s}$ have coproducts obtained by forming the disjoint union of their events using injections in_i, placing two events in conflict if they occur in different components of the coproduct.

Proposition 15. *Let $(ES)_{i \in I}$ be a family of event structures indexed by I, where $ES_i = (E_i, \leq_i, \#_i)$. A coproduct of these event structures in the category **ES** and also in the category $\mathbf{ES}_\mathrm{s}$ is the event structure $\sum_{i \in I} ES_i = (E, \leq, \#)$ with events $E = \{\mathsf{in}_i e \mid i \in I \;\&\; e \in E_i\}$ and relations*

$$\begin{array}{rcl} \mathsf{in}_i e \;\leq\; \mathsf{in}_j e' & \iff & i = j \;\&\; e \leq_i e' \\ \mathsf{in}_i e \;\#\; \mathsf{in}_j e' & \iff & i \neq j \text{ or } (i = j \;\&\; e \#_i e') \end{array}$$

For each $j \in I$, the function $\mathsf{in}_j : ES_j \to \sum_{i \in I} ES_i$ defined as $\mathsf{in}_i(e) = \mathsf{in}_i e$ is the associated injection into the coproduct.

The category of event structures also has pullbacks. Their construction is given in Appendix C of [23]. Their direct construction is

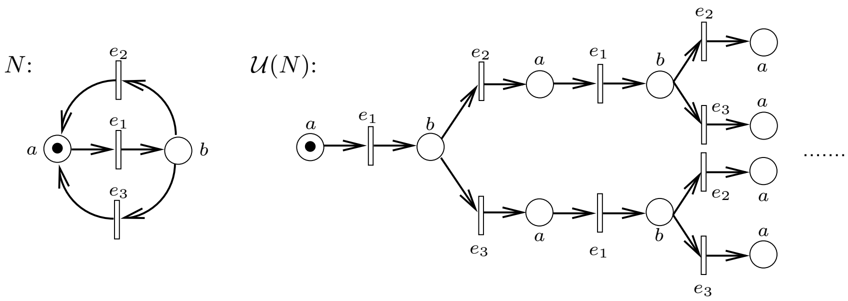


Figure 3: The unfolding of a safe net

hard, being most easily seen in the category of stable families, so we shall not present it here.

5 Results on singly-marked nets

We briefly recount some results on singly-marked nets. We shall first describe the coreflection between singly-marked safe nets and occurrence nets and shall then give the coreflection between singly-marked occurrence nets and event structures. The constructions used in the second coreflection shall be used in the description of the coreflection between multiply-marked occurrence nets and event structures.

5.1 Singly-marked occurrence nets and safe nets

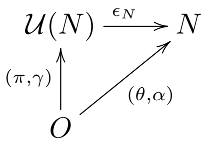
The category **Occ** is a coreflective subcategory of **Safe**:

$$\mathbf{Occ} \underset{\mathcal{U}}{\overset{\hookrightarrow}{\ \perp\ }} \mathbf{Safe}$$

The left adjoint is the inclusion of the category of occurrence nets into the category of safe nets. The right adjoint *unfolds* the safe net to an occurrence net. An example is presented in Figure 3, with events and conditions in the unfolding labelled by the corresponding events and conditions in the original net. The occurrence net unfolding of a safe net, first defined in [12], describes how the event occurrences of the safe net causally depend on and conflict with each other due to their effect on the holding of conditions. We shall not describe the unfolding concretely here, though its definition occurs in Section 6 in the more general setting of multiply-marked P/T nets.

The unfolding $\mathcal{U}(N)$ of a safe net N is equipped with a morphism $\epsilon_N : \mathcal{U}(N) \to N$ in the category **Safe** relating the unfolding back to the original net. The adjunction between safe nets and occurrence nets [20, 21] is proved by showing that $\mathcal{U}(N)$ and ϵ_N are *cofree* over N.

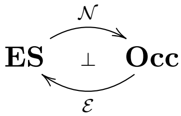
That is, for any safe net N and morphism of nets $(\theta, \alpha) : O \to N$ from an occurrence net, there is a unique morphism of occurrence nets $(\pi, \gamma) : O \to \mathcal{U}(N)$ such that the following diagram commutes:



It is a standard result of category theory that this is sufficient to give the adjunction [10]. The adjunction is a coreflection because the inclusion **Occ** $\hookrightarrow$ **Safe** is full and faithful. The coreflection also goes through for synchronous morphisms to show that $\mathbf{Occ}_{\mathrm{s}}$ is a coreflective subcategory of $\mathbf{Safe}_{\mathrm{s}}$.

5.2 Event structures and singly-marked occurrence nets

There is a coreflection that embeds the category of event structures into the category of singly-marked occurrence nets.



The functor $\mathcal{N}$ constructs an occurrence net from an event structure, saturating the events of the event structure with as many conditions as possible that are consistent with the the relations of causal dependency and conflict in the original event structure. The functor $\mathcal{E}$ strips away the conditions from the occurrence net to reveal the underlying causal dependency and conflict relations on events. Since we shall use the constructions in forming a coreflection between event structures and multiply-marked occurrence nets, we now give the constructions $\mathcal{E}$ and $\mathcal{N}$ concretely. A coreflection can also be obtained via the category of asynchronous transition systems as in [25].

The functor $\mathcal{E} : \mathbf{Occ} \to \mathbf{ES}$

The functor $\mathcal{E}$ takes an occurrence net to an event structure by interpreting causal dependency on the events of the occurrence net as the transitive closure of the flow relation and obtaining the conflict relation as in Definition 4.

Definition 16. *Let $O = (B, E, F, \mathbb{M})$ be an occurrence net. The event structure $\mathcal{E}(O) = (E, \leq, \#)$ has the same events as O, inherits conflict from O as in Definition 4 and has $e \leq e'$ iff $e\ F^*\ e'$.*

It is an immediate consequence of the definitions that $\mathcal{E}(O)$ is an event structure for any occurrence net O. Recalling that a morphism between occurrence nets O and O' is a pair (η, β) of which $\eta : E \rightarrow_* E'$ is a partial function on their underlying sets of events, we obtain the operation of the functor on morphisms.

Proposition 17. *Let* $(\eta, \beta) : O \rightarrow O'$ *be a morphism in* **Occ**. *Then* $\eta : \mathcal{E}(O) \rightarrow \mathcal{E}(O')$ *is a morphism in* **ES**.

It is straightforward to see that defining $\mathcal{E}(\eta, \beta) = \eta$ yields an operation that preserves identities and composition, so $\mathcal{E} : \mathbf{Occ} \rightarrow \mathbf{ES}$ is a functor. This is easily seen to restrict to categories with synchronous morphisms, so also $\mathcal{E} : \mathbf{Occ}_{\mathrm{s}} \rightarrow \mathbf{ES}_{\mathrm{s}}$.

The functor $\mathcal{N} : \mathbf{ES} \rightarrow \mathbf{Occ}$

We now consider how to form an occurrence net from an event structure. As stated earlier, the essential idea is to form an occurrence net with the same events as the original event structure, adding as many conditions as possible that are consistent with the causal dependency and conflict relations of the original event structure.

Definition 18. *Let* $ES = (E, \leq, \#)$ *be a event structure. The net* $\mathcal{N}(ES)$ *is defined as* $(B, E, F, \{M\})$, *where*

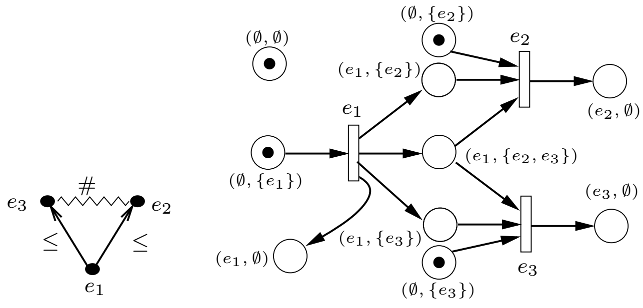
$$\begin{array}{rcl} M & = & \{(\emptyset, A) \mid A \subseteq E \;\&\; (\forall a, a' \in A : a \asymp a')\} \\ B & = & M \cup \begin{array}{c} \{(e, A) \mid e \in E \;\&\; A \subseteq E \;\&\; (\forall a, a' \in A : a \asymp a') \\ \& \; (\forall a \in A : e < a)\} \end{array} \\ F & = & \quad \{(e, (e, A)) \mid (e, A) \in B\} \\ & & \cup\{((x, A), e) \mid (x, A) \in B \;\&\; e \in A\} \end{array}$$

The net is formed with conditions (e, A) indicating that all the events in A are in conflict with each other and all causally depend on e. There are conditions $(\emptyset, A)$ to indicate just that the events in A are in conflict with each other but might not causally depend on some other event. The net formed is *condition-extensional* in the sense that any two conditions with precisely the same beginning- and end-events are identified. The occurrence net of an example event structure is presented in Figure 4.

Proposition 19. *The net* $\mathcal{N}(ES)$ *is an occurrence net. Furthermore, for any event structure* ES *we have* $\mathcal{E}(\mathcal{N}(ES)) = ES$.

Freeness and morphisms

In order to obtain a coreflection, this time it is easier to show a *freeness* result. This is sufficient, also, to show how the operation $\mathcal{N}$ extends to a functor [10].



(a) Event structure ES

(b) Occurrence net $\mathcal{N}(ES)$

Figure 4: An event structure with its associated occurrence net

Proposition 20. *For any event structure in* **ES**, *the net* $\mathcal{N}(ES)$ *and morphism* $\mathsf{id}_{ES} : ES \to \mathcal{E}(\mathcal{N}(ES))$ *is free over* ES *with respect to* $\mathcal{E}$. *That is, for any occurrence net* O *and morphism* $\eta : ES \to \mathcal{E}(O)$ *in* **ES** *there is a unique morphism* $(\pi, \gamma) : \mathcal{N}(ES) \to O$ *in* **Occ** *such that the triangle in the following diagram commutes:*

$$\begin{array}{ccccc} \mathcal{N}(ES) & \quad & \mathcal{E}(\mathcal{N}(ES)) & \xleftarrow{\mathsf{id}_{ES}} & ES \\ \Big\downarrow {\scriptstyle (\pi,\gamma)} & & \Big\downarrow {\scriptstyle \mathcal{E}(\pi,\gamma)=\pi} & \swarrow {\scriptstyle \eta} & \\ O & & \mathcal{E}(O) & & \end{array}$$

Hence the functor $\mathcal{N}$: **ES** $\to$ **Occ** is left-adjoint to the functor $\mathcal{E}$: **Occ** $\to$ **ES**. Since the unit of the adjunction is a natural isomorphism (in this case, the identity), the adjunction is a coreflection.

Proposition 20 also applies using the categories $\mathbf{ES}_{\mathrm{s}}$ and $\mathbf{Occ}_{\mathrm{s}}$ in place of **ES** and **Occ**, so a coreflection is obtained for the categories with synchronous morphisms. We shall to use the same symbols to represent the functors $\mathcal{N} : \mathbf{ES}_{\mathrm{s}} \to \mathbf{Occ}_{\mathrm{s}}$ and $\mathcal{E} : \mathbf{Occ}_{\mathrm{s}} \to \mathbf{ES}_{\mathrm{s}}$.

6 Relating occurrence nets and P/T nets

In this section, we consider an adjunction between the category of P/T nets and its full subcategory of occurrence nets.

$$\mathbf{Occ}^{\sharp} \underset{\mathcal{U}}{\overset{\hookrightarrow}{\bot}} \mathbf{PT}^{\sharp}$$

The left adjoint is the inclusion arising from every occurrence net being a P/T net; the right adjoint is the functor $\mathcal{U}$ which unfolds a P/T net to an occurrence net. The adjunction is a coreflection, with

$\mathbf{Occ}^\sharp$ being a coreflective subcategory of $\mathbf{PT}^\sharp$. We shall work within the broader categories of occurrence and P/T nets with morphisms that can be partial on events, although the coreflection cuts down to yield a coreflection between the subcategories of occurrence nets and P/T nets with synchronous morphisms. The proof of the coreflection follows that in [20] for safe nets and occurrence nets, with a little generalization to account for multiple markings.

We shall give a direct construction of the coreflection between multiply-marked nets. The coreflection shall be seen to restrict to giving a coreflection between singly-marked occurrence and P/T nets. The adjunction between the categories of multiply-marked nets cannot be obtained algebraically from the adjunction between singly-marked nets in an analogous manner to the adjunction in the following section between event structures and occurrence nets because, in general, a multiply-marked P/T net cannot be expressed as the coproduct of singly-marked P/T nets.

An important fact when dealing with occurrence nets is that any element of an occurrence net occurs at a unique *depth*. For a condition b and an event e of an occurrence net $O = (B, E, F, \mathbb{M})$, depth is defined as:

$$\begin{array}{rcll} \mathrm{depth}(b) & = & 0 & \text{if } \exists M \in \mathbb{M} : (b \in M) \\ \mathrm{depth}(b) & = & \mathrm{depth}(e) & \text{if } e\,F\,b \end{array}$$

$$\mathrm{depth}(e) = 1 + \max\{\mathrm{depth}(b) \mid b \in B \ \&\ b\,F\,e\}$$

The occurrence net unfolding of a P/T net N is defined inductively: The unfolding to depth zero $\mathcal{U}_0(N)$ is defined first and then the unfolding $\mathcal{U}_n(N)$ to depth n is used to define the unfolding $\mathcal{U}_{n+1}(N)$ to depth $n+1$. The unfolding $\mathcal{U}(N)$ is obtained as a colimit of this sequence. We shall not present details here, but shall characterize the unfolding uniquely.

Theorem 21. *Let $N = (P, T, F_N, \mathbb{M}_N)$ be a P/T net. The unfolding $\mathcal{U}(N) = (B, E, F, \mathbb{M})$ is the unique occurrence net to satisfy*

$$\begin{array}{rcl} B & = & \{(M, p) \mid M \in \mathbb{M}_N \ \&\ p \in M\} \\ & & \cup\{(\{e\}, p) \mid e \in E \ \&\ p \in \eta(e)^\bullet\} \\ E & = & \{(A, t) \mid A \subseteq B \ \&\ t \in T \ \&\ \mathrm{co}\, A \ \&\ \beta A = {}^\bullet\eta(t)\} \\ & & (X, p) \ \ F \ \ (A, t) \iff (X, p) \in A \\ & & (A, t) \ \ F \ \ (X, p) \iff X = \{(A, t)\} \\ \mathbb{M} & = & \{\{(M, p) \mid p \in M\} \mid M \in \mathbb{M}_N\} \end{array}$$

where co *is the concurrency relation on $\mathcal{U}(N)$ and*

$$\eta(A, t) = t \qquad \beta((X, p), p') \iff p = p'.$$

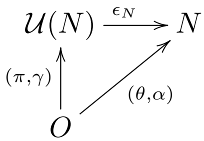
Furthermore, $\epsilon_N = (\eta, \beta) : \mathcal{U}(N) \to N$ is a morphism in the category $\mathbf{PT}^\sharp$.

Proof. Existence of the net and morphism follows from the inductive definition. Uniqueness follows from the fact that any element of an occurrence net occurs at unique depth. ∎

6.1 A coreflection

Now that we have presented the operation of unfolding a P/T net to form an occurrence net, we characterize the operation as being right adjoint to the inclusion of the category of occurrence nets into the category of P/T nets. To do so, we show that the construction is cofree with respect to N.

Theorem 22. *For any P/T net N and occurrence net O, if there is a morphism $(\theta, \alpha) : O \to N$ in the category $\mathbf{PT}^\sharp$ then there is a unique morphism $(\pi, \gamma) : O \to \mathcal{U}(N)$ in the category $\mathbf{Occ}^\sharp$ such that the following diagram commutes:*



As standard [10], this cofreeness result implies that the operation $\mathcal{U}(N)$ extends to being a functor $\mathcal{U} : \mathbf{PT}^\sharp \to \mathbf{Occ}^\sharp$ which is right adjoint to the forgetful functor $\mathbf{Occ}^\sharp \hookrightarrow \mathbf{PT}^\sharp$. Since $\mathbf{Occ}^\sharp$ is a full and faithful subcategory of $\mathbf{PT}^\sharp$, it follows that the adjunction is a coreflection.

The coreflection is readily seen to restrict to give a coreflection between the categories with synchronous morphisms $\mathbf{Occ}^\sharp_s$ and $\mathbf{PT}^\sharp_s$ and to categories of singly-marked nets **Occ** and **PT**.

6.2 Symmetry

An important class of net, *causal nets*, is often encountered in describing paths of nets. They shall form a path category from which open maps of nets may be obtained.

Definition 23. *A causal net is an occurrence net with empty conflict relation and with at most one initial marking.*

We denote the category of causal nets with net morphisms **Caus**. We shall not go into detail here, but the requirements for a net morphism to be **Caus**-open are stronger than those required for it to be $\mathcal{N}^\sharp$**Elem**-open, requiring that the morphism is a bijection when restricted to any reachable marking, in addition to the path lifting condition required for $\mathcal{N}^\sharp$**Elem**-bisimilarity.

We wish to show that the coreflection above extends to give an adjunction between the categories enriched with symmetry

$$\mathcal{S}_{\mathbf{Caus}}\mathbf{Occ}^{\sharp} \underset{\mathcal{SU}}{\overset{\hookrightarrow}{\perp}} \mathcal{S}_{\mathbf{Caus}}\mathbf{PT}^{\sharp}$$

The requirement that the inclusion $\mathbf{Occ}^{\sharp} \hookrightarrow \mathbf{PT}^{\sharp}$ and the functor $\mathcal{U} : \mathbf{PT}^{\sharp} \to \mathbf{Occ}^{\sharp}$ preserve open maps follows from the earlier coreflection and results on preservation of open maps through coreflections [9]. The functor $\mathcal{U}$ preserves limits since it is a right adjoint, so all that remains is to show that the inclusion $\mathbf{Occ}^{\sharp} \hookrightarrow \mathbf{PT}^{\sharp}$ preserves pullbacks of **Caus**-open maps.

It can be proved that all **Caus**-open morphisms of the category $\mathbf{Occ}^{\sharp}$ are folding morphisms (morphisms that are functions on conditions). That is, any **Caus**-open morphism in $\mathbf{Occ}^{\sharp}$ occurs in the category $\mathbf{Occ}^{\sharp}_{\mathrm{f}}$. The category $\mathbf{Occ}^{\sharp}_{\mathrm{f}}$ gives us a handle on pullbacks of open maps, since pullbacks in the category $\mathbf{Occ}^{\sharp}_{\mathrm{f}}$ are known to exist and are relatively straightforwardly characterized; they coincide with the pullbacks taken in $\mathbf{PT}^{\sharp}$.

Lemma 24. *The inclusions*

$$\begin{array}{rcl} \mathbf{Occ}^{\sharp}_{\mathrm{f}} & \hookrightarrow & \mathbf{Occ}^{\sharp} \\ \mathbf{Occ}^{\sharp}_{\mathrm{f}} & \hookrightarrow & \mathbf{PT}^{\sharp}_{\mathrm{f}} \\ \mathbf{PT}^{\sharp}_{\mathrm{f}} & \hookrightarrow & \mathbf{PT}^{\sharp} \end{array}$$

preserve pullbacks.

Consequently, the inclusion $\mathbf{Occ}^{\sharp} \hookrightarrow \mathbf{PT}^{\sharp}$ preserves pullbacks of **Caus**-open morphisms, despite the fact that the inclusion does not preserve pullbacks of *all* morphisms.

As an immediate consequence of Lemma 24 and Proposition 2, a coreflection between categories enriched with symmetry can now be demonstrated.

Theorem 25. *The unfolding functor* $\mathcal{SU} : \mathcal{S}_{\mathbf{Caus}}\mathbf{PT}^{\sharp} \to \mathcal{S}_{\mathbf{Caus}}\mathbf{Occ}^{\sharp}$ *is right adjoint to the inclusion* $\mathcal{S}_{\mathbf{Caus}}\mathbf{Occ}^{\sharp} \hookrightarrow \mathcal{S}_{\mathbf{Caus}}\mathbf{PT}^{\sharp}$. *Furthermore, the adjunction is a coreflection.*

The definition of symmetry in Section 2 used to define the categories with symmetry here is different from that in [23, 24] since it requires a symmetry to be a span that is a *pseudo* equivalence rather than an equivalence. As such, the maps l and r of an object with symmetry need not be jointly monic according to the definition here. In Figure 5, we give a symmetry that happens to be jointly monic in $\mathbf{Occ}^{\sharp}$. When the morphisms l and r are considered in the category $\mathbf{PT}^{\sharp}$ (or in $\mathbf{Safe}^{\sharp}$), however, the morphisms are not jointly

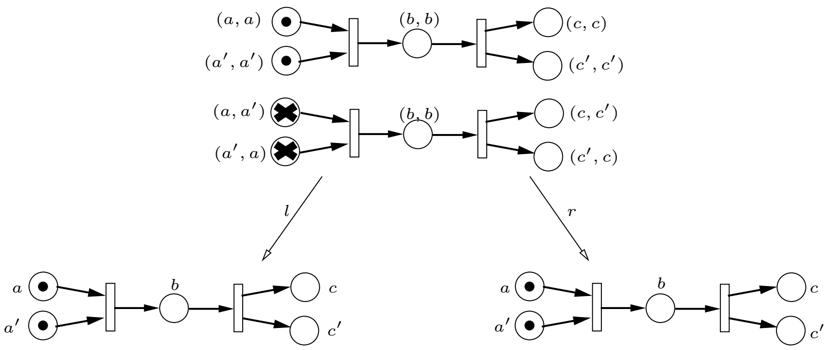


Figure 5: A symmetry $(N; l, r)$ with (folding) morphisms $l(x, y) = x$ and $r(x, y) = y$.

monic. With the restriction to jointly monic maps, a symmetry of occurrence nets would not be a symmetry of P/T nets by virtue of the fact that any occurrence net is a P/T net. Let the symmetry in Figure 5 be denoted $(N; l, r)$ spanning from the net S. In fact, it can be seen that there is no corresponding jointly monic symmetry in the category $\mathbf{PT}^\sharp$ since the image of the net S in $N \times N$, taking the product in $\mathbf{PT}^\sharp$, has more behaviour than S. Consequently, we would fail to obtain a coreflection if we were to restrict attention to jointly monic maps.

7 Relating event structures and occurrence nets

We now progress to consider a coreflection between event structures and the multiply-marked occurrence nets presented in Section 3.2. To obtain an adjunction, it is necessary to restrict attention to categories of occurrence nets and event structures with synchronous morphisms (morphisms that are total on events). We shall briefly mention how partiality could be recovered at the end of this section.

We first define how a multiply-marked occurrence net forms an event structure, giving rise to a functor $\mathcal{E}^\sharp_{\mathrm{s}} : \mathbf{Occ}^\sharp_{\mathrm{s}} \to \mathbf{ES}_{\mathrm{s}}$.

The construction of the event structure $\mathcal{E}^\sharp_{\mathrm{s}}(O)$ from a multiply-marked occurrence net O is similar to that presented in Section 5.2. The events of $\mathcal{E}^\sharp_{\mathrm{s}}(O)$ are simply the events of O; causal dependency of events is obtained from the flow relation F; and the conflict relation on events is obtained from the conflict relation of O. Recall that the conflict relation on the occurrence net places two events in conflict if they are given rise to by different initial markings.

Definition 26. *Let* $O = (B, E, F, \mathbb{M})$ *be an occurrence net. The* *event structure* $\mathcal{E}^\sharp_{\mathrm{s}}(N)$ *is* $(E, \leq, \#)$ *where*

$$e \leq e' \iff e\ F^*\ e'$$

and $\#$ *is the conflict relation on the occurrence net* O *in Definition 4.*

The operation $\mathcal{E}^\sharp_{\mathrm{s}}$ extends to a functor $\mathcal{E}^\sharp_{\mathrm{s}} : \mathbf{Occ}^\sharp_{\mathrm{s}} \to \mathbf{ES}_{\mathrm{s}}$ by taking a morphism of occurrence nets $(\eta, \beta) : O \to O'$ to

$$\mathcal{E}^\sharp_{\mathrm{s}}(\eta, \beta) = \eta.$$

It is relatively straightforward to show that $\eta : \mathcal{E}^\sharp_{\mathrm{s}}(O) \to \mathcal{E}^\sharp_{\mathrm{s}}(O')$ is a morphism of event structures and that $\mathcal{E}^\sharp_{\mathrm{s}}$ satisfies the requirements for being a functor.

The specification of a functor from event structures to occurrence nets with multiple initial markings is less straightforward. The generalization of occurrence nets to allow them to possess more than one initial marking gives rise to two distinct ways in which their events may be in conflict.

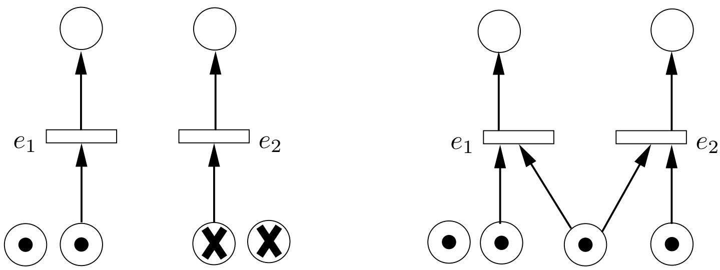
'Early' conflict Any event in an occurrence net can occur in a marking reachable from precisely one initial marking. The events may conflict if they arise from distinct initial markings.

'Late' conflict As with singly-marked occurrence nets, two events e_1 and e_2 might be in conflict because they either share a precondition or there might exist events e_1' and e_2' that share a common precondition for which e_1 causally depends on e_1' and e_2 causally depends on e_2'.

Quite clearly, all conflict in singly-marked occurrence nets is late conflict. The old functor $\mathcal{N} : \mathbf{ES}_{\mathrm{s}} \to \mathbf{Occ}_{\mathrm{s}}$ from event structures to singly-marked occurrence nets therefore uses late conflict to represent conflict in the event structure.

In the category $\mathbf{Occ}^\sharp_{\mathrm{s}}$, early conflict embeds into late conflict. Consider, for instance, the nets in Figure 6. There is a morphism preserving events from N to N'. Late conflict, however, does not embed into early conflict; there is no morphism preserving events from N' to N.

The old functor $\mathcal{N}$ can be seen, as a result, not to be left adjoint to the new functor $\mathcal{E}^\sharp_{\mathrm{s}}$. If it were, there would have to be a morphism preserving events of the form $\mathcal{N}(\mathcal{E}^\sharp_{\mathrm{s}}(N)) \to N$. It is easy to see that the event structures $\mathcal{E}^\sharp_{\mathrm{s}}(N)$ and $\mathcal{E}^\sharp_{\mathrm{s}}(N')$ are equal, comprising two events e_1 and e_2 that are in conflict. The net $\mathcal{N}(\mathcal{E}^\sharp_{\mathrm{s}}(N))$ is isomorphic to N', so the required morphism does not exist. The problem is that in constructing the net $\mathcal{N}(\mathcal{E}^\sharp_{\mathrm{s}}(N))$, early conflict is replaced by late conflict. We must therefore define a new functor $\mathcal{N}^\sharp_{\mathrm{s}} : \mathbf{ES}_{\mathrm{s}} \to \mathbf{Occ}^\sharp_{\mathrm{s}}$



(a) Net N with early conflict

(b) Net N' with late conflict

Figure 6: Nets with early conflict and late conflict

that ensures that if two events of a net N are in early conflict then they remain in early conflict in the net $\mathcal{E}_s^\sharp(\mathcal{N}_s^\sharp(N))$.

The functor $\mathcal{N}_s^\sharp$ will involve the *compatibility* relation to distinguish which pairs of events in $\mathcal{E}_s^\sharp(N)$ could possibly have been in early conflict in the net N. Let $ES = (E, \leq, \#)$ be an event structure. The compatibility relation $\frown \subseteq E \times E$ is defined as:

$$e \frown e' \stackrel{\triangle}{\iff} \neg(e \# e')$$

Two events are compatible if there exists a configuration containing them both. The compatibility relation is symmetric and reflexive, so its transitive closure $\frown^+$ is an equivalence relation. The event structure ES can be partitioned into a family $(ES_c)_{c\in C}$ of $\frown^+$-equivalence classes. Each $\frown^+$-equivalence class ES_c is an event structure with conflict and causal dependency inherited from ES. Any event of ES_c is in conflict with every event of ES_d in the event structure ES if $c \neq d$.

Lemma 27. *Let ES be an event structure and let the $\frown^+$-equivalence classes contained in ES form the family $(ES_c)_{c\in C}$ for some indexing set C. Each equivalence class ES_c is an event structure and $ES \cong \sum_{c\in C} ES_c$ through an isomorphism natural in ES, taking the coproduct in the category $\mathbf{ES}_s$ defined in Proposition 15.*

Two events of the event structure $\mathcal{E}_s^\sharp(N)$ cannot be in early conflict if they are $\frown^+$-related. Given an event structure ES, we are now able to define the occurrence net $\mathcal{N}_s^\sharp(ES)$.

Definition 28. *Let ES be an event structure generated by the family of $\frown^+$-equivalence classes $(ES_c)_{c\in C}$. Define*

$$\mathcal{N}_s^\sharp(ES) \triangleq \sum_{c\in C} \mathcal{N}(ES_c).$$

It is clear that $\mathcal{N}^\sharp_{\mathrm{s}}(ES)$ is an occurrence net since $\mathcal{N}(ES_c)$ is known to be a (singly-marked) occurrence net for each $c \in C$ and the coproduct of occurrence nets is itself an occurrence net.

An important observation when considering the coreflection between event structures and multiply-marked occurrence nets will be that morphisms of event structures preserve the relation $\smile^+$.

Lemma 29. *Let* $ES_1 = (E_1, \leq_1, \#_1)$ *and* $ES_2 = (E_2, \leq_2, \#_2)$ *be event structures with compatibility relations* $\smile_1$ *and* $\smile_2$ *respectively. Let* $\eta : ES_1 \to ES_2$ *be a morphism in* $\mathbf{ES}_{\mathrm{s}}$*. For any* $e, e' \in E_1$*, if* $e \smile_1^+ e'$ *then* $\eta(e) \smile_2^+ \eta(e')$*.*

The previous lemma relies on the fact that morphisms are synchronous, *i.e.* total on events. It need not be the case that if $e_1 \smile_1^+ e_2$ and $\eta(e_1)$ and $\eta(e_2)$ are defined for some non-synchronous morphism η then $\eta(e_1) \smile_2^+ \eta(e_2)$.

Let the $\smile_1^+$-equivalence classes of ES_1 be the family $(ES_c)_{c \in C}$ and the $\smile_2^+$-equivalence classes of ES_2 be the family $(ES_d)_{d \in D}$, and suppose that there is a synchronous morphism $\eta : ES_1 \to ES_2$ in $\mathbf{ES}_{\mathrm{s}}$. As a consequence of the previous lemma, for all $c \in C$ and $d \in D$:

$$\begin{aligned} & \exists e \in E_c : \ E_d = \{e_2 \mid \eta(e) \smile_2^+ e_2\} \\ \iff \quad & \forall e \in E_c : \ E_d = \{e_2 \mid \eta(e) \smile_2^+ e_2\}. \end{aligned}$$

We may therefore define a function $\hat{\eta} : C \to D$ as $\hat{\eta}(c) = d$ iff $\exists e \in E_c : \ E_d = \{e_2 \mid \eta(e) \smile_2^+ e_2\}$, which informs that the event structure ES_c within ES_1 is taken by η to $ES_{\hat{\eta}(c)}$ in ES_2. The morphism $\eta : ES_1 \to ES_2$ therefore restricts to a morphism $\eta_c : ES_c \to ES_{\hat{\eta}(c)}$. Applying the old functor $\mathcal{N} : \mathbf{ES}_{\mathrm{s}} \to \mathbf{Occ}_{\mathrm{s}}$ from event structures to singly-marked occurrence nets, we obtain a morphism $\mathcal{N}(\eta_c) : \mathcal{N}(ES_c) \to \mathcal{N}(ES_d)$. We therefore have a morphism

$$\mathsf{in}_c \circ \mathcal{N}(\eta_c) : \mathcal{N}(ES_c) \to \mathcal{N}^\sharp(ES_2)$$

for each $c \in C$. Since $\mathcal{N}^\sharp(ES_1) = \sum_{c \in C} \mathcal{N}(ES_c)$ is a coproduct, we have a morphism

$$\mathcal{N}^\sharp_{\mathrm{s}}(\eta) : \mathcal{N}^\sharp(ES_1) \to \mathcal{N}^\sharp_{\mathrm{s}}(ES_2).$$

This construction is straightforwardly shown to form a functor $\mathcal{N}^\sharp_{\mathrm{s}} : \mathbf{ES}_{\mathrm{s}} \to \mathbf{Occ}^\sharp_{\mathrm{s}}$.

We now proceed to show that $\mathcal{N}^\sharp_{\mathrm{s}}$ is left adjoint to the functor $\mathcal{E}^\sharp_{\mathrm{s}}$ giving the coreflection

$$\mathbf{ES}_{\mathrm{s}} \underset{\mathcal{E}^\sharp_{\mathrm{s}}}{\overset{\mathcal{N}^\sharp_{\mathrm{s}}}{\rightleftarrows}} \mathbf{Occ}^\sharp_{\mathrm{s}} \quad (\bot)$$

This will also specify the action of the operation $\mathcal{N}_s^\sharp$ on morphisms of event structures, yielding a functor $\mathcal{N}_s^\sharp : \mathbf{ES}_s \to \mathbf{Occ}_s^\sharp$.

Theorem 30. *The functors $\mathcal{E}_s^\sharp$ and $\mathcal{N}_s^\sharp$ form a coreflection: There is an isomorphism of hom-sets*

$$\phi_{ES,N} : \mathbf{ES}_s(ES, \mathcal{E}_s^\sharp(N)) \cong \mathbf{Occ}_s^\sharp(\mathcal{N}_s^\sharp(ES), N),$$

natural in ES and N and, furthermore, the functor $\mathcal{N}_s^\sharp$ is full and faithful.

PROOF. Let the occurrence net $O = (B, E, F, \mathbb{M})$ have marking decomposition $(O_M)_{M\in\mathbb{M}}$. Suppose that the $\frown^+$-decomposition of the event structure ES is $(ES_c)_{c\in C}$. We have the following chain of isomorphisms.

$$\begin{array}{rcll}
\mathbf{ES}_s(ES, \mathcal{E}_s^\sharp(O)) & \cong & \mathbf{ES}_s(\sum_{c\in C} E_c, \sum_{M\in\mathbb{M}} \mathcal{E}(O_M)) & (1)\\
& \cong & \prod_{c\in C} \mathbf{ES}_s(ES_c, \sum_{M\in\mathbb{M}} \mathcal{E}(O_M)) & (2)\\
& \cong & \prod_{c\in C} \sum_{M\in\mathbb{M}} \mathbf{ES}_s(ES_c, \mathcal{E}(O_M)) & (3)\\
& \cong & \prod_{c\in C} \sum_{M\in\mathbb{M}} \mathbf{Occ}_s(\mathcal{N}(ES_c), O_M) & (4)\\
& \cong & \prod_{c\in C} \mathbf{Occ}_s^\sharp(\mathcal{N}(ES_c), \sum_{M\in\mathbb{M}} O_M) & (5)\\
& \cong & \mathbf{Occ}_s^\sharp(\sum_{c\in C} \mathcal{N}(ES_c), \sum_{M\in\mathbb{M}} O_M) & (6)\\
& \cong & \mathbf{Occ}_s^\sharp(\mathcal{N}_s^\sharp(ES), O) & (7)
\end{array}$$

Isomorphism (1) is an immediate consequence of Lemma 27 and the definition of $\mathcal{E}_s^\sharp(O)$. Isomorphisms (2) and (6) are from the universal characterization of coproduct. Isomorphism (3) is a straightforward consequence of Lemma 29. Isomorphism (4) arises from the old adjunction between singly-marked occurrence nets and event structures. Isomorphism (5) is a straightforward consequence of Proposition 9. Finally, isomorphism (7) follows from Proposition 11 and the definition of $\mathcal{N}_s^\sharp(ES)$. We omit the proofs that the isomorphisms are natural.

The functor $\mathcal{N}_s^\sharp$ is easily seen to be full and faithful as a consequence of the functor $\mathcal{N}$ being full and faithful, due to the existing coreflection between singly-marked occurrence nets and event structures. ∎

The account so far has been restricted to categories of nets and event structures with synchronous morphisms. To lift this restriction and still obtain an adjunction, event structures may presumably be extended to record information on early conflict, essentially by considering families of event structures. We shall not, however, go further into the precise definition of the structure of "event structures with early conflict" in the present paper.

7.1 Symmetry

We would now wish to show that the coreflection above between event structures and occurrence nets with synchronous morphisms extends to the categories with symmetry. Unfortunately, for reasons that we shall now explain, this is problematic.

To attach symmetry to these categories, we would have to choose a path category such that open maps are preserved by the adjunctions. The appropriate paths of event structures in this situation seem to be elementary event structures and the appropriate paths of occurrence nets are the images under the functor $\mathcal{N}_s^\sharp : \mathbf{ES}_s \to \mathbf{Occ}_s^\sharp$ of elementary event structures. Write $\mathcal{N}_s^\sharp\mathbf{Elem}_s$ for the subcategory of $\mathbf{Occ}_s^\sharp$ formed as the image of $\mathbf{Elem}_s$ under the functor $\mathcal{N}_s^\sharp$. Open maps of safe nets, and hence occurrence nets, with elementary event structures as paths were studied in [13].

A general result about open maps presented in [9] shows that the functors $\mathcal{E}_s^\sharp$ and $\mathcal{N}_s^\sharp$ preserve open maps as defined here. The adjunction between the categories with symmetry is, however, stymied by the fact that the functor $\mathcal{N}_s^\sharp$ does not preserve pullbacks of $\mathbf{Elem}_s$-open maps. This can be seen by considering the event structures ES, an event structure with two events e_1 and e_2 that are in conflict, and the event structure ES', an event structure with one event, e. The morphism $\eta : ES \to ES'$ defined as

$$\eta(e_1) = \eta(e_2) = e$$

is $\mathbf{Elem}_s$-open. It can be shown that the pullback Q of the morphism η taken against itself is equal to the event structure with events

$$\{(e_1, e_1), (e_1, e_2), (e_2, e_1), (e_2, e_2)\},$$

all of which are in conflict with each other. This is not preserved as a pullback by the functor $\mathcal{N}_s^\sharp$.

8 Conclusion and related work

In this paper, we have shown how symmetry can be applied to two forms of Petri net, P/T nets and occurrence nets, and that this necessitates extending the definition of nets to allow them to have multiple initial markings. A coreflection connecting these two categories was given and this was shown to extend to a coreflection between the categories with symmetry. A coreflection between categories of event structures and occurrence nets was given, but this was shown not to extend to categories with symmetry.

An alternative, explicit notion of symmetry on Petri nets has been used in [16]. The use of coreflections to connect models for concurrency is described in [25]. The unfolding of safe nets was defined in [12]. The operation was extended to unfolding P/T nets by Engelfriet in [3], where the unfolding was characterized as a limit within a lattice of partial unfoldings. In [11] a coreflection between occurrence nets and (singly-marked) *semi-weighted* nets is given. Semi-weighted nets generalize P/T nets net in that they allow conditions to occur with multiplicity greater than one as preconditions to events, though the postconditions of any event must form a set as must the initial marking of the net. Presumably the coreflection extends to one between multiply-marked nets.

The failure to obtain a coreflection between the categories of event structures and occurrence nets with symmetry highlights an interesting point, that not all coreflections between categories of models will extend to give coreflections between the categories with symmetry. Even had there been a coreflection, it would not have been directly connected to the coreflection between P/T nets with symmetry and occurrence nets with symmetry due to the choice of different path categories. Further study will be needed to consider how this situation might be ameliorated.

More generally, we note that the addition of symmetry seems to answer a call from net theory for more general event structures with which to understand the unfolding of nets [8], though here, to be completely compelling, we would need also to address the collective token game — work for the future.

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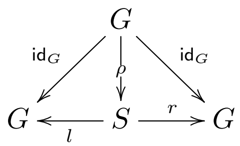
A Equivalences

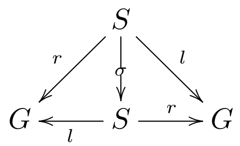
Assume a category $\mathcal{C}$ with pullbacks. Let $l, r : S \to G$ be a pair of morphisms in $\mathcal{C}$. They form a *pseudo equivalence* iff they satisfy:

Reflexivity there is a map ρ such that



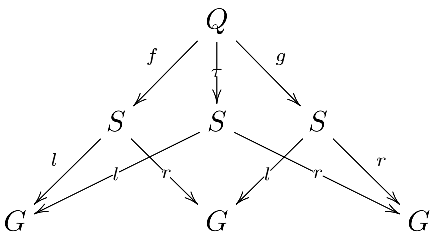
commutes;

Symmetry there is a map σ such that



commutes; and

Transitivity there is a map τ such that



commutes, where Q, f, g is a pullback of r, l.

If, furthermore, the maps l and r are jointly monic, then they form an *equivalence*.