Logic and Proof - Supervision 1

Hugo Paquet

Lent Term 2017

1. Prove the following two statements, using any method you like.

(a) $p \to q \not\simeq q \to p$

(b) $(p \to q) \to ((\neg p \to q) \to q) \simeq \top$

2. Reduce each of the following formulas to NNF, CNF, and DNF.

(a) $(q \to \neg p) \to p$

(b) $((p \wedge q) \vee r) \wedge \neg(p \vee r)$

3. For each formula φ below, either prove that it is valid, or give an interpretation satisfying $\neg\varphi$:

(a) $(p \to r) \wedge (q \to r) \to (p \vee q \to r)$

(b) $((p \to q) \to r) \to (p \to (q \to r))$

(c) $(\neg p \vee (q \to p)) \to (\neg p \wedge q)$

4. Prove the following sequents, using the sequent calculus rules given in the notes:

(a) $\neg\neg p \Rightarrow p$

(b) $(p \wedge q) \wedge r \Rightarrow p \wedge (q \wedge r)$

(c) $(p \vee q) \wedge (p \vee r) \Rightarrow p \vee (q \wedge r)$

(d) $\neg(p \vee q) \Rightarrow \neg p \wedge \neg q$

5. Suppose $\mathcal{L}$ is a first-order language with symbols F and M, of arity 2, and Ed, of arity 0. Let $F(x, y)$ mean that x is the father of y, and let $M(x, y)$ mean that x is the mother of y. The constant Ed denotes the person Ed. Write for each of the sentences below, a formula of $\mathcal{L}$ with the same meaning:

(a) Everybody has a mother.

(b) Everybody has a mother and a father.

(c) Whoever has a mother has a father.

(d) Ed is a grandfather.

(e) Nobody's grandmother is anybody's father.

6. Consider the formula

$$\varphi = \forall x \exists y \exists z (P(x, y) \wedge P(z, y) \wedge (P(x, z) \rightarrow P(z, x))).$$

Which of the following models satisfy φ? Justify briefly.

(a) $\mathcal{M} = (\mathbb{N}, I)$ where $I[P] = \{(m, n) \mid m, n \in \mathbb{N} \wedge m < n\}$

(b) $\mathcal{M}' = (\mathbb{N}, I)$ where $I[P] = \{(m, 2 * m) \mid m \in \mathbb{N}\}$

(c) $\mathcal{M}'' = (\mathbb{N}, I)$ where $I[P] = \{(m, n) \mid m, n \in \mathbb{N} \wedge m < n + 1\}$