Introduction to

Sequentiality

References:

P.-L. Curien "Categorical Combinator, Sequential Algo-

rithms & Functional Prog."

Birkhäuser '93.

P.-L. Curien "Sequentiality & Full abstraction"

-previous Landouf

T. Ehrhard "Hypercoherence" MSCS, Dec. '93

A. Bucciarelli's Ph.D. thesis '93
A continuous function

\( f : \mathbb{N}^m \to \mathbb{N} \)  \( (m \geq 1) \)

is sequential \((\text{Vuillemon})\)

iff

\[ \forall \vec{x} \in \mathbb{N} : \exists \vec{y} \neq \vec{x} : f(\vec{y}) \neq f(\vec{x}) \]

\( \Rightarrow \) \( \forall i : y_i \neq x_i \)

A continuous function

\( f : \mathbb{N}^m \to \mathbb{N}^q \)  \( (m \geq 1) \)

is sequential \((\text{Vuillemon})\) iff each composition \( \Pi_j \circ f \), \( j = 1, \ldots, q \),
is sequential as above.
The Gustave–function (Berry) $\overline{\mu}$ a stable but non-sequential function

$$G : \mathbb{B}^3 \rightarrow \{0, 1\}$$

minimum s.t.

$$G(\top, \bot, \bot) = T$$

$$G(\bot, \top, f) = T$$

$$G(f, \bot, \top) = T$$

[One reason why restricting to stable functions does not give full abstraction]
A **sequential structure** consists of

\[ S = (C, E, i, \triangle) \]

- \( C \) - cells
- \( E \) - events, initial event \( i \in E \),
- \( \triangle \) - accessibility relation
- \( \triangle^* \) a p.o.

\((\text{Cell }/\text{Event}) \text{ Occurrences}) \]
non-empty sequences \( e_0, c, e_1, c, e_2, \ldots \)

for which \( e_0 = i \) & \( \ldots e_k \triangle c_{k+1} \triangle e_{k+1} \ldots \).

A **configuration** of \( S \) consists of \( x \) a subset of occurrences s.t.

- **nonempty**:
  - \( i \in x \)
- **prefix-closed**:
  - \( se \in x \Rightarrow s \in x \), \( d \in CuE \)
- **event-determined**:
  - \( sc \in x \& c \in C \Rightarrow JeeE. sc \in x \)
- **consistent**:
  - \( se_1, se_2 \in x \& e_1, e_2 \in E \Rightarrow e_1 = e_2 \)

Write \( P(S) \) for set of configurations.
Alternatively, could define configurations as "partial strategies,"

\[ x \subseteq \text{Occurrences} \]

nonempty: \( \exists i \in x \)

prefix-closed: \( sd \in x \implies s \in x \)

cell-closed: \( s \in x \land sca \implies sc \in x \)

consistent: \( se_1, se_2 \in x \land e_1, e_2 \in E \implies e_1 = e_2 \).
Notation

For a seq. structure $S$, write
\[ C^*, E^*, D^* \] for all, event, all occurrences.
Write $\leq$ for the order of extension or occurrence. $D^*$ forms a tree, and root $i$, and has meet $\wedge$.

For $x, y \in P(S)$, $c \in C^*$
\[ x \prec y \text{ means } x \cup \{ c : c \in y \} = y \quad \text{and} \quad x \not\supseteq y, \text{ for some } c \in E. \]
\[ x \subseteq y \text{ means } x \prec z \subseteq y, \text{ for some } z \in P(S). \]

Prop. A config. $u$ of $S$ is determined by its event occurrence $u \cap E^*$.

$(P(S), \subseteq)$ is a partial concrete domain.

Let $S, S'$ be seq. structures.

A continuous function
\[ f : (\mathcal{P}(S), \subseteq) \rightarrow (\mathcal{P}(S'), \subseteq) \]

is sequential (Kalmar-Plotkin)

iff \( \forall x \in \mathcal{P}(S) \).

\( \forall c' \in C' \)

if \( \exists y \geq x . \ f(y) \geq_c f(x) \)

then \( \exists c \in C^* . \ f(y) \geq_c f(x) \Rightarrow y \geq_c x \)
From here there are two routes to sequentiality at higher types.

1. Berry & Gierz's sequential algorithm
   [including model]

2. Bucciarelli & Erhard's strongly stable functors wr. tolerance
   (via their representation via "hyper-tolerance")
   [extension model]

We first look at 1.
Affine function space

$S_0, S_1$ seq. structures

$S_0 \xrightarrow{\cdot} S_1 = (E_0 \times C_1, E_0 \times E_1 \cup C_0 \times C_1, (i_0, i_1), \Delta)$

where

$(d_0, d_1) \Delta (d'_0, d'_1) \iff (d_0 \triangleleft d'_0 \land d_1 = d'_1)$ or

$(d_0 = d'_0 \land d_1 \triangleleft d'_1)$

Morphisms are to be configurations of the affine part space. In order to compose them we use the following characterization.
Let $\alpha \in \Gamma \left( S_0 \mapsto S_1 \right)$. Then $\alpha \cap E^*$ has two kinds of event occurrences:

$$s(e_0, e_1) \downarrow (s_0, e_0, s_1, e_1) \in E_0^* \times E_1^*$$

$$s(c_0, c_1) \downarrow (s_0, c_0, s_1, c_1) \in C_0^* \times C_1^*$$

Config. $\alpha$ are in 1-1 correspondence with

$$\overline{\alpha} \subseteq E_0^* \times E_1^* \cup C_0^* \times C_1^*$$

s.t.

1. $(e_0, e_1) \in \alpha$
2. (a) $(d_0, d_1) \in \overline{\alpha}$ & $d_0 \geq c_0 \in C_0^* \Rightarrow \exists! e_0 \in E_0^*: (c_0, e_0) \in \overline{\alpha}$
   (b) $(d_0, d_1) \in \overline{\alpha}$ & $d_0 \geq e_1 \in E_1^* \Rightarrow \exists! e_0 \in E_0^*: (e_0, e_1) \in \overline{\alpha}$
3. If $(d_0, d_1), (d_0', d_1') \in \overline{\alpha}$ then
   (a) $d_0 \leq d_0' \& d_0 \in E_0^* \Rightarrow d_1 \land d_1' \in E_1^*$, and
   (b) $d_1 \leq d_1' \& d_1 \in C_1^* \Rightarrow d_0 \land d_0' \in C_0^*$.

Composition in the category is given by relational composition on $\overline{\alpha}$'s. Identifiers correspond to identity relations on occurrences.
Tensor

\[ S_0 \otimes S_1 = (C_0 \times E_1 \cup E_0 \times C_1, E_0 \times E_1, (i_0, i_1), \Delta) \]

where \((d_0, d_1) \Delta (d'_0, d'_1) \iff (d_0 \leq d'_0 \land d_1 = d'_1)\) or \((d_0 = d'_0 \land d_1 < d'_1)\).

Have:

\[ S_0 \otimes S_1 \rightarrow S_2 \equiv S_0 \rightarrow (S_1 \rightarrow S_2) \]

Product x disjoint junta position.

Claim: \(!S_0 \otimes !S_1 \equiv !(S_0 \times S_1)\)

for def. of exponential below.
Exponential

$$S = (C_i, E_1, i_i, i_i)$$ where

$$C_i = \{ x \in I | x \not\prec_i \}$$

$$E_1 = \Pi(S)^\circ \quad \text{(finite contg. of } S \text{)}$$

$$A_i$$ is least rel. s.t:

$$xc \prec_A y \iff x \not\prec_i y$$

$$xc \prec_A xc$$

Berry & Curien's sequential algorithm conv. to contg.

$$\preceq_0 \rightarrow \preceq_1 = (C, E_i, i, \prec_i)$$ where

$$C = \Pi(S_0)^\circ \times C_i,$$

$$E = \Pi(S_0)^\circ \times E_1 \cup \{(x_0c_0, c_i) | x_0 \not\prec_{c_0} c_i \in C_i \},$$

$$i = (\{i_0\}, i_i),$$

$$x_0e_i \prec x_0c_i \quad \text{if } e_i \not\prec_i c_i, \quad x_0c_i \prec x_0e_i \quad \text{if } c_i \not\prec_i e_i,$$

$$x_0c_i \prec x_0c_0c_i \quad \text{if } x_0 \not\prec_{c_0} c_i, \quad x_0c_0c_i \prec y_0c_1 \quad \text{if } x_0 \not\prec_{c_0} y_0.$$