RELATIVE PSEUDOMONADS, KLEISLI BICATEGORIES, AND SUBSTITUTION MONOIDAL STRUCTURES

M. FIORE, N. GAMBINO, M. HYLAND, AND G. WINSKEL

Abstract. We introduce the notion of a relative pseudomonad, which generalizes the notion of a pseudomonad, and define the Kleisli bicategory associated to a relative pseudomonad. We then present an efficient method to define pseudomonads on the Kleisli bicategory of a relative pseudomonad. The results are applied to define several pseudomonads on the bicategory of profunctors in an homogeneous way, thus providing a uniform approach to the definition of bicategories that are of interest in operad theory, mathematical logic, and theoretical computer science.

1. **Introduction**

Just as classical monad theory provides a general approach to study algebraic structures on objects of a category (see [5] for example), 2-dimensional monad theory offers an elegant way to investigate algebraic structures on objects of a 2-category [10, 29, 33, 36, 55, 57]. Even if the strict notion of a 2-monad is sufficient to develop large parts of the theory, the strictness requirements that are part of its definition are too restrictive for some applications, for which it is necessary to work with the notion of a pseudomonad [12], in which the diagrams expressing the associativity and unit axioms for a 2-monad commute up to specified invertible modifications, rather than strictly. In recent years, pseudomonads have been studied extensively by several researchers [15, 39, 50, 51, 53, 54, 63].

Our general aim here is to develop further the theory of pseudomonads. In particular, we introduce relative pseudomonads, which generalize pseudomonads, define the associated Kleisli bicategory of a relative pseudomonad, and describe a method to extend a 2-monad on a 2-category to a pseudomonad on the Kleisli bicategory of a relative pseudomonad. We use this method to show how several 2-monads on the 2-category **Cat** of small categories and functors can be extended to pseudomonads on the bicategory **Prof** of small categories and profunctors (also known as bimodules or distributors) [8, 46, 61]. This result has applications in the theory of variable binding [23, 56, 62], concurrency [14], species of structures [22], models of the differential λ-calculus [21], and operads and multicategories [18, 24] (see also [16, 17]).

For these applications, one would like to regard the bicategory of profunctors as a Kleisli bicategory and then use the theory of pseudo-distributive laws [33, 50, 51], i.e. the 2-dimensional counterpart of Beck's fundamental work on distributive laws [6] (see [59] for an abstract treatment). In order to carry out this idea, one is naturally led to try to consider the presheaf construction, which sends a small category $\mathbb{X}$ to its category of presheaves $P(\mathbb{X}) =_{\mathrm{def}} [\mathbb{X}^{\mathrm{op}}, \mathbf{Set}]$, as a pseudomonad. Indeed, a profunctor $F \colon \mathbb{X} \to \mathbb{Y}$, i.e. a functor $F \colon \mathbb{Y}^{\mathrm{op}} \times \mathbb{X} \to \mathbf{Set}$, can be identified with a functor $F \colon \mathbb{X} \to P(\mathbb{Y})$. However, the presheaf construction fails to be a pseudomonad for size reasons, since it sends small categories to locally small ones, making it impossible to define a multiplication. Although some aspects of the theory can be developed restricting the attention to small presheaves [20], which support the structure of a pseudomonad,

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some of our applications involve naturally general presheaves and thus require us to deal not only with coherence but also size issues.

In order to do so, we introduce the notion of a relative pseudomonad (Definition 3.1), which is based on the notions of a relative monad [1, Definition 2.1] and of a no-iteration pseudomonad [54, Definition 2.1]. These notions are, in turn, inspired by Manes' notion of a Kleisli triple [48], which is equivalent to that of a monad, but better suited to define Kleisli categories (see also [52, 64]). Fixed an inclusion of bicategories $J\colon \mathcal{C} \to \mathcal{D}$ (which in our main example is the inclusion $J\colon \mathbf{Cat} \to \mathbf{CAT}$ of the 2-category of small categories into the 2-category of locally small categories), the core of the data for a relative pseudomonad T over J consists of an object $TX \in \mathcal{D}$ for every $X \in \mathcal{C}$, a morphism $i_X\colon JX \to TX$ for every $X \in \mathcal{C}$, and a morphism $f^*\colon TX \to TY$ for every $f\colon JX \to TY$ in $\mathcal{D}$. This is essentially as in a relative monad, but the equations for a relative monad are replaced in a relative pseudomonad by families of invertible 2-cells satisfying appropriate coherence conditions, as in a no-iteration pseudomonad. As we will see in Theorem 3.4, these coherence imply that every relative pseudomonad T over $J\colon \mathcal{C} \to \mathcal{D}$ has an associated Kleisli bicategory $\mathrm{Kl}(T)$, defined analogously to the one-dimensional case. In our main example, the presheaf construction gives rise to a relative pseudomonad over the inclusion $J\colon \mathbf{Cat} \to \mathbf{CAT}$ in a natural way and it is then immediate to identify its Kleisli bicategory with the bicategory of profunctors. It should be noted here that the presheaf construction is neither a no-iteration pseudomonad (because of size issues) nor a relative monad (because of strictness issues).

As part of our development of the theory of relative pseudomonads, we show how relative pseudomonads generalize no-iteration pseudomonads (Proposition 3.7) and hence (by the results in [54]) pseudomonads. We then introduce relative pseudoadjunctions, which are related to relative pseudomonads just as adjunctions are connected to monads. In particular, we show that every relative pseudoadjunction gives rise to a relative pseudomonad (Theorem 4.4) and that the Kleisli bicategory associated to a relative pseudomonad fits in a relative pseudoadjunction (Theorem 4.6).

Furthermore, we introduce and study the notion of a lax idempotent relative pseudomonads, which appears to be the appropriate counterpart in our setting of the notion of a lax idempotent 2-monad (i.e. often called Kock-Zöberlein doctrines) [36, 38, 65] and pseudomonad [49, 53, 61]. Indeed, we will show in Proposition 5.7 that a 2-monad is lax idempotent as a 2-monad in the usual sense only if it is lax idempotent as a relative pseudomonad in our sense. This notion is of interest since it allows us to exhibit examples of relative pseudomonads by reducing the verification of the coherence axioms for a relative pseudomonad to the verification of certain universal properties. In particular, we shall construct the relative pseudomonad of presheaves in this way.

We then consider the question of when a 2-monad on the 2-category $\mathbf{Cat}$ of small categories can be extended to a pseudomonad on the bicategory $\mathbf{Prof}$ of profunctors. Rather than adapting the theory of distributive laws to relative pseudomonads along the lines of what has been done for no-iteration monads [52], which would involve complex calculations with coherence conditions, we establish directly that, for an inclusion $J\colon \mathcal{C} \to \mathcal{D}$ of 2-categories, a 2-monad $S\colon \mathcal{D} \to \mathcal{D}$ restricting to $\mathcal{C}$, and a relative pseudomonad T over J, if T admits a lifting to 2-categories of strict algebras or pseudoalgebras for S, then S admits an extension to the Kleisli bicategory of T (Theorem 6.3). We do so bypassing the notion of a pseudodistributive law in a counterpart of Beck's result.

This result is well-suited to our applications, where the structure that manifests itself most naturally is that of a lift of the relative pseudomonad of presheaves to various 2-categories of categories equipped with algebraic structures, often via forms of Day's convolution monoidal structure [19, 30]. In particular, our results will imply that the 2-monads for several important notions (categories with terminal object, categories with finite products, categories with finite limits, monoidal categories, symmetric monoidal categories, unbiased monoidal categories, unbiased symmetric monoidal categories strict monoidal categories, and symmetric strict monoidal categories) can be extended to pseudomonads on the bicategory of profunctors. A reason for interest in this result is that the composition in the Kleisli bicategories of these pseudomonads can be understood as variants of the substitution monoidal structure that can be used to characterize the notion of an operad [35, 58].

As an illustration of the applications of our theory, we discuss our results in the special case of the 2-monad S for symmetric strict monoidal categories, showing how it can be extended to a pseudomonad on the bicategory of profunctors. This result is the cornerstone of the understanding of the bicategory of generalized species of structures defined in [22] as a 'categorified' version of the relational model of linear logic [28, 27] and leads to showing that the substitution monoidal structure that gives rise to the notion of a coloured operad [4] is a special case of the composition in the Kleisli bicategory. The results presented here are intended to make these ideas precise by dealing with both size and coherence issues in a conceptually clear way.

Organization of the paper. Section 2 reviews some background material on 2-monads, pseudomonads and their algebras. Our development starts in Section 3, where we introduce relative pseudomonads, relate them to no-iteration pseudomonads and to ordinary pseudomonads, and define the Kleisli bicategory associated to a relative pseudomonad. Section 4 introduces relative pseudoadjunctions and establishes a connection between them and relative pseudomonads. In Section 5 we study lax idempotent relative pseudomonads. Section 6 show that an extension of a relative pseudomonad T to 2-categories of strict algebras or pseudoalgebras for a 2-monad S induces an extension of S to the Kleisli bicategory of T, as well as a 'composite' relative pseudomonad TS. We conclude the paper in Section 7 by discussing applications of our theory and showing how several 2-monads can be extended from **Cat** to **Prof**.

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2. Background

2-categories and 2-monads. We assume that readers are familiar with the fundamental aspects of the theory of 2-categories and of bicategories (as presented, for example, in [11, 41]) and confine ourselves to review some facts that will be used in the following and to fix notation and conventions.

For a 2-category $\mathcal{C}$ and a pair of objects $X, Y \in \mathcal{C}$, we write $\mathcal{C}[X, Y]$ for the hom-category of morphisms $f \colon X \to Y$ and 2-cells between them, which we denote with lower-case Greek letters, $\phi \colon f \to f'$. Two parallel morphisms $f\, f' \colon X \to Y$ are said to be *isomorphic* if they are isomorphic as objects of $\mathcal{C}[X, Y]$, and we write $f \cong f'$ in this case. We write **CAT** for the 2-category of locally small categories, functors, and natural transformations. Its full sub-2-category spanned by small categories will be written **Cat**. We then have an inclusion $J \colon \mathbf{Cat} \to \mathbf{CAT}$. We use the terms *pseudofunctor*, *pseudonatural transformation*, and *pseudoadjunction* rather than homomorphism, strong natural transformation, and biadjunction, respectively.

Let us now review some aspects of 2-dimensional monad theory [10]. By a 2*-monad* on a 2-category $\mathcal{C}$ we mean a 2-functor $S \colon \mathcal{C} \to \mathcal{C}$ equipped with 2-natural transformations $m \colon S^2 \to S$ and $e \colon 1_{\mathcal{C}} \to S$, called the *multiplication* and *unit* of the 2-monad, respectively, satisfying the usual axioms for a monad in a strict sense. As usual, we often refer to a 2-monad by mentioning only its underlying 2-functor, leaving implicit the rest of its data. Similar conventions will be used for other kinds of structures considered in the rest of the paper.

For a 2-category $\mathcal{C}$ and 2-monad $S\colon \mathcal{C} \to \mathcal{C}$, we write Ps-$S$-Alg$_{\mathcal{C}}$ (or Ps-S-Alg if no confusion arises) for the 2-category of pseudoalgebras, pseudomorphisms and algebra 2-cells. Here, by a *pseudoalgebra* we mean an object $A \in \mathcal{C}$, called the *underlying object* of the algebra, equipped with a morphism $a\colon SA \to A$, called the *structure map* of the algebra, and invertible 2-cells

$$\begin{tikzcd} S^2A \arrow[r, "Sa"] \arrow[d, "m_A"'] \arrow[dr, phantom, "\Downarrow \bar{a}"] & SA \arrow[d, "a"] \\ SA \arrow[r, "a"'] & A\,, \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "e_A"] \arrow[dr, "1_A"'] \arrow[r, phantom, shift right=3ex, "\overset{\tilde{a}}{\Longrightarrow}"] & SA \arrow[d, "a"] \\ & A\,, \end{tikzcd}$$

called the *associativity* and *unit* 2-cells of the algebra, subject to two coherence axioms [60]. We have a *strict algebra* when the associativity and unit 2-cells are identities, in which case (as in analogous cases below) the coherence conditions are satisfied trivially. For pseudoalgebras A and B (and in particular for strict algebras), a pseudomorphism from A to B consists of a morphism $f\colon A \to B$ and an invertible 2-cell

$$\begin{tikzcd} SA \arrow[r, "Sf"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow \bar{f}"] & SB \arrow[d, "b"] \\ A \arrow[r, "f"'] & B\,, \end{tikzcd}$$

required to satisfy two coherence axioms [10, 60]. For pseudomorphisms $f, g\colon A \to B$, an *algebra 2-cell* between them is a 2-cell $\alpha : f \to g$ that satisfies one coherence axiom [10]. We have a forgetful 2-functor $U\colon$ Ps-S-Alg $\to \mathcal{C}$ with a left pseudoadjoint $F\colon \mathcal{C} \to$ Ps-S-Alg, defined by mapping an object $X \in \mathcal{C}$ to the *free algebra* on X, which is the strict algebra having SX as its underlying object and $m_X\colon S^2X \to SX$ as its structure map. The components of the unit of the pseudoadjunction are the components of the unit of the monad. These are universal in the sense that, for $X \in \mathcal{C}$ and $A \in$ Ps-S-Alg, there is an adjoint equivalence of hom-categories

$$\begin{tikzcd} \mathcal{C}[X, A] \arrow[r, shift left, "(-)^\sharp"] \arrow[r, phantom, "\perp"] & \text{Ps-}S\text{-Alg}[SX, A]\,. \arrow[l, shift left, "U(-)\,e_X"] \end{tikzcd} \tag{2.1}$$

In particular, given $f\colon X \to A$, the pseudomorphism $f^\sharp\colon SX \to A$ is given by $f^\sharp =_{\mathrm{def}} a\,S(f)$, with an evident structure 2-cell. The component of unit of the adjunction in (2.1) associated to $f\colon X \to A$ is a 2-cell $\eta_f\colon f \to f^\sharp e_X$, thus fitting into the diagram

$$\begin{tikzcd} X \arrow[r, "e_X"] \arrow[dr, bend right, "f"'] \arrow[r, phantom, shift right=3ex, "\overset{\eta_f}{\Rightarrow}"] & SX \arrow[d, "f^\sharp"] \\ & A\,. \end{tikzcd} \tag{2.2}$$

We write S-Alg$_{\mathcal{C}}$ (or S-Alg) for the full sub-2-category of Ps-S-Alg$_{\mathcal{C}}$ spanned by strict algebras, so that we have a full inclusion of 2-categories $J\colon S$-Alg $\to$ Ps-S-Alg. There is another forgetful 2-functor $U\colon S$-Alg $\to \mathcal{C}$, which has a left pseudoadjoint $F\colon \mathcal{C} \to S$-Alg, defined essentially as above. But now, for $X \in \mathcal{C}$ and $A \in S$-Alg, the adjoint equivalence in (2.1) becomes a *retract* adjoint equivalence

$$\begin{tikzcd} \mathcal{C}[X, A] \arrow[r, shift left, "(-)^\sharp"] \arrow[r, phantom, "\perp"] & S\text{-Alg}[SX, A]\,, \arrow[l, shift left, "U(-)\,e_X"] \end{tikzcd} \tag{2.3}$$

with the 2-cell η_f in (2.2) an identity.

Traditionally, the focus of 2-dimensional monad theory is on S-Alg rather than on Ps-S-Alg. Indeed, for most notions of a category with structure (such as that of a monoidal category) there is a 2-monad S whose *strict* algebras are exactly the categories with the structure under consideration, even when that structure involves coherent isomorphisms (as in a monoidal category). Furthermore, for every finitary 2-monad $S\colon \mathcal{C} \to \mathcal{C}$ there is another 2-monad $S'\colon \mathcal{C} \to \mathcal{C}$, called the *flexible* 2-monad associated to S, such that there is an isomorphism of 2-categories S'-Alg $\cong$ Ps-S-Alg (see [9, Remark 7.2]). However, for some of our applications (most importantly the one based on the 2-monad for symmetric strict monoidal categories), we will deal simultaneously with both strict algebras and pseudoalgebras for a 2-monad S, and it will be simpler to do so considering just S rather than both S and its flexible replacement S'. For further details on the theory of flexibility, see [9, 10, 33, 37, 40].

In our applications, we will consider several 2-monads on **CAT** (restricting to **Cat** in an evident way), for which we invite the readers to consult [10, 45, 47]. Among them, the 2-monads for (strict) monoidal categories, symmetric (strict) monoidal categories, categories with finite limits, categories with finite products, and categories with a terminal object.

Bicategories and pseudomonads. For a bicategory $\mathcal{C}$, we write the associativity and unit isomorphisms as natural families of invertible 2-cells

$$(h\,g)\,f \xrightarrow{\cong} h\,(g\,f)\ , \qquad 1_Y f \xrightarrow{\cong} f\ , \qquad f \xrightarrow{\cong} f\,1_X\,. \tag{2.4}$$

which we leave unnamed. The coherence diagrams for a bicategory can then be written as follows:

$$\begin{tikzcd}
& ((k\,h)\,g)\,f \arrow[dl] \arrow[dr] & \\
(k\,(h\,g))\,f \arrow[d] & & (k\,h)\,(g\,f) \arrow[d] \\
k\,((h\,g)\,f) \arrow[rr] & & k\,(h\,(g\,f))\,,
\end{tikzcd}
\qquad
\begin{tikzcd}
g\,f \arrow[r] \arrow[ddr, bend right, "1"'] & (g\,1_Y)\,f \arrow[d] \\
& g\,(1_Y\,f) \arrow[d] \\
& g\,f\,.
\end{tikzcd} \tag{2.5}$$

By the coherence theorem for bicategories [44] (which also follows from the bicategorical Yoneda lemma [61], see [25]), every bicategory is biequivalent to a 2-category. In virtue of this, we shall often treat bicategories as if they were 2-categories.

Convention. We adopt the convention of leaving unnamed the associativity and unit isomorphisms of the base bicategories over which we work, labelling the relevant 2-cells only with the symbol for isomorphism.

Remark. The directions of the invertible 2-cells in (2.4) that we chose is consonant with the one of the 2-cells that are part of the definition of a relative pseudomonad (introduced in Definition 3.1 below), which are the natural ones in our examples. Other directions for these 2-cells are generally given in the literature [7, 13]. Note that while the lax notion of bicategory of Burroni differs in having $f \to 1_Y\,f$ where we have $1_Y\,f \to f$, this may well be an accident as in his leading example the map he takes is an isomorphism; so he might as well have chosen our direction. Yet a different direction appears in the lax notion of bicategory in the work of Grandis on directed homotopy [26]. However, our directions are not new: they appear in notes by Street from around 1970 developing an enriched category theory with what we now call weighted limits on the basis of a lax monoidal category.

Example 2.1. Fundamental to our applications is the bicategory **Prof** of profunctors [8, 46, 61]. Its objects are small categories; and for small categories $\mathbb{X}$ and $\mathbb{Y}$, the hom-category $\mathbf{Prof}[\mathbb{X}, \mathbb{Y}]$ is defined to be $\mathbf{CAT}[\mathbb{Y}^{\mathrm{op}} \times \mathbb{X}, \mathbf{Set}]$. The composite of profunctors $F \colon \mathbb{X} \to \mathbb{Y}$ and $G \colon \mathbb{Y} \to \mathbb{Z}$ is given by the profunctor $G \circ F \colon \mathbb{X} \to \mathbb{Z}$ defined by the coend formula

$$(G \circ F)(z, x) =_{\mathrm{def}} \int^{y \in \mathbb{Y}} G(z, y) \times F(y, x)\,. \tag{2.6}$$

For a small category $\mathbb{X}$, the identity profunctor $\mathrm{Id}_{\mathbb{X}} \colon \mathbb{X} \to \mathbb{X}$ is defined by letting

$$\mathrm{Id}_{\mathbb{X}}(x, y) =_{\mathrm{def}} \mathbb{X}[x, y]\,. \tag{2.7}$$

The verification that these definitions give rise to a bicategory is often left as an exercise to the reader in the literature. We shall give a conceptual account of why this is the case in Section 3 by describing **Prof** as the Kleisli bicategory associated to the relative pseudomonad of presheaves.

By definition, a *pseudomonad* on a bicategory $\mathcal{C}$ is given by a pseudofunctor $T \colon \mathcal{C} \to \mathcal{C}$, pseudonatural transformations $n \colon T^2 \to T$ and $i \colon 1_{\mathcal{C}} \to T$, called the *multiplication* and *unit* of the pseudomonad, respectively, and invertible modifications α, ρ, and λ , called the *associativity*, *right unit*, and *left unit*, respectively, of T, fitting in the diagrams

$$\begin{tikzcd} T^3 \arrow[r, "Tn"] \arrow[d, "nT"'] \arrow[dr, phantom, "\Downarrow \alpha"] & T^2 \arrow[d, "n"] \\ T^2 \arrow[r, "n"'] & T\,, \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "iT"] \arrow[dr, "1"'] & T^2 \arrow[d, "n"] \arrow[l, phantom, "\overset{\lambda}{\Rightarrow}"] \arrow[r, phantom, "\overset{\rho}{\Rightarrow}"] & T \arrow[l, "Ti"'] \arrow[dl, "1"] \\ & T\,, & \end{tikzcd} \tag{2.8}$$

and subject to two coherence conditions [39]. The notions of a strict algebra and pseudoalgebra, of strict morphism and pseudomorphism, and of algebra 2-cell make sense also for pseudomonads, giving rise to bicategories Ps-S-Alg and S-Alg. When $\mathcal{C}$ is a 2-category, these are again 2-categories.

Every pseudomonad has also an associated Kleisli bicategory [15], which can be defined in complete analogy with the one-dimensional case, but we do not spell this out since we will give an alternative account of the Kleisli construction in Section 3. Importantly, in constrast with the situation for algebras discussed above, the Kleisli construction for a pseudomonad T produces a genuine bicategory even when $\mathcal{C}$ is a 2-category, with the associativity and unit isomorphisms of T used to give the associativity and unit isomorphisms of the Kleisli bicategory (see also Theorem 3.4 below).

Remark. The directions of the modifications in (2.8) are the same as in definition of a lax monad in [12]. There, only 2-categories are considered, but with the added generality that T is what we would now call a colax functor or comorphism. Different directions are considered in [39], but since the 2-cells are invertible there is no difference of substance.

3. **Relative pseudomonads**

In ordinary category theory, the notion of a monad has an equivalent alternative presentation, via the notion of a Kleisli triple [48], which is particularly convenient to define Kleisli categories. The notion of a Kleisli triple admits a natural generalization, given by the notion of a relative monad [1], which is obtained by allowing the the underying mapping on objects of the Kleisli triple to be defined relatively to a fixed functor (see [1] for details). Similarly, in 2-dimensional category theory, the notion of a pseudomonad can be rephrased equivalently as the notion of a no-iteration pseudomonad [54], which is the 2-dimensional analogue of the notion of a Kleisli triple. Here, we introduce relative pseudomonads, which generalize no-iteration pseudomonads in essentially the same way as relative monads generalize Kleisli triples, i.e. by allowing the mapping on objects that is part of a no-iteration pseudomonad to be defined relatively to a fixed pseudofunctor between bicategories. In contrast with the situation for relative monads, here we assume that this pseudofunctor is an inclusion, since this is the case in all our examples. However, we expect that all our results carry over to the more general situation of a arbitrary pseudofunctor. So, from now until the end of this section, we consider a fixed inclusion of bicategories $J \colon \mathcal{C} \to \mathcal{D}$.

Definition 3.1. A *relative pseudomonad* T over $J \colon \mathcal{C} \to \mathcal{D}$ consists of

- an object $TX \in \mathcal{D}$, for every $X \in \mathcal{C}$;
- a family of functors $(-)^*_{X,Y} \colon \mathcal{D}[JX, TY] \to \mathcal{D}[TX, TY]$ for $X, Y \in \mathcal{C}$;
- a family of morphisms $i_X \colon JX \to TX$ in $\mathcal{D}$ for $X \in \mathcal{C}$;
- a natural family of invertible 2-cells $\mu_{g,f} \colon (g^* \, f)^* \to g^* \, f^*$, for $f \colon JX \to TY$, $g \colon JY \to TZ$;
- a natural family of invertible 2-cells $\eta_f \colon f \to f^* \, i_X$, for $f \colon JX \to TY$ in $\mathcal{D}$;
- a family of invertible 2-cells $\theta_X \colon {i_X}^* \to 1_{TX}$, for $X \in \mathcal{C}$;

such that

- the diagram

$$
\begin{tikzcd}
 & ((h^* \, g)^* f)^* \arrow[dl, "(\mu_{h,g} \, f)^*"'] \arrow[dr, "\mu_{h^* g, f}"] & \\
((h^* g^*) f)^* \arrow[d, "\cong"'] & & (h^* g)^* f^* \arrow[d, "\mu_{h,g} \, f^*"] \\
(h^* (g^* f))^* \arrow[d, "\mu_{h, g^* f}"'] & & (h^* g^*) f^* \arrow[d, "\cong"] \\
h^* (g^* f)^* \arrow[rr, "h^* \, \mu_{g,f}"'] & & h^* (g^* f^*)
\end{tikzcd}
\tag{3.1}
$$

  commutes for every $f \colon JX \to TY$, $g \colon JY \to TZ$, $h \colon TZ \to TV$;
- the diagram

$$
\begin{tikzcd}
f^* \arrow[r, "(\eta_f)^*"] \arrow[rrd, bend right, "\cong"'] & (f^* \, i_X)^* \arrow[r, "\mu_{f, i_X}"] & f^* \, {i_X}^* \arrow[d, "f^* \, \theta_X"] \\
 & & f^* \, 1_{TX}
\end{tikzcd}
\tag{3.2}
$$

  commutes for every $f \colon JX \to TY$.

We introduce some terminology which will be used in the following. We refer to the family of morphisms $i_X \colon JX \to TX$, for $X \in \mathcal{C}$, as the *unit* of the relative pseudomonad, and to the morphism $f^* \colon TX \to TY$ as the *Kleisli extension* of $f \colon JX \to TY$. The family of 2-cells μ, η, and θ are called the *associativity*, *right unit*, and *left unit*, respectively. Finally, we then refer to the axioms in (3.1) and (3.2) as the *associativity* and *unit axioms*, respectively.

Convention. In order to simplify notation, we omit the subscripts on the functors $(-)^*_{X,Y}$ and we adopt the convention of writing X rather than i_X in a subscript of μ and θ. So, for example, we have

$$
\mu_{f,X} \colon (f^* i_X)^* \to f^* \, {i_X}^* \, ,
$$
$$
\eta_X \colon i_X \to {i_X}^* \, i_X \, .
$$

Furthermore, in the interest of readability, we sometimes omit the detailed definition of some 2-cells in diagrams, labelling arrows only with the main 2-cell involved in its definition, and omitting subscripts for readability. In all such cases, the precise definition of the 2-cell can be easily deduced by pattern matching of its domain and codomain.

Remark 3.2. As we will see in Proposition 3.7 below, the notion of a no-iteration pseudomonad [54] is closely related to that of a relative pseudomonad. The directions of the 2-cells corresponding to the 2-cells named here μ, η and θ in a no-iteration pseudomonad in [54, Definition 2.1] are the opposites of the ones considered here. Since the 2-cells are invertible, this is not essential. The situation for the coherence conditions, however, is more delicate, c.f. Proposition 3.7.

As we will show in Remark 3.8, every pseudomonad can be viewed as a relative pseudomonad. But, as the next example shows, there are relative pseudomonads which are not pseudomonads.

Example 3.3. We indicate how the presheaf construction determines a relative pseudomonad[1] over the inclusion $J\colon \mathbf{Cat} \to \mathbf{CAT}$. For $\mathbb{X} \in \mathbf{Cat}$, we define $P(\mathbb{X})$ to be the category of presheaves on $\mathbb{X}$, $P(\mathbb{X}) =_{\mathrm{def}} [\mathbb{X}^{\mathrm{op}}, \mathbf{Set}]$, which is (in a precise sense to be discussed later) the completion of $\mathbb{X}$ under small colimits. The components of the unit are the Yoneda embeddings $\mathrm{y}_{\mathbb{X}}\colon \mathbb{X} \to P(\mathbb{X})$. For a functor $F\colon \mathbb{X} \to P(\mathbb{Y})$, we define $F^*\colon P(\mathbb{X}) \to P(\mathbb{Y})$ to be the left Kan extension of F along the Yoneda embedding, defined by the coend formula

$$F^*(p)(y) =_{\mathrm{def}} \int^{x \in \mathbb{X}} F(x)(y) \times p(x)\,. \tag{3.3}$$

The invertible 2-cells η_F fit into the diagram

$$\begin{tikzcd} \mathbb{X} \arrow[r, "\mathrm{y}_X"] \arrow[dr, bend right, "F"'] & P(\mathbb{X}) \arrow[d, "F^*"] \\ \arrow[ur, phantom, "\overset{\eta_F}{\Rightarrow}"] & P(\mathbb{Y}) \end{tikzcd}$$

The 2-cells $\mu_{F,G}$ and $\theta_{\mathbb{X}}$ are uniquely determined by the universal property of left Kan extensions, but we postpone giving a precise account of the 2-dimensional structure until Section 4.

A variant of this example is obtained by considering the completion of a small category under filtered colimits, also known as the Ind-completion [2]. For $\mathbb{X} \in \mathbf{Cat}$, we define $D(\mathbb{X}) \in \mathbf{CAT}$ to be the full subcategory of $P(\mathbb{X})$ spanned by filtered colimits of representables.

We now introduce the Kleisli bicategory of relative pseudomonad, extending to the 2-dimensional setting the definition of the Kleisli category of a relative monad [1, Section 2.3]. We work with a fixed relative pseudomonad T over $J\colon \mathcal{C} \to \mathcal{D}$. We define the Kleisli bicategory $\mathrm{Kl}(T)$ as follows. The objects of $\mathrm{Kl}(T)$ are the objects of $\mathcal{C}$. For $X, Y \in \mathcal{C}$, define their hom-category by letting

$$\mathrm{Kl}(T)[X, Y] =_{\mathrm{def}} \mathcal{D}[JX, TY]\,.$$

To define composition, let $f\colon JX \to TY$ and $g\colon JY \to TZ$. We define $g \circ f\colon JX \to TZ$ as the composite in $\mathcal{D}$

$$JX \xrightarrow{f} TY \xrightarrow{g^*} TZ\,.$$

This obviously extends to 2-cells, so as to obtain the required composition functors. For $X \in \mathcal{C}$, the identity morphism on X in $\mathrm{Kl}(T)$ is $i_X\colon JX \to TX$. For the associativity isomorphisms, let $f\colon JX \to TY$, $g\colon JY \to TZ$ and $h\colon JZ \to TV$. Since

$$(h \circ g) \circ f = (h^*\, g)^* f\,, \quad h \circ (g \circ f) = h^*\,(g^*\, f)\,,$$

[1]The idea of regarding the presheaf construction as a relative pseudomonad has been considered also by Sam Staton, c.f. [1, Example 2.7].

we define the associativity isomorphism $\alpha_{h,g,f} \colon (h \circ g) \circ f \to h \circ (g \circ f)$ to be the composite 2-cell

$$(h^*\, g)^*\, f \xrightarrow{\mu_{h,g}\, f} (h^*\, g^*)\, f \xrightarrow{\cong} h^*\, (g^*\, f)\,.$$

For the right and left unit, let $f \colon JX \to TY$. Since $f \circ i_X = f^*\, i_X$, we define $\rho_f \colon f \circ i_X \to f$ to be the 2-cell $\eta_f \colon f \to f^*\, i_X$. Since $i_Y \circ f = {i_Y}^*\, f$, we define $\lambda_f \colon i_Y \circ f \to f$ to be the composite 2-cell

$${i_Y}^*\, f \xrightarrow{\theta_Y\, f} 1_{TY}\, f \xrightarrow{\cong} f\,.$$

Note that the canonical directions of the associativity and unit 2-cells of $\mathrm{Kl}(T)$ are those described in (2.4) and that the directions of the associativity and unit 2-cells of $\mathcal{D}$ match those of the Kleisli structure, in that we do not need to invert any 2-cells to construct the associativity and unit 2-cells of $\mathrm{Kl}(T)$. Our next theorem establishes that these natural isomorphisms satisfy the required coherence conditions.

Theorem 3.4. *Let T be a relative pseudomonad over $J \colon \mathcal{C} \to \mathcal{D}$. Then* $\mathrm{Kl}(T)$ *is a bicategory.*

Proof. We give the proof making explicit the bicategorical structure of $\mathcal{C}$ and $\mathcal{D}$. We only need to show that the associativity, left unit, and right unit isomorphisms satisfy the coherence conditions expressed by the diagrams in (2.5). The coherence axiom for associativity is obtained via the following diagram:

$$\begin{tikzcd}
 & & ((k^*\, h)^*\, g)^*\, f \arrow[dl, "(\mu\, g)^*\, f"'] \arrow[dr, "\mu\, f"] & & \\
 & (k^*\, h^*)\, g)^*\, f \arrow[dl, "\cong"'] & & ((k^* h)^*\, g^*)\, f \arrow[d, "(\mu\, g^*)\, f"'] \arrow[dr, "\cong"] & \\
(k^*(h^*\, g))^* f \arrow[d, "\mu\, f"'] & & & ((k^*\, h^*)\, g^*)\, f \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & (k^*\, h)^*(g^*\, f) \arrow[d, "\mu\, (g^*\, f)"] \\
(k^*(h^*\, g)^*)\, f \arrow[rr, "(k^*\, \mu)\, f"] \arrow[d, "\cong"'] & & (k^*(h^*\, g^*))\, f \arrow[d, "\cong"] & & (k^*\, h^*)\, (g^*\, f) \arrow[d] \\
k^*\, ((h^*\, g)^*\, f) \arrow[rr, "k^*\, (\mu\, f)"'] & & k^*((h^*\, g^*) f) \arrow[rr, "\cong"'] & & k^*\, (h^*(g^*\, f))\,,
\end{tikzcd}$$

where, starting from the square on the right-hand side and proceeding clockwise, we use naturality of the associativity in $\mathcal{D}$, coherence of associativity in $\mathcal{D}$, naturality of the associativity in $\mathcal{D}$ again, and finally the associativity coherence axiom for a relative pseudomonad in (3.1). The coherence axiom for the units is instead obtained via the following diagram, where we use similar conventions as above:

$$
\begin{tikzcd}
g^* f \arrow[r, "{\eta_g}^* f"] \arrow[drr, "\cong"'] \arrow[ddrrr, bend right, "1"'] & (g^* i_Y)^* f \arrow[r, "\mu_{g,Y} f"] & (g^* {i_Y}^*) f \arrow[r, "\cong"] \arrow[d, "(g^* \theta_Y) f"] & g^* ({i_Y}^* f) \arrow[d, "g^* (\theta_Y f)"] \\
 & & (g^* 1_{TY}) f \arrow[r, "\cong"] & g^* (1_{TY} f) \arrow[d, "\cong"] \\
 & & & g^* f \, ,
\end{tikzcd}
$$

where, starting from the triangle on the top left-hand side, we used the coherence axiom for units of the relative pseudomonad in (3.2), naturality of the associativity in $\mathcal{D}$, and the coherence axiom for units of $\mathcal{D}$ in (2.5). □

Note that, as mentioned in Section 2 for ordinary pseudomonads, $\mathrm{Kl}(T)$ is a genuine bicategory even if $\mathcal{C}$ and $\mathcal{D}$ are 2-categories.

Example 3.5. It is straightforward to identify the bicategory of profunctors of Example 2.1 with the Kleisli bicategory associated to the relative pseudomonad of presheaves P of Example 3.3. First of all, both bicategories have small categories as objects. Secondly, for small categories $\mathbb{X}$ and $\mathbb{Y}$ we have

$$\mathbf{Prof}[\mathbb{X}, \mathbb{Y}] = [\mathbb{Y}^{\mathrm{op}} \times \mathbb{X}, \mathbf{Set}] \, , \qquad \mathrm{Kl}(P)[\mathbb{X}, \mathbb{Y}] = \mathbf{CAT}[\mathbb{X}, P(\mathbb{Y})] \, .$$

Thus, we have a canonical isomorphism of hom-categories

$$\tau \colon \mathbf{Prof}[\mathbb{X}, \mathbb{Y}] \to \mathrm{Kl}(P)[\mathbb{X}, \mathbb{Y}] \, ,$$

given by adjoint transposition. Furthermore, these isomorphisms are compatible with composition and identities. For composition, it suffices to observe that, for profunctors $F \colon \mathbb{X} \to \mathbb{Y}$ and $G \colon \mathbb{Y} \to \mathbb{Z}$, there is a natural isomorphism

$$\tau(G \circ F) \cong (\tau G) \circ (\tau F) \, ,$$

where the composition on the left-hand side is that of **Prof**, as defined in (2.6), while the composition on the right is the one of $\mathrm{Kl}(P)$, which is given by the functorial composite of $\tau(F) \colon \mathbb{X} \to P(\mathbb{Y})$ and $(\tau G)^\dagger \colon P(\mathbb{Y}) \to P(\mathbb{Z})$, the latter being defined by the formula for left Kan extensions in (3.3). For identities, simply note that, for a small category $\mathbb{X}$, the adjoint transpose of the identity profunctor on $\mathbb{X}$, as defined in (2.7), is exactly the Yoneda embedding, which is the identity on $\mathbb{X}$ in $\mathrm{Kl}(P)$.

We now make precise in what sense relative pseudomonads are a generalization of no-iteration pseudomonads [54, Definition 2.1]. This will be useful in order to relate relative pseudomonads and ordinary pseudomonads.

Lemma 3.6. *Let T be a relative pseudomonad over $J \colon \mathcal{C} \to \mathcal{D}$.*

(i) For every $f \colon JX \to TY$ and $g \colon JY \to TZ$, the diagram

$$
\begin{tikzcd}
g^* f \arrow[r, "\eta_{g^* f}"] \arrow[dr, bend right, "g^* \eta_f"'] & (g^* f)^* i_X \arrow[d, "\mu_{g,f}"] \\
 & g^* f^* i_X
\end{tikzcd}
$$

commutes.

(ii) For every $f\colon JX \to TY$*, the diagram*

$$\begin{tikzcd}
(i_Y{}^* \, f)^* \arrow[r, "\mu_{Y,f}"] \arrow[dr, bend right, "(\theta_Y \, f)^*"'] & i_Y{}^* \, f^* \arrow[d, "\theta_Y \, f^*"] \\
 & f^*
\end{tikzcd}$$

commutes.

(iii) For every $X \in \mathcal{C}$*, the diagram*

$$\begin{tikzcd}
i_X \arrow[r, "\eta_X"] \arrow[dr, bend right, "1"'] & i_X{}^* \, i_X \arrow[d, "\theta_X \, i_X"] \\
 & i_X
\end{tikzcd}$$

commutes.

Proof. The proof is a modified version of the proof of the redundancy of three axioms in the original definition of a monoidal category [32] (see also [31]), which has a version also for pseudomonads [49, Proposition 8.1]. □

Proposition 3.7. *A no-iteration pseudomonad is the same thing as a relative pseudomonad over the identity.*

Proof. The two notions involve exactly the same data, except for the direction of the invertible 2-cells. Then, using the numbering of axioms for a no-iteration pseudomonad in [54, Definition 2.1], the equivalence between axioms for a relative pseudomonad and those for a no-iteration pseudomonad are given as follows:

Relative pseudomonads		*No-iteration pseudomonads*
Naturality of μ	$\Leftrightarrow$	Axioms 6 and 7
Naturality of η	$\Leftrightarrow$	Axiom 4
Associativity axiom	$\Leftrightarrow$	Axiom 8
Unit axiom	$\Leftrightarrow$	Axiom 2
Lemma 3.6, part (i)	$\Leftrightarrow$	Axiom 5
Lemma 3.6, part (ii)	$\Leftrightarrow$	Axiom 3
Lemma 3.6, part (iii)	$\Leftrightarrow$	Axiom 1.

Note that it follows that the axioms (1), (3) and (5) for a no-iteration pseudomonad in [54, Definition 2.1] are redundant, in that they can be derived from the others. □

We now use Proposition 3.7 and the analysis of the relationship between ordinary pseudomonads and no-iteration pseudomonads in [54] to show how a pseudomonad can be regarded as a relative pseudomonads over the identity $1_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$ and, conversely, how every relative pseudomonad over the identity determines a pseudomonad.

Remark 3.8 (From pseudomonads to relative pseudomonads)*.* The combination of [54, Theorem 6.1] and Proposition 3.7 shows that every pseudomonad induces a relative pseudomonad over the identity $1_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$. We discuss this explicitly for use in the rest of the paper. Let us fix a pseudomonad $T\colon \mathcal{C} \to \mathcal{C}$ with data as in Section 2. We define a relative pseudomonad over the identity $1_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C}$ as follows. For $X \in \mathcal{C}$, we already have $TX \in \mathcal{C}$ and a morphism $i_X \colon X \to TX$ as part of the pseudomonad structure. For a morphism $f \colon X \to TY$, we define $f^* \colon TX \to TY$ by letting $f^* =_{\mathrm{def}} n_Y \, T(f)$. The three families of invertible 2-cells μ, η, θ are then obtained as follows. For $f \colon X \to TY$ and $g \colon Y \to TZ$, we need $\mu_{g,f} \colon (g^* f)^* \to g^* f^*$. Since we have

$$(g^\dagger f)^\dagger = n_Z \, T\big(n_Z \, T(g) \, f\big) \, , \quad g^\dagger f^\dagger = n_Z \, T(g) \, n_Y \, T(f) \, .$$

we can define $\mu_{g,f}$ to be the composite 2-cell

$$n_Z \, T(n_Z \, T(g) \, f) \xrightarrow{\cong} n_Z \, T(n_Z) \, T^2(g) \, T(f) \xrightarrow{\alpha} n_Z \, n_{TZ} \, T^2(g) \, T(f) \xrightarrow{\cong} n_Z \, T(g) \, n_Z \, T(f) \, ,$$

where the unnamed isomorphism 2-cells are the pseudofunctoriality of T and pseudonaturality of n. For $f \colon X \to TY$, we let $\eta_f \colon f \to f^* i_X$ be the composite 2-cell

$$f \xrightarrow{\lambda} n_Y \, i_{TY} \, f \xrightarrow{\cong} n_Y \, T(f) \, i_X \, .$$

where the unnnamed isomorphism 2-cell is a pseudonaturality of i. Finally, for $X \in \mathcal{C}$, we define $\theta_X \colon {i_X}^* \to 1_{TX}$ to be $\rho_X \colon n_{TX} \, T(i_X) \to 1_{TX}$. For the verification of the coherence conditions, see [54, Theorem 6.1]. Note how the direction of the invertible modifications of a pseudomonad chosen in (2.8) is consonant with the one of the 2-cells of a relative pseudomonad in Definition 3.1, in that we do not need to invert any of the former to obtain the latter.

Remark 3.9 (From relative pseudomonad over the identity to pseudomonads). The combination of Proposition 3.7 and [54, Theorem 3.6] shows that every relative pseudomonad over an identity pseudofunctor induces a pseudomonad. Fix a relative pseudomonad T over $1_{\mathcal{C}} \to \mathcal{C}$ with data as in Definition 3.1, i.e. a no-iteration pseudomonad on $\mathcal{C}$. We define a pseudomonad on $\mathcal{C}$ as follows. By [54, Proposition 3.1] (or Proposition 3.10 below) we have a pseudofunctor $T \colon \mathcal{C} \to \mathcal{C}$ and by [54, Proposition 3.2] (or Proposition 3.11 below) we have a pseudonatural transformation $i \colon 1_{\mathcal{C}} \to T$, providing the unit of the pseudomonad. For $X \in \mathcal{C}$, the component of the multiplication $n_X \colon T^2 X \to TX$ is given by letting $n_X =_{\mathrm{def}} (1_{TX})^*$ and it is not difficult to make n into a pseudonatural transformation. For the associativity modification in (2.8), we need to define, for $X \in \mathcal{C}$, an invertible 2-cell $\alpha_X \colon n_X \, n_{TX} \to n_X \, T(n_X)$. Since

$$n_X \, n_{TX} = (1_{TX})^* \, (1_{T^2X})^* \, , \quad n_X \, T(n_X) = {1_{TX}}^* (i_{TX} \, {1_{TX}}^*)^* \, ,$$

we can define α_X to be the composite

$$(1_{TX})^* (1_{T^2X})^* \xrightarrow{\mu^{-1}} ((1_{TX})^*)^* \xrightarrow{\eta} ({1_{TX}}^* \, i_X \, {1_{TX}}^*)^* \xrightarrow{\mu} {1_{TX}}^* (i_{TX} \, {1_{TX}}^*)^* \, .$$

For the left unit modification in (2.8), we need invertible 2-cells $\lambda_X \colon 1_{TX} \to n_X \, i_{TX}$. Since

$$n_X \, i_{TX} = {1_{TX}}^* \, i_{TX} \, ,$$

we can define λ_X to be $\eta_{1_{TX}} \colon 1_{TX} \to {1_{TX}}^* \, i_{TX}$. For the right unit modification in (2.8), we need invertible 2-cell $\rho_X \colon n_X \, T(i_X) \to 1_{TX}$. Since

$$n_X \, T(i_X) = {1_{TX}}^* \, (i_{TX} \, i_X)^* \, ,$$

we can define ρ_X to be the composite

$$(1_{TX})^* \, (i_{TX} \, i_X)^* \xrightarrow{\mu^{-1}} \big((1_{TX})^* \, i_{TX} \, i_X\big)^* \xrightarrow{\eta^{-1}} (i_X)^* \xrightarrow{\theta} 1_{TX} \, .$$

These definitions are a bit involved, and therefore checking the coherence diagrams directly is not straightforward [54, Theorem 3.6], but we shall outline a more conceptual account of the construction of a pseudomonad from a relative pseudomonad in Remark 4.7.

Next, we generalize and extend some results on no-iteration pseudomonads to relative pseudomonads. We begin by generalizing [54, Proposition 3.1].

Proposition 3.10. *Let T be a relative pseudomonad over $J\colon \mathcal{C} \to \mathcal{D}$. Then the function mapping $X \in \mathcal{C}$ to $TX \in \mathcal{D}$ admits the structure of a pseudofunctor $T\colon \mathcal{C} \to \mathcal{D}$.*

Proof. For $f\colon X \to Y$, we define $T(f)\colon TX \to TY$ by $Tf =_{\mathrm{def}} (i_Y J(f))^*$. We then define the pseudofunctoriality 2-cells. First, we need an invertible 2-cells $\tau_{g,f}\colon T(g\,f) \to T(g)\,T(f)$, for $f\colon X \to Y$ and $g\colon Y \to Z$. By definition, we have

$$T(g\,f) = (i_Z\,J(g)\,J(f))^*\,, \quad T(g)\,T(f) = (i_Z\,J(g))^*\,(i_Y\,J(f))^*\,.$$

We then define $\tau_{g,f}$ as the composite 2-cell

$$(i_Z\,J(g)\,J(f))^* \xrightarrow{\quad\eta\quad} ((i_Z\,J(g))^*\,i_Y\,J(f))^* \xrightarrow{\quad\mu\quad} (i_Z\,J(g))^*\,(i_Y\,J(f))^*$$

Secondly, we need invertible 2-cells $\tau_X\colon T(1_X) \to 1_{TX}$ for $X \in \mathcal{C}$. But since $T(1_X) = {i_X}^*$ by definition, we let $\tau_X =_{\mathrm{def}} \theta_X$, the component of the left unit of the relative pseudomonad.

One should check the three coherence laws for a pseudofunctor. The coherence law for $\tau_{g,f}$ involves a pasting of the associativity condition in (3.1), part *(i)* of Lemma 3.6 and all the naturaliy conditions for the families 2-cells of a relative pseudomonad. One of the coherence laws for τ_X comes from the unit condition in (3.2), while the other is from part *(ii)* of Lemma 3.6. □

The next proposition generalizes [54, Proposition 3.2].

Proposition 3.11. *Let T be a relative pseudomonad over $J\colon \mathcal{C} \to \mathcal{D}$. Then the family of morphisms $i_X\colon JX \to TX$, for $X \in \mathcal{C}$, admits the structure of a pseudonatural transformation $i\colon J \to T$.*

Proof. The required pseudonaturality 2-cell for $f\colon X \to Y$ fits into the diagram

$$\begin{tikzcd} JX \arrow[r, "J(f)"] \arrow[d, "i_X"'] \arrow[dr, phantom, "\Downarrow \bar{i}_f"] & JY \arrow[d, "i_Y"] \\ TX \arrow[r, "T(f)"'] & TY \end{tikzcd}$$

Since $T(f) = (i_Y\,J(f))^*$, we can simply let

$$\bar{i}_f =_{\mathrm{def}} \eta_{i_Y\,J(f)}\,. \tag{3.4}$$

We should check two coherence conditions for pseudonatural transformations. The composition condition involves a pasting of a naturality of η to a diagram coming from part *(i)* of Lemma 3.6. The identity condition is just part *(iii)* of Lemma 3.6 . □

We conclude this short series of propositions with a result whose counterpart for no-iteration pseudomonads does not seem to have been noticed yet.

Proposition 3.12. *Let T be a relative pseudomonad over $J\colon \mathcal{C} \to \mathcal{D}$. Then the family of functions*

$$(-)^*\colon \mathcal{D}[JX, TY] \to \mathcal{D}[TX, TY]$$

for $X, Y \in \mathcal{C}$, admits the structure of a pseudonatural transformation.

Proof. To see that this is pseudonatural in X, take $u\colon X' \to X$, observe that $f^*\,T(u) = f^*\,(i_X\,u)^*$ and note the 2-cell

$$(f\,u)^* \xrightarrow{\ \eta\ } (f^*\,i_X\,u)^* \xrightarrow{\ \mu\ } f^*\,(i_X\,u)^*\,.$$

To see that $(-)^*$ is pseudonatural in Y, take $v\colon Y \to Y'$, observe that $(T(v)\, f)^* = ((i_{Y'}\, v)^*\, f)^*$ and $T(v)\, f^* = (i_{Y'}\, v)^*\, f^*$, and note the 2-cell

$$\big((i_{Y'}\, v)^*\, f\big)^* \xrightarrow{\ \mu\ } (i_{Y'}\, v)^*\, f^* \, .$$

There are coherence conditions to check, but they are straightforward. □

4. **Relative pseudoadjunctions**

We introduce a generalization of the notion of pseudoadjunction between bicategories [12, 61], extending to the 2-categorical setting the notion of a relative adjunction studied in [1, Section 2.2]. We fix again an inclusion of bicategories $J\colon \mathcal{C} \to \mathcal{D}$.

Definition 4.1. Let $G\colon \mathcal{E} \to \mathcal{D}$ be a pseudofunctor. A *relative left pseudoadjoint* F to G over $J\colon \mathcal{C} \to \mathcal{D}$, denoted

$$\begin{tikzcd} & \mathcal{E} \arrow[d, "G"] \\ \mathcal{C} \arrow[ur, "F"] \arrow[r, "J"'] & \mathcal{D} \, , \end{tikzcd}$$

consists of

- an object $FX \in \mathcal{D}$, for every object $X \in \mathcal{C}$;
- a family of morphisms $i_X\colon JX \to GFX$, for $X \in \mathcal{C}$;
- a family of adjoint equivalences

$$\begin{tikzcd} \mathcal{D}[JX, GA] \arrow[r, shift left=1.5ex, "(-)^\sharp"] \arrow[r, phantom, "\perp"] & \mathcal{E}[FX, A] \, , \arrow[l, shift left=1.5ex, "G(-)\, i_X"] \end{tikzcd} \tag{4.1}$$

for $X \in \mathcal{C}$, $A \in \mathcal{E}$.

For a relative pseudoadjunction as in Definition 4.1, the components of the unit and counit of the adjoint equivalences in (4.1) will be written

$$\eta_f\colon f \to G(f^\sharp)\, i_X \, , \quad \varepsilon_u\colon (G(u)\, i_X)^\sharp \to u \, ,$$

respectively, where $f\colon JX \to GA$ and $u\colon FX \to A$. As we will see in Theorem 4.4, the direction of the adjoint equivalence in (4.1) has been chosen so as to be consonant with the notion of a relative pseudomonad. Note that a relative pseudoadjunction over the identity $1_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$ is the same thing as a pseudoadjunction in the usual sense [34, 61]. However, as our next example shows, relative pseudoadjunctions are more general than pseudoadjunctions.

Example 4.2. There is a relative pseudoadjunction of the form

$$\begin{tikzcd} & \mathbf{COC} \arrow[d, "U"] \\ \mathbf{Cat} \arrow[ur, "P"] \arrow[r, "J"'] & \mathbf{CAT} \, , \end{tikzcd}$$

where **COC** is the 2-category of locally small cocomplete categories, cocontinuous functors, and natural transformations, and $U\colon \mathbf{COC} \to \mathbf{CAT}$ is the evident forgetful functor. The category $P(\mathbb{X})$ of presheaves over a small category $\mathbb{X}$ is the colimit completion of $\mathbb{X}$ in the sense that, for every cocomplete $\mathbb{A}$, composition with the Yoneda embedding $\mathrm{y}_{\mathbb{X}} \colon \mathbb{X} \to P(\mathbb{X})$ induces an equivalence of categories

$$\mathbf{CAT}[\mathbb{X}, \mathbb{A}] \xleftarrow{U(-)\,\mathrm{y}_{\mathbb{X}}} \mathbf{COC}[P(\mathbb{X}), \mathbb{A}]\,.$$

Thus P provides a relative left pseudoadjoint to U. Similarly, there is a relative pseudoadjunction

$$\begin{tikzcd} & \mathbf{FIL} \arrow[d, "U"] \\ \mathbf{Cat} \arrow[ur, "D"] \arrow[r, "J"'] & \mathbf{CAT}\,, \end{tikzcd}$$

where **FIL** is the 2-category of locally small categories with filtered colimits, functors preserving such colimits, and all natural transformations [2].

We begin to explore some consequences of the definition of a relative pseudoadjunction, with a view towards establishing that every relative pseudoadjunction determines a relative pseudomonad.

Lemma 4.3. *Let*

$$\begin{tikzcd} & \mathcal{E} \arrow[d, "G"] \\ \mathcal{C} \arrow[ur, "F"] \arrow[r, "J"'] & \mathcal{D} \end{tikzcd}$$

be a relative pseudoadjunction. Then the function sending $X \in \mathcal{C}$ to $FX \in \mathcal{E}$ admits the structure of a pseudofunctor $F \colon \mathcal{C} \to \mathcal{E}$.

Proof. For a morphism $f \colon X \to Y$ in $\mathcal{C}$, we define $F(f) =_{\mathrm{def}} (i_Y\, J(f))^\sharp$. The unit of the adjoint equivalence in (4.1) gives us invertible 2-cells

$$\psi_f \colon i_Y\, J(f) \to GF(f)\, i_X\,,$$

for $f \colon X \to Y$. For $f \colon X \to Y$, $g \colon Y \to Z$, we need an invertible 2-cell $\phi_{g,f} \colon F(g\, f) \to F(g)\, F(f)$. By the definition, we have

$$F(g\, f) = (i_Z\, J(g\, f))^\sharp = (i_Z\, J(g)\, J(f))^\sharp\,, \quad F(g)\, F(f) = \big(G\big(F(g)\, F(f)\big)\, i_X\big)^\sharp\,.$$

So we can define $\phi_{g,f}$ as the composite

$$(i_Z\, J(g)\, J(f))^\sharp \xrightarrow{\psi} \big(GF(g)\, i_Y\, J(f)\big)^\sharp \xrightarrow{\psi} \big(GF(g)\, GF(f)\, i_X\big)^\sharp \xrightarrow{\cong} \big(G\big(F(g)\, F(f)\big)\, i_X\big)^\sharp\,,$$

where the unnamed isomorphism is given by the pseudofunctoriality of G. For $X \in \mathcal{C}$, we need an invertible 2-cell $\phi_X \colon F(1_X) \to 1_{FX}$. By definition, $F(1_X) = (i_X)^\sharp = (G(1_{FX})\, i_X)^\sharp$, and so we define ϕ_X to be $\varepsilon_{1_{FX}} \colon (G(1_{FX})\, i_X)^\sharp \to 1_{FX}$. The proof of the coherence conditions is routine (cf. [34, 61]). $\square$

We now establish a variant of a standard fact that an pseudoadjunction of bicategories gives rise to pseudomonad [34, 61], namely that a relative pseudoadjunction determines a relative pseudomonad.

Theorem 4.4. *Let*

$$\begin{tikzcd} & \mathcal{E} \arrow[d, "G"] \\ \mathcal{C} \arrow[ur, "F"] \arrow[r, "J"'] & \mathcal{D} \end{tikzcd}$$

be a relative pseudoadjunction. Then the function sending $X \in \mathcal{C}$ to $GF(X) \in \mathcal{D}$ admits the structure of a relative pseudomonad over J.

Proof. For $X \in \mathcal{C}$, define $TX =_{\text{def}} GFX$. The relative pseudoadjunction gives morphisms $i_X : X \to TX$, for $X \in \mathcal{C}$, and adjoint equivalences

$$\begin{tikzcd} \mathcal{D}[JX, GFY] \arrow[r, shift left, "(-)^\sharp"] \arrow[r, phantom, "\perp"] & \mathcal{E}[FX, FY] \arrow[l, shift left, "G(-)\, i_X"] \end{tikzcd} \tag{4.2}$$

for $X, Y \in \mathcal{C}$. We then define $(-)^* : \mathcal{D}[JX, TY] \to \mathcal{D}[TX, TY]$ by letting $f^* =_{\text{def}} G(f^\sharp)$. It now remains to define the families of invertible 2-cells μ, η and θ. For $\mu_{g,f} : (g^* f)^* \to g^* f^*$, observe that

$$(g^* f)^* = G(G(g^\sharp)\, f)^\sharp \,, \quad g^* f^* = G(g^\sharp)\, G(f^\sharp) \,,$$

and so we define $\mu_{g,f}$ to be the composite

$$G\big(G(g^\sharp)\, f\big)^\sharp \xrightarrow{\eta} G\big(G(g^\sharp)\, G(f^\sharp)\, i_X\big)^\sharp \xrightarrow{\cong} G(G(g^\sharp\, f^\sharp)\, i_X)^\sharp \xrightarrow{\varepsilon} G(g^\sharp\, f^\sharp) \xrightarrow{\cong} G(g^\sharp)\, G(f^\sharp) \,.$$

The 2-cells $\eta_f : f \to f^* \, i_X$ are given by the units of the adjunction (4.2), which satisfy the required naturality condition. For $\theta_X : {i_X}^* \to 1_{TX}$, recall that ${i_X}^* = G({i_X}^\sharp)$, so we define θ_X to be the composite 2-cell

$$G({i_X}^\sharp) \xrightarrow{\eta} G\big((G(1_{FX})\, i_X)^\sharp\big) \xrightarrow{\varepsilon} G(1_{FX}) \xrightarrow{\cong} 1_{GFX} \,.$$

It now remains to establish the coherence conditions. While it is possible to show this directly, it is more illuminating to argue in terms of universal properties. Simply restating the adjunction in (4.2), we observe that, given $f : JX \to GA$ in $\mathcal{D}$ and $u : FX \to A$ in $\mathcal{E}$, for every 2-cell $\phi : f \to G(u)\, i_X$, there is a unique 2-cell $\psi : f^\sharp \to u$, the adjoint transpose, such that the diagram

$$\begin{tikzcd} f \arrow[r, "\eta_f"] \arrow[dr, bend right, "\phi"'] & G(f^\sharp)\, i_X \arrow[d, "G(\psi)\, i_X"] \\ & G(u)\, i_X \end{tikzcd}$$

commutes. Accordingly, we can characterize $\mu_{g,f}$ and θ_X as follows. There are 2-cells

$$\tilde{\kappa}_{g,f} : G(G(g^\sharp)\, f)^\sharp \to G(g^\sharp\, f^\sharp) \,, \qquad \tilde{\kappa}_X : G({i_X}^\sharp) \to G(1_{FX})$$

being the image under G of the unique 2-cells $(G(g^\sharp)\, f)^\sharp \to g^\sharp\, f^\sharp$ and ${i_X}^\sharp \to 1_{FX}$ such that the diagrams

$$\begin{tikzcd} G(g^\sharp)\, f \arrow[r, "\eta"] \arrow[d, "\eta"'] & G(G(g^\sharp)\, f)^\sharp\, i_X \arrow[d, "\tilde{\kappa}_{g,f}\, i_X"] \\ G(g^\sharp)\, G(f^\sharp)\, i_X \arrow[r, "\cong"'] & G(g^\sharp\, f^\sharp)\, i_X \end{tikzcd}$$

and

$$\begin{tikzcd} i_X \arrow[r, "\eta"] \arrow[d, "\cong"'] & G({i_X}^\sharp)\, i_X \arrow[d, "\tilde{\kappa}_X\, i_X"] \\ 1_{GFX}\, i_X \arrow[r, "\cong"'] & G(1_{FX}) i_X \end{tikzcd}$$

commute, respectively. The 2-cells $\mu_{g,f}$ and θ_X then arise by composing these 2-cells with pseudofunctoriality 2-cells of G. The coherence diagrams follow readily, and we give details in the 2-categorical case, where the characterizing diagrams for $\mu_{g,f}$ and θ_X reduce to the diagrams

$$
\begin{tikzcd}
i_X \arrow[r, "\eta"] \arrow[dr, bend right, "1"'] & {i_X}^* \, i_X \arrow[d, "\theta_X \, i_X"] \\
 & i_X \, ,
\end{tikzcd}
\qquad
\begin{tikzcd}
g^* \, f \arrow[r, "\eta"] \arrow[dr, bend right, "g^* \, \eta_f"'] & (g^* \, f)^* \, i_X \arrow[d, "\mu_{g,f} \, i_X"] \\
 & g^* \, f^* \, i_X \, .
\end{tikzcd}
$$

For the associativity condition in (3.1), we have commuting diagrams

$$
\begin{tikzcd}
(h^* \, g)^* \, f \arrow[r, "\eta"] \arrow[ddd, "\mu_{h,g} \, f"'] \arrow[dr, "(h^* \, g)^* \, \eta"'] & \big((h^* \, g)^* \, f\big)^* \, i_X \arrow[d, "\mu_{h^* \, g, f} \, i_X \, ,"] \\
 & (h^* \, g)^* \, f^* \, i_X \arrow[dd, "\mu_{h,g} \, f^* \, i_X"] \\
h^* \, g^* \, f \arrow[dr, "h^* \, g^* \, \eta"'] & \\
 & h^* \, g^* \, f^* \, i_X
\end{tikzcd}
\qquad
\begin{tikzcd}
(h^* \, g)^* \, f \arrow[r, "\eta"] \arrow[d, "\mu_{g,h} \, f"'] & \big((h^* \, g)^* \, f\big)^* \, i_X \arrow[d, "(\mu_{h,g} \, f)^* \, i_X"] \\
h^* \, g^* \, f \arrow[r, "\eta"] \arrow[dr, "h^* \, \eta"'] \arrow[ddr, bend right, "h^* \, g^* \, \eta"'] & (h^* \, g^* \, f)^* \, i_X \arrow[d, "\mu_{h, g^* \, f} \, i_X"] \\
 & h^* \, (g^* \, f)^* \, i_X \arrow[d, "h^* \, \mu_{f,g} \, i_X"] \\
 & h^* \, g^* \, f^* \, i_X \, .
\end{tikzcd}
$$

Note that the triangles in these diagrams commute by part *(i)* of Lemma 3.6. Since both of the composites on the right-hand side of the diagrams lie in the image of G, we deduce by universality that they are equal, as required. For the unit condition in (3.2) we have a commuting diagram

$$
\begin{tikzcd}
f \arrow[r, "\eta_f"] \arrow[d, "\eta_f"'] & f^* \, i_X \arrow[d, "{\eta_f}^* \, i_X"] \\
f^* \, i_X \arrow[r, "\eta_{f^* \, i_X}"] \arrow[dr, "f^* \, \eta_{i_X}"'] \arrow[ddr, bend right, "1"'] & (f^* \, i_X)^* \, i_X \arrow[d, "\mu_{f, i_X} \, i_X"] \\
 & f^* \, {i_X}^* \, i_X \arrow[d, "f^* \, \theta_X \, i_X"] \\
 & f^* \, i_X \, .
\end{tikzcd}
$$

Here, the triangles commute by part *(i)* and *(iii)* of Lemma 3.6. Again, since the composite of $(f^* \, \theta_X)\,(\mu_{f,i_X})\,({\eta_f}^*)$ lies in the image of G we deduce by universality that it equals the identity, as required. ☐

In one-dimensional category theory, every monad determines two adjunctions relating the base category with the category of Eilenberg-Moore algebras and the Kleisli category for the monad. For pseudomonads, the construction of the bicategory of pseudoalgebras is well-known and it has been considered for no-iteration pseudomonads in [54, Section 4], but we do not need its counterpart for relative pseudomonads. We focus instead on the counterpart of the Kleisli adjunction, which has not been considered for no-iteration pseudomonads yet. The first step is the following lemma.

Lemma 4.5. *Let T be a relative pseudomonad over $J\colon \mathcal{C} \to \mathcal{D}$. Then the function sending $X \in \mathrm{Kl}(T)$ to $TX \in \mathcal{D}$ admits the structure of a pseudofunctor $G^T\colon \mathrm{Kl}(T) \to \mathcal{D}$.*

Proof. For $X, Y \in \mathrm{Kl}(T)$, we define the functor $G^T_{X,Y}\colon \mathrm{Kl}(T)[X,Y] \to \mathcal{D}[G^T X, G^T Y]$ to be

$$(-)^*\colon \mathcal{D}[JX, TY] \longrightarrow \mathcal{D}[TX, TY]\,.$$

By inspection of the definitions, we can define the pseudofunctoriality 2-cells of G^T to be exactly some of the 2-cells that are part of the data of a relative pseudomonad, namely

$$\mu_{g,f}\colon G^T(g \circ f) \to G^T(g)\, G^T(f)\,, \quad \theta_X\colon G^T(i_X) \to 1_{G^T X}\,.$$

In order to have a pseudofunctor, we need to verify that the following three coherence diagrams commute:

$$\begin{tikzcd}
& G^T\big((h\circ g)\circ f\big) \arrow[dl, "G^T(\mu_{f,g,h})"'] \arrow[dr, "\mu_{h\circ g,f}"] & \\
G^T\big(h\circ(g\circ f)\big) \arrow[d, "\mu_{h,g\circ f}"'] & & G^T(h\circ g)\, G^T(f) \arrow[d, "\mu_{h,g}\, G^T(f)"] \\
G^T(h)\, G^T(g\circ f) \arrow[rr, "G^T(h)\, \mu_{g,f}"'] & & G^T(h)\, G^T(g)\, G^T(f)\,,
\end{tikzcd}$$

$$\begin{tikzcd}
G^T(f) \arrow[r, "G^T(\rho_f)"] \arrow[rrd, bend right, "1_{G^T(f)}"'] & G^T(f\circ i_X) \arrow[r, "\mu_{f,X}"] & G^T(f)\, G^T(1_X) \arrow[d, "G^T(f)\, \theta_X"] \\
& & G^T(f)\,,
\end{tikzcd}$$

$$\begin{tikzcd}
G^T(i_Y\circ f) \arrow[r, "\mu_{Y,f}"] \arrow[rd, bend right, "G^T(\lambda_f)"'] & G^T(i_Y)\, G^T(f) \arrow[d, "\theta_Y\, G^T(f)"] \\
& G^T(f)\,.
\end{tikzcd}$$

The first and second diagram follow at once from the coherence conditions in (3.1) and (3.2) that are part of the definition of a relative pseudomonad. The third is part *(ii)* of Lemma 3.6. ☐

By analogy with the one-dimensional case, we expect that the pseudofunctor $G^T\colon \mathrm{Kl}(T) \to \mathcal{D}$ has some form of left adjoint. The next result makes this precise.

Theorem 4.6. *Let $T\colon \mathcal{C} \to \mathcal{D}$ be a relative pseudomonad. Then $G^T\colon \mathrm{Kl}(T) \to \mathcal{D}$ has a relative left pseudoadjoint,*

$$\begin{tikzcd}
& \mathrm{Kl}(T) \arrow[d, "G^T"] \\
\mathcal{C} \arrow[ur, "F^T"] \arrow[r, "J"'] & \mathcal{D}\,.
\end{tikzcd}$$

Proof. For $X \in \mathcal{C}$, we define $F^T X =_{\text{def}} X$. Then we have $G^T F^T X = TX$, so the relative pseudomonad provides a morphism $i_X : JX \to G^T F^T X$. For these to act like the components of the unit of a relative pseudoadjunction one needs to show that the functor

$$\mathrm{Kl}(T)[F^T X, Y] \xrightarrow{G^T(-)\, i_X} \mathcal{D}[JX, G^T Y]$$

is a adjoint equivalence. But $\mathrm{Kl}(T)[F^T X, Y] = \mathcal{D}[JX, TY] = \mathcal{D}[JX, G^T Y]$ and the functor $G^T(-)\, i_X$ is naturally isomorphic to the identity. This is because, for $f : JX \to TY$, we have $G^T(f) \circ i_X = f^*\, i_X$ and there is a 2-cell $\eta_f : f \to f^*\, i_X$. □

Remark 4.7. Theorem 4.6 allows us to give a more conceptual account of the construction of a pseudomonad from a relative pseudomonad over the identity $1_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ in Remark 3.9. Given a relative pseudomonad T over $1_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$, we can construct a pseudoadjunction between $\mathcal{C}$ and $\mathrm{Kl}(T)$ as in Theorem 4.6. Then, the pseudomonad associated to this pseudoadjunction is exactly the pseudomonad described in Remark 3.9. So we have established indirectly the coherence conditions for a pseudomonad.

Remark 4.8. Note that if we start with a relative pseudomonad T over $J : \mathcal{C} \to \mathcal{D}$, form the associated Kleisli relative pseudoadjunction (as in Theorem 4.6), and take the induced relative pseudomonad (as in Theorem 4.4), then we then retrieve the original relative pseudomonad. The only issues arise at the 2-cell level and we leave the verifications to the interested readers. Conversely, suppose that we start with a relative pseudoadjunction

$$\begin{tikzcd} & \mathcal{E} \arrow[d, "G"] \\ \mathcal{C} \arrow[ur, "F"] \arrow[r, "J"'] & \mathcal{D}, \end{tikzcd}$$

form the induced relative pseudomonad $T = GF$ over $J : : \mathcal{C} \to \mathcal{D}$ (as in Theorem 4.4), and then take the induced relative pseudoadjunction

$$\begin{tikzcd} & \mathrm{Kl}(T) \arrow[d, "G^T"] \\ \mathcal{C} \arrow[ur, "F^T"] \arrow[r, "J"'] & \mathcal{D}, \end{tikzcd}$$

as in Theorem 4.6. We expect a comparison and indeed we have a pseudofunctor $C : \mathrm{Kl}(T) \to \mathcal{E}$ defined on objects by letting $C(X) =_{\text{def}} FX$, for $X \in \mathcal{C}$. On hom-categories, for $X, Y \in \mathcal{C}$, we define

$$C_{X,Y} : \mathcal{D}[JX, TY] \to \mathcal{E}[FX, FY]$$

by letting $C(f) =_{\text{def}} f^\sharp$, where we used that $TY = GF(Y)$.

In ordinary category theory one can compose adjunctions, and similarly one can compose pseudoadjunctions between bicategories. It does not make sense to compose relative pseudoadjunctions, but one can form the composite of a relative pseudoadjunction and a pseudoadjunction, as the next proposition shows.

Proposition 4.9. *Let*

$$\begin{tikzcd} & \mathcal{E} \arrow[d, "G"] \\ \mathcal{C} \arrow[ur, "F"] \arrow[r, "J"'] & \mathcal{D}, \end{tikzcd} \qquad \begin{tikzcd} \mathcal{E} \arrow[r, shift left, "F'"] \arrow[r, phantom, "\perp"] & \mathcal{E}' \arrow[l, shift left, "G'"] \end{tikzcd}$$

be a relative pseudoadjunction and a pseudoadjunction, respectively. Then there exists a relative pseudoadjunction of the form

$$\begin{tikzcd} & \mathcal{E}' \arrow[d, "GG'"] \\ \mathcal{C} \arrow[ur, "F'F"] \arrow[r, "J"'] & \mathcal{D}\,. \end{tikzcd}$$

Proof. The construction is evident and left to the readers. □

5. Lax idempotency

We isolate a special class of relative pseudomonads, which appears to be the appropriate generalization to our setting of the notion of a lax idempotent 2-monad (or Kock-Zöberlein 2-monad, or KZ-doctrine). The original treatments in [38, 65]. An extensive analysis of these 2-monads, with useful equivalent formulations, was given in the course of a study of general property-like 2-monads in [36]. For pseudomonads, the more general notion of lax idempotent pseudomonad on a bicategory already appears in [61], with yet another characterisation of the notion. Lax idempotent pseudomonads have been studied further in [49, 53]. We fix an inclusion $J\colon \mathcal{C} \to \mathcal{D}$.

Definition 5.1. A *lax idempotent relative pseudomonad* over J is a relative pseudomonad T over $J\colon \mathcal{C} \to \mathcal{D}$ equipped with a further family of 2-cells $\varepsilon_u : (u\, i_X)^* \to u$, for $u\colon TX \to TY$, such that

- the 2-cells η_f and ε_u are the components of the unit and counit of adjunctions

$$\begin{tikzcd} \mathcal{D}[JX, TY] \arrow[r, shift left, "(-)^*"] \arrow[r, phantom, "\perp"] & \mathcal{D}[TX, TY] \arrow[l, shift left, "(-)\, i_X"] \end{tikzcd}$$

  for $X, Y \in \mathcal{C}$.
- the diagram

$$\begin{tikzcd} (g^*\, f)^* \arrow[r, "(g^*\, \eta_f)^*"] \arrow[dr, bend right, "\mu_{f,g}"'] & (g^*\, f^*\, i_X)^* \arrow[d, "\varepsilon_{g^*\, f^*}"] \\ & g^*\, f^* \end{tikzcd}$$

  commutes for all $f\colon JX \to TY$, $g\colon JY \to TZ$;
- $\theta_X = \varepsilon_{1_{TX}}$ for all $X \in \mathcal{C}$.

Note that in lax idempotent relative pseudomonad, the 2-cells $\mu_{f,g}$ and θ_X that are part of the underlying relative pseudomonad are completely determined by the axioms, even if one needs to check that they are invertible and that they satisfy the appropriate coherence conditions. Our next goal is to give an alternative characterization of lax idempotent relative pseudomonads, which we will use to discuss further the relative pseudomonad of presheaves. For this, we introduce the auxiliary notion of a lax local adjunction in Definition 5.2 below.

Definition 5.2. A *lax local adjunction* over $J\colon \mathcal{C} \to \mathcal{D}$ consists of:

- an object $TX \in \mathcal{D}$ for every object $X \in \mathcal{C}$;
- a family of morphisms $i_X : JX \to TX$, for $X \in \mathcal{C}$;
- a family of adjunctions

$$\begin{tikzcd} \mathcal{D}[JX, TY] \arrow[r, shift left, "(-)^*"] \arrow[r, phantom, "\perp"] & \mathcal{D}[TX, TY] \arrow[l, shift left, "(-)\, i_X"] \end{tikzcd}$$

for $X, Y \in \mathcal{C}$, whose unit and counit are written

$$\eta_f : f \to f^*\, i_X\,, \quad \varepsilon_u : (u\, i_X)^* \to u$$

Note that the 2-cells η_f and ε_u that are part of a lax local adjunction are not necessarily invertible.

Example 5.3. There are adjunctions

$$\begin{tikzcd} \mathbf{CAT}[\mathbb{X}, P(\mathbb{Y})] \arrow[r, shift left, "(-)^*"] \arrow[r, phantom, "\perp"] & \mathbf{CAT}[P(\mathbb{X}), P(\mathbb{Y})] \arrow[l, shift left, "(-)\, \mathrm{y}_X"] \end{tikzcd} \tag{5.1}$$

with unit $\eta_F : F \to F^*\, \mathrm{y}_{\mathbb{X}}$ and counit $\varepsilon_U : (U\, \mathrm{y}_{\mathbb{X}})^* \to u$ which derive from the universal property of left Kan extensions. Note that the unit η_f is always an isomorphism. A similar example arises by considering the Ind-completion.

We want to establish under what conditions a lax local adjunction determines a lax idempotent relative pseudomonad. We begin in Lemma 5.4 below by constructing the missing 2-cells and establishing the necessary coherence conditions.

Lemma 5.4. *Let $T : \mathcal{C} \to \mathcal{D}$ be a lax local adjunction over J. Then, the families of 2-cells $\mu_{g,f}$, η_f, and θ_X, where*

$$\mu_{g,f} =_{\mathrm{def}} \varepsilon_{g^*\, f^*}\, (g^* \eta_f)^*\,, \quad \theta_X =_{\mathrm{def}} \varepsilon_{1_X}\,,$$

satisfy the associativity and unit coherence conditions for a relative pseudomonad.

Proof. For the associativity condition first note that that the diagram

$$\begin{tikzcd} (h^*\, u\, i_X)^* \arrow[r, "(h^*\, \eta_{u\, i_X})^*"] \arrow[dr] & (h^*\, (u\, i_X)^*\, i_X)^* \arrow[r, "\varepsilon_{h^*\, (u\, i_X)^*}"] \arrow[d, "(h^*\, \varepsilon_u\, i_X)^*"] & h^*\, (u\, i_X)^* \arrow[d, "h^*\, \varepsilon_u"] \\ & (h^*\, u\, i_X)^* \arrow[r, "\varepsilon_{h^*\, u}"'] & h^*\, u \end{tikzcd} \tag{5.2}$$

commutes, by a triangle identity and the naturality of ε. Then the associativity coherence condition is given by the diagram

$$\begin{tikzcd}
 & ((h^* g)^* f)^* \arrow[dl, "((h^* \eta_g)^* f)^*"'] \arrow[dr, "((h^* g)^* \eta_f)^*"] & \\
((h^* g^* i)^* f)^* \arrow[dr, "((h^* g^* i)^* \eta_f)^*"] \arrow[dd, "(\varepsilon_{h^* g^*} f)^*"'] & & ((h^* g)^* f^* i)^* \arrow[dl, "((h^* \eta_g)^* f^* i)^*"'] \arrow[dd, "\varepsilon_{(h^* g)^* f^*}"] \\
 & ((h^* g^* i)^* f^* i)^* \arrow[dd, "(\varepsilon_{h^* g^*} f^* i)^*"'] \arrow[ddddr, "\varepsilon_{(h^* g^* i)^* f^*}"] & \\
(h^* g^* f)^* \arrow[dr, "(h^* g^* \eta_f)^*"] \arrow[dd, "(h^* \eta_{g^* f})^*"'] & & (h^* g)^* f^* \arrow[dd, "(h^* \eta_g)^* f^*"] \\
 & (h^* g^* f^* i)^* \arrow[dd, "(h^* \eta_{g^* f^* i})"'] \arrow[dddr, "\varepsilon_{h^* g^* f^*}"] & \\
(h^* (g^* f)^* i)^* \arrow[dr, "(h^* (g^* \eta_f)^* i)"] \arrow[dd, "\varepsilon_{h^* (g^* f)^*}"'] & & (h^* g^* i)^* f^* \arrow[dd, "\varepsilon_{h^* g^*} f^*"] \\
 & (h^* (g^* f^* i)^* i)^* \arrow[dd, "\varepsilon_{h^* (g^* f^* i)^*}"'] & \\
h^* (g^* f)^* \arrow[dr, "h^* (g^* \eta_f)^*"'] & & h^* g^* f^* \\
 & h^* (g^* f^* i)^* \arrow[ur, "h^* \varepsilon_{g^* f^*}"'] &
\end{tikzcd}$$

where, starting from the top in a clockwise direction, we use interchange, two naturalities of ε, the diagram in (5.2), a naturality of ε, a naturality of η, and finally an interchange again.

The unit condition is given by the following diagram:

$$\begin{tikzcd}
f^* \arrow[r, "{\eta_f}^*"] \arrow[d, "{\eta_f}^*"] \arrow[dd, bend right=60, "1"'] & (f^* i_X)^* \arrow[dl, "1"'] \arrow[d, "(f^* \eta_{i_X})^*"] \\
(f^* i_X)^* \arrow[d, "\varepsilon_{f^*}"] & (f^* (i_X)^* i_X)^* \arrow[l, "(f^* \varepsilon i_X)^*"] \arrow[d, "\varepsilon_{f^* i_X^*}"] \\
f^* & f^* i_X{}^* \arrow[l, "f^* \varepsilon_{1_{TX}}"]
\end{tikzcd}$$

where we have two uses of the triangle identities and a naturality of ε. $\square$

Thus, the data of a lax local adjunction determines all the structure of a lax idempotent relative pseudomonad and the necessary coherence conditions, but we need to ensure that the 2-cells $\mu_{f,g}$, η_f, and θ_X are invertible. The following proposition provides equivalent conditions for this to be the case.

Proposition 5.5. *Let T be a local lax adjunction. Then the following conditions are equivalent:*

(i) T is a lax idempotent relative pseudomonad,

(ii) For every $f\colon JX \to TY$, the 2-cells η_f and ε_{f^*} are invertible, and we have isomorphisms

$$g^* f^* \cong (g^* f)^* \qquad i_X{}^* \cong 1_{TX}\,;$$

(iii) For every $f\colon JX \to TY$, the 2-cells η_f and ε_{f^*} are invertible and the bicategory $\mathcal{E}$ with

- *objects TX, for $X \in \mathcal{C}$,*
- *hom-category $\mathcal{E}[TX,TY]$, for $X,Y \in \mathcal{C}$, given by the full subcategory of $\mathcal{D}[TX,TY]$ spanned by the morphisms $u\colon TX \to TY$ such that $u \cong f^*$ for some $f\colon JX \to TY$;*

is a sub-bicategory of $\mathcal{C}$.

(iv) The 2-cells η_f are invertible and there is a sub-bicategory $\mathcal{E}$ of $\mathcal{D}$ through which $(-)^*$ factors and with the property that ε_u is invertible for all $u \in \mathcal{E}$.

Proof. For $(i) \Rightarrow (ii)$, we immediately have the invertibility of η_f and the given isomorphisms. Furthermore one easily checks that ε_{f^*} is the composite

$$(f^*\, i_X)^* \xrightarrow{\mu_{f,i_X}} f^*\, {i_X}^* \xrightarrow{f^*\theta_X} f^*$$

so it also is invertible as θ_X and μ_{f,i_X} are. For $(ii) \Rightarrow (iii)$, the given isomorphisms show that $\mathcal{E}$ as defined contains identities and is closed under composition. The implication $(iii) \Rightarrow (iv)$ is obvious. For $(iv) \Rightarrow (i)$, since $(-)^*$ sends morphisms of $\mathcal{E}$ to morphisms of $\mathcal{E}$, we see that the 2-cells $\varepsilon_{1_{TX}}$ and $\varepsilon_{g^* f^*}$ are invertible. We then see from the axioms of a lax idempotent relative pseudomonad that θ_X and $\mu_{g,f}$ are invertible. $\square$

Example 5.6. The relative pseudomonad of presheaves is lax idempotent since the left adjoints in (5.1) factor through **COC** the sub-2-category of cocomplete locally small categories and cocontinuous functors, and if $U\colon P(\mathbb{X}) \to P(\mathbb{Y})$ is cocontinuous, then ε_U is an isomorphism. The relative pseudomonad of presheaves derives from this by Lemma 5.4 and Proposition 5.5.

Similarly, the Ind-completion relative pseudomonad $\mathcal{D}$ is lax idempotent. This is because the corresponding left adjoint factors through **FIL** the sub-2-category of Ind-complete categories and functors preserving filtered colimits; and if $U\colon D(\mathbb{X}) \to D(\mathbb{Y})$ preserves filtered colimits then ε_U is an isomorphism. The Ind-completion relative pseudomonad derives from this by Lemma 5.4 and Proposition 5.5.

A detailed study of lax idempotent 2-monads on a 2-category appears in [36]. We use some results therein to show that our notion of a lax idempotent relative pseudomonad relates to the notion of a lax idempotent 2-monad. Suppose, as in [36] that we have a 2-category $\mathcal{C}$ and a 2-monad T on it. As a special case of Remark 3.8, the 2-monad gives rise to a canonical relative pseudomonad over the identity on $\mathcal{C}$, although we will not need 2-cells $\mu_{f,g}$ and θ_X here.

Proposition 5.7. *Let $T\colon \mathcal{C} \to \mathcal{C}$ be a 2-monad. Then T is lax idempotent as a 2-monad only if it is lax idempotent as a relative pseudomonad.*

Proof. Suppose first that T is lax idempotent as a 2-monad. We wish to show that the pair of functors

$$\begin{tikzcd} \mathcal{C}[X,TY] \arrow[r, shift left, "n_Y\, T(-)"] \arrow[r, phantom, "\perp"] & \mathcal{C}[TX,TY] \arrow[l, shift left, "(-)\, i_X"] \end{tikzcd}$$

are adjoint. We take the unit of the adjunction to be the identity. To define the counit,

$$\varepsilon_u\colon n_Y T(i_X u) \to u\,,$$

we use the modification $\delta\colon T\, i \to i\, T$ from the equivalent condition *(iv)* in [36, Theorem 6.2], whose characteristic property is that $\delta\, i = 1$ and $m\, \delta = 1$. For $u\colon TX \to TY$ we let ε_u be given by

$$n_Y\, T(u\, i_X) = n_Y\, T(u)\, T(i_X) \xrightarrow{n_Y T(u)\delta_X} n_Y\, T(u)\, i_{TX} = n_Y\, i_{TY}\, u = u\,.$$

We need to prove the triangle identities, which amount to $\varepsilon_u\, i_X = 1_{u\, i_X}$ and $\varepsilon_{n_Y\, T(f)} = 1_{n_Y\, T(f)}$. For the first, simply observe that $\varepsilon_u\, i_X = n_Y\, T(u)\, \delta_X\, i_X = 1_{u\, i_X}$. For the second, observe that

$$
\begin{aligned}
\varepsilon_{n_Y T(f)} &= n_Y\, T(n_Y\, T(f))\, \delta_X \\
&= n_Y\, T(n_Y)\, T^2(f)\, \delta_X \\
&= n_Y\, n_{TY}\, T^2(f)\delta_X \\
&= n_Y\, T(f)\, n_X\, \delta_X \\
&= 1_{n_Y\, T(f)}\,.
\end{aligned}
$$

Conversely, suppose that we have a counits ε_u with units identities as above. Since

$$
\begin{aligned}
n_{TX}\, T(i_{TX}\, i_X) &= n_{TX}\, T(Ti_X\, i_X) \\
&= n_{TX}\, T^2(i_X)\, T(i_X) \\
&= T(i_X)\, n_X\, T(i_X) \\
&= T(i_X)\,,
\end{aligned}
$$

we define $\delta_X \colon T(i_X) \to i_{TX}$ by letting $\delta_X =_{\mathrm{def}} \varepsilon_{i_{TX}}$. Now, using one triangle identity, we see that

$$\delta_X\, i_X = \varepsilon_{i_{TX}}\, i_X = 1_{i_{TX}\, i_X}\,,$$

so we get one condition on δ. For the other, note that if $v \colon TY \to TZ$ is a map of the corresponding free T-algebras, or equivalently $v = n_Z\, T(v\, i_Y)$, then the diagram

$$
\begin{tikzcd}
\mathcal{D}[X, TY] \arrow[r, shift left, "n_Y\, T(-)"] \arrow[r, phantom, "\perp"] \arrow[d, "{\mathcal{D}[X,v]}"'] & \mathcal{D}[TX, TY] \arrow[l, shift left, "(-)\, i_X"] \arrow[d, "{\mathcal{D}[TX,v]}"] \\
\mathcal{D}[X, TY] \arrow[r, shift left, "n_Z\, T(-)"] \arrow[r, phantom, "\perp"] & \mathcal{D}[TX, TZ] \arrow[l, shift left, "(-)\, i_X"]
\end{tikzcd}
$$

commutes serially. Then since composition with v preserves the units (which are identities), it preserves the counits, that is, we have $v\, \varepsilon_u = \varepsilon_{vu}$. Now we apply this to n_X which is a strict map of T-algebras. We deduce

$$n_X\, \varepsilon_{i_{TX}} = \varepsilon_{n_X\, i_{TX}} = \varepsilon_{1_{TX}}\,.$$

But $1_{TX} = n_X\, T(i_X)$ is in the image of the left adjoint and so $\varepsilon_{1_{TX}} = 1_{1_{TX}}$ giving the second condition on δ. ☐

Observe that, by the equivalence of Proposition 5.7, Proposition 5.5 adds a further equivalent characterisation of lax idempotent 2-monads to those of [36].

Lax idempotent relative pseudomonads play an important role for us. So we need to recognize when a relative pseudoadjunction gives rise to one. We note the following simple result.

Proposition 5.8. *Let*

$$
\begin{tikzcd}
 & \mathcal{E} \arrow[d, "G"] \\
\mathcal{C} \arrow[ur, "F"] \arrow[r, "J"'] & \mathcal{D}
\end{tikzcd}
$$

be a relative pseudoadjunction. Suppose that G is locally full and faithful. Then the relative pseudomonad $T =_{\mathrm{def}} GF$ over $J \colon \mathcal{C} \to \mathcal{D}$ is lax idempotent.

Proof. The relative pseudoadjunction gives us, for $X, Y \in \mathcal{C}$, an adjoint equivalence

$$\begin{tikzcd} \mathcal{D}[X, GFY] \arrow[r, shift left, "(-)^\sharp"] \arrow[r, phantom, "\perp"] & \mathcal{E}[FX, FY] \arrow[l, shift left, "G(-)\, i_X"] \end{tikzcd}$$

with unit $\eta_f \colon f \to G(f^\sharp) i_X$ and counit $\varepsilon_v \colon (G(v)\, i_X)^\sharp \to v$. Now G is locally full and faithful, that is,

$$G_{FX,FY} \colon \mathcal{E}[FX, FY] \to \mathcal{D}[GFX, GFY]$$

is an isomorphism of categories, and that immediately induces the required adjunction. We have $T =_{\mathrm{def}} GF$. For $f \colon JX \to TY$ and $u \colon TX \to TY$, we let $v \colon FX \to FY$ be the unique morphism such that $G(v) = u$. Then, since

$$\mathcal{D}[JX, TY](f, u\, i_X) = \mathcal{D}[JX, GFY](f, G(v)\, i_X)\,,$$

we have natural isomorphisms

$$\mathcal{D}[JX, TY](f, u\, i_X) \cong \mathcal{E}[FX, FY](f^\sharp, v) \cong \mathcal{E}[TX, TY](G(f^\sharp), u)\,.$$

Note that the unit at f is $\eta_f \colon f \to f^* i_X = G(f^\sharp) i_X$ is as above and the counit is

$$G(\varepsilon_v) \colon (u i_X)^* = G((G(v)\, i_X)^\sharp) \to G(v) = u\,. \qquad \square$$

Example 5.9. Proposition 5.8 allows us to obtain another proof that the relative pseudomonad of presheaves is lax idempotent. Indeed, the presheaf construction P lies in a relative pseudoadjunction

$$\begin{tikzcd} & \mathbf{COC} \arrow[d, "U"] \\ \mathbf{Cat} \arrow[ur, "P"] \arrow[r, "J"'] & \mathbf{CAT} \end{tikzcd}$$

and the relative pseudomonad corresponding to it is exactly the relative pseudomonad of presheaves. Since the forgetful 2-functor $U \colon \mathbf{COC} \to \mathbf{CAT}$ is locally full and faithful, Proposition 5.8 applies. Similar considerations apply to the relative pseudomonad of the Ind-completion.

6. Liftings, extensions, and compositions

We now discuss a general method to extend a 2-monad to the Kleisli bicategory of a relative pseudomonad, which we will apply in Section 7 to extend several 2-monads from the 2-category **Cat** of small categories and functors to the bicategory **Prof** of small categories and profunctors.

Let us begin by introducing the setting in which we will be working. We fix an inclusion of 2-categories $J \colon \mathcal{C} \to \mathcal{D}$, a relative pseudomonad T over J, with data as in Definition 3.1, and a 2-monad $S \colon \mathcal{D} \to \mathcal{D}$, with data as in Section 2. We assume that S restricts along J, in the sense that we have a commuting diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "J"] \arrow[d, "S"'] & \mathcal{D} \arrow[d, "S"] \\ \mathcal{C} \arrow[r, "J"'] & \mathcal{D}\,. \end{tikzcd}$$

This implies that the inclusion $J\colon \mathcal{C} \to \mathcal{D}$ can be lifted to inclusions $J\colon \text{Ps-}S\text{-Alg}_{\mathcal{C}} \to \text{Ps-}S\text{-Alg}_{\mathcal{D}}$ and $J\colon S\text{-Alg}_{\mathcal{C}} \to S\text{-Alg}_{\mathcal{D}}$ making the following diagram commute:

$$\begin{tikzcd} S\text{-Alg}_{\mathcal{C}} \arrow[r, "J"] \arrow[d] & S\text{-Alg}_{\mathcal{D}} \arrow[d] \\ \text{Ps-}S\text{-Alg}_{\mathcal{C}} \arrow[r, "J"] \arrow[d] & \text{Ps-}S\text{-Alg}_{\mathcal{D}} \arrow[d] \\ \mathcal{C} \arrow[r, "J"'] & \mathcal{D}\,, \end{tikzcd} \tag{6.1}$$

where the top vertical arrows are inclusions and the bottom ones are forgetful 2-functors. We shall deal with two types of liftings, one involving only strict algebras (Definition 6.1) and another involving both strict algebras and pseudoalgebras (Definition 6.2). We begin by defining the simpler type of lifting, involving only strict algebras.

Definition 6.1. A *lifting of T to strict algebras for S*, denoted

$$\begin{tikzcd} S\text{-Alg}_{\mathcal{C}} \arrow[r, "\bar{T}"] \arrow[d, "U"'] & S\text{-Alg}_{\mathcal{D}} \arrow[d, "U"] \\ \mathcal{C} \arrow[r, "T"'] & \mathcal{D}\,, \end{tikzcd}$$

consists of

- a strict algebra structure on TA, for every $A \in S\text{-Alg}_{\mathcal{C}}$;
- a pseudomorphism structure on $f^*\colon TA \to TB$, for every pseudomorphism $f\colon JA \to TB$;
- a pseudomorphism structure on $i_A\colon JA \to TA$, for every $A \in S\text{-Alg}_{\mathcal{C}}$;

such that

- $\mu_{f,g} : (g^*\, f)^* \to g^*\, f^*$ is an algebra 2-cell for every pair of pseudomorphisms $f\colon JA \to TB$ and $g\colon JB \to TC$;
- $\eta_f\colon f \to f^*\, i_A$ is an algebra 2-cell for every pseudomorphism $f\colon JA \to TB$;
- $\theta_A\colon {i_A}^* \to 1_{TA}$ is an algebra 2-cell for $A \in S\text{-Alg}_{\mathcal{C}}$.

Note that a lifting of T to strict algebras gives immediately a relative pseudomonad $\bar{T}$ over the inclusion $J\colon S\text{-Alg}_{\mathcal{C}} \to S\text{-Alg}_{\mathcal{D}}$ such that applying the forgetful 2-functors to the data of $\bar{T}$ returns the corresponding data of T. We shall give several examples of liftings of the relative monad of presheaves to strict algebras in Section 7. However, it is not useful to work with liftings to categories of strict algebras for other 2-monads, since for a strict algebra A there is no evident structure of strict algebra structure on TA. In order to address this situation, we introduce the following definition.

Definition 6.2. A *lifting of T to pseudoalgebras for S*, denoted

$$\begin{tikzcd} S\text{-Alg}_{\mathcal{C}} \arrow[r, "\bar{T}"] \arrow[d] & \text{Ps-}S\text{-Alg}_{\mathcal{D}} \arrow[d] \\ \mathcal{C} \arrow[r, "T"'] & \mathcal{D} \end{tikzcd}$$

consists of the following data:

- a pseudoalgebra structure on TA, for every $A \in S\text{-Alg}_{\mathcal{C}}$;

- a pseudomorphism structure on $f^* \colon TA \to TB$, for every pseudomorphism $f \colon JA \to TB$;
- a pseudomorphism structure on $i_A \colon A \to TA$, for every $A \in S\text{-Alg}_{\mathcal{C}}$;

such that

- $\mu_{f,g} \colon (g^* f)^* \to g^* f^*$ is an algebra 2-cell for every pair of pseudomorphisms $f \colon JA \to TB$ and $g \colon JB \to TC$;
- $\eta_f \colon f \to f^* i_A$ is an algebra 2-cell for every pseudomorphism $f \colon JA \to TB$;
- $\theta_A \colon {i_A}^* \to 1_{TA}$ is an algebra 2-cell for every $A \in S\text{-Alg}_{\mathcal{C}}$.

Similarly to what happened for liftings to strict algebras, a lifting of T to pseudoalgebras gives now a relative pseudomonad $\bar{T}$, but now over the inclusion $J \colon S\text{-Alg}_{\mathcal{C}} \to \text{Ps-}S\text{-Alg}_{\mathcal{D}}$, again suitably related to $\bar{T}$ via the appropriate forgetful 2-functors. Note here that for the inclusion $J \colon \mathbf{Cat} \to \mathbf{CAT}$, the corresponding inclusion $J \colon S\text{-Alg}_{\mathbf{Cat}} \to \text{Ps-}S\text{-Alg}_{\mathbf{CAT}}$ is not merely about size distinction, but involves both strict algebras and pseudoalgebras. Indeed, the notion of a relative pseudomonad was designed to encompass these situations as well.

We now want to show how a lifting of a relative pseudomonad T gives rise to pseudomonad on the Kleisli bicategory of T. In the one-dimensional situation, such a step involves passing via a distributive law [6]. In our setting, where we are dealing with both coherence and size issues, such approach would be rather complicated, as one would have to adapt the theory of pseudo-distributive laws [33, 50, 51] to relative pseudomonad. However, it is possible to take a more direct approach, as the next theorem shows.

Theorem 6.3. *Assume that T has a lifting to either strict algebras or pseudoalgebras for S. Then S has an extension to a pseudomonad $\tilde{S} \colon \mathrm{Kl}(T) \to \mathrm{Kl}(T)$ on the Kleisli bicategory of T.*

Proof. We only deal with the case of a lifting to pseudoalgebras, since the case of lifting to strict algebras is completely analogous. First, we consider the relative pseudomonad $\bar{T}$ over $J \colon S\text{-Alg}_{\mathcal{C}} \to \text{Ps-}S\text{-Alg}_{\mathcal{D}}$ and its Kleisli bicategory $\mathrm{Kl}(\bar{T})$. The objects of $\mathrm{Kl}(\bar{T})$ are strict algebras with underlying object in $\mathcal{C}$, and its hom-categories given by

$$\mathrm{Kl}(\bar{T})[A, B] = \text{Ps-}S\text{-Alg}_{\mathcal{D}}[JA, TB]\,.$$

Secondly, we observe that there is a forgetful pseudofunctor $U \colon \mathrm{Kl}(\bar{T}) \to \mathrm{Kl}(T)$, defined on objects by sending a strict algebra to its underlying object. To define the action on hom-categories, let $A, B \in S\text{-Alg}_{\mathcal{C}}$. Then, the required functor is determined by the diagram

$$\begin{tikzcd}
\mathrm{Kl}(\bar{T})[A,B] \arrow[r, "U_{A,B}"] \arrow[d, equal] & \mathrm{Kl}(T)[A,B] \arrow[d, equal] \\
\text{Ps-}S\text{-Alg}[JA,TB] \arrow[r, "U_{A,B}"'] & \mathcal{D}[JA,TB]\,.
\end{tikzcd}$$

We claim that U has a left pseudoadjoint. The action of the left pseudoadjoint on objects is defined by sending X to SX, the free pseudoalgebra on X (which is in fact a strict algebra since S is a 2-monad). Next, for $X \in \mathcal{C}$, we define morphisms $\tilde{e}_X \colon X \to SX$ in $\mathrm{Kl}(T)$ as the composite in $\mathcal{D}$

$$X \xrightarrow{e_X} SX \xrightarrow{i_{SX}} TSX\,.$$

We wish to show that these are suitably universal. For this, let us observe that the diagram

$$\begin{tikzcd}
\mathrm{Kl}(T)[X,A] \arrow[d, equal] & & \mathrm{Kl}(\bar{T})[SX,A] \arrow[ll, "U(-)\circ\tilde{e}_X"'] \arrow[d, equal] \\
\mathcal{D}[JX,TA] \arrow[d, equal] & & \text{Ps-}S\text{-Alg}_{\mathcal{D}}[JSX,TA] \arrow[d, equal] \\
\mathcal{D}[JX,TA] & & \text{Ps-}S\text{-Alg}_{\mathcal{D}}[SJX,TA] \arrow[ll, "U(-)\, e_X"]
\end{tikzcd} \tag{6.2}$$

commutes up to natural isomorphism, since if $f\colon SJX \to TA$ is a pseudomorphism, then

$$f \circ \tilde{e}_X = f^*\, \tilde{e}_X = f^*\, i_{SX}\, e_X \cong f\, e_X\,.$$

Since the horizontal arrow at the bottom of (6.2) is an equivalence, we have the desired universality of the morphism $\tilde{e}_X$. Now that we have pseudoadjunction

$$\begin{tikzcd}
\mathrm{Kl}(T) \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathrm{Kl}(\bar{T}) \arrow[l, shift left, "U"]
\end{tikzcd},$$

we obtain the desired extension $\tilde{S}$ as the pseudomonad associated to this pseudoadjunction. □

We conclude this section by showing that we can define a 'composite' relative pseudomonad.

Theorem 6.4. *Assume that T admits a lifting to pseudoalgebras of S. Then the function sending $X \in \mathcal{C}$ to $TS(X) \in \mathcal{D}$ admits the structure of a relative pseudomonad over $J\colon \mathcal{C} \to \mathcal{D}$.*

Proof. First, recall that, by Theorem 4.6, we have a relative pseudoadjunction

$$\begin{tikzcd}
 & \mathrm{Kl}(T) \arrow[d, "G^T"] \\
\mathcal{C} \arrow[ur, "F^T"] \arrow[r, "J"'] & \mathcal{D}\,.
\end{tikzcd} \tag{6.3}$$

Secondly, let us consider the pseudomonad $\tilde{S}\colon \mathrm{Kl}(T) \to \mathrm{Kl}(T)$ constructed in the proof of Theorem 6.3 and its associated Kleisli bicategory $\mathrm{Kl}(\tilde{S})$. Applying again Theorem 4.6, this time in the case of an ordinary pseudomonad, we have a pseudoadjunction

$$\begin{tikzcd}
\mathrm{Kl}(T) \arrow[r, shift left, "F^S"] \arrow[r, phantom, "\perp"] & \mathrm{Kl}(\tilde{S}) \arrow[l, shift left, "G^S"]
\end{tikzcd}. \tag{6.4}$$

By Proposition 4.9, we can then compose with the pseudoadjunctions in (6.3) and (6.4) so as to obtain a new relative pseudoadjunction

$$\begin{tikzcd}
 & \mathrm{Kl}(\tilde{S}) \arrow[d, "G^T G^S"] \\
\mathcal{C} \arrow[ur, "F^S F^T"] \arrow[r, "J"'] & \mathcal{D}\,.
\end{tikzcd}$$

By Theorem 4.4 we then obtain a relative pseudomonad over $J\colon \mathcal{C} \to \mathcal{D}$. Unfolding the definitions, one readily checks that the underlying function of this relative pseudomonad sends $X \in \mathcal{C}$ to $TS(X) \in \mathcal{D}$, as required. □

7. Applications

We now apply our results to obtain a homogeneous method for extending several 2-monads from the 2-category **Cat** of small categories and functors to the bicategory **Prof** of small categories and profunctors, encompassing all the examples of such 2-monads considered in the theory of variable binding [23, 56, 62], concurrency [14], species of structures [22], models of the differential λ-calculus [21], and operads [24].

The simplest examples liftings of the relative pseudomonad for presheaves are with respect to the 2-monads on **CAT** whose strict algebras are locally small categories equipped with suitable classes of limits, so that the 2-monads are co-lax, as discussed in [36]. The specific examples of 2-monads that we consider are those for categories with terminal object, categories with finite products, categories with finite limits. Each of these 2-monads is flexible in the sense of [9, 10] and restricts along the inclusion $J\colon \mathbf{Cat} \to \mathbf{CAT}$ to a 2-monad on the 2-category **Cat** of small categories, so as to determine a situation as in (6.1). We speak of small (or locally small) strict algebras to indicate small (or locally small) categories equipped with a strict algebra structure.

We make some preliminary observations about the pseudomorphisms in the cases under consideration. Since limits are determined up to a unique isomorphism, the pseudomorphisms are exactly the functors preserving the specified limits in the usual up to isomorphism sense: the coherence conditions for a pseudomorphism are automatic [36]. Similarly, one sees directly that any 2-cell between functors preserving the the relevant limits is an algebra 2-cell. In the terminology of [36], these 2-monads are fully property-like. It follows in particular that $S\text{-}\mathrm{Alg}_{\mathbf{CAT}}[\mathbb{A}, \mathbb{B}]$ is a full subcategory of $\mathbf{CAT}[\mathbb{A}, \mathbb{B}]$. All this is in fact an abstract consequence of the fact that the S in question are all co-lax monads. That fact is evident and the general theory appears in [36].

Theorem 7.1. *Let $S\colon \mathbf{CAT} \to \mathbf{CAT}$ be the 2-monad for categories with terminal object, or categories with finite products, or categories with finite limits. Then the relative pseudomonad of presheaves $P\colon \mathbf{Cat} \to \mathbf{CAT}$ has a lifting to strict S-algebras,*

$$\bar{P}\colon S\text{-}\mathrm{Alg}(\mathbf{Cat}) \to S\text{-}\mathrm{Alg}(\mathbf{CAT})\,.$$

Proof. Let us begin by observing that we have a choice of limits in **Set**, so for any $\mathbb{X} \in \mathbf{Cat}$, $P(\mathbb{X})$ has chosen limits defined pointwise. Thus, there is a canonical strict S-algebra structure on $P(\mathbb{X})$. Furthermore, the Yoneda embedding $\mathrm{y}_{\mathbb{X}}\colon \mathbb{X} \to P(\mathbb{X})$ preserves such limits. Hence, if $\mathbb{A} \in S\text{-}\mathrm{Alg}_{\mathbf{Cat}}$ is a small strict algebra, then $\mathrm{y}_{\mathbb{A}}\colon \mathbb{A} \to P(\mathbb{A})$ is a pseudomorphism of S-algebras in a canonical fashion. Composition with $\mathrm{y}_{\mathbb{A}}$ thus gives us a functor

$$S\text{-}\mathrm{Alg}_{\mathbf{CAT}}[\mathbb{A}, P(\mathbb{B})] \xleftarrow[(-)\,\mathrm{y}_{\mathbb{A}}]{} S\text{-}\mathrm{Alg}_{\mathbf{CAT}}[P(\mathbb{A}), P(\mathbb{B})]\,.$$

Now, suppose that $F\colon \mathbb{A} \to P(\mathbb{B})$ is a pseudomorphism, that is to say, F preserves the relevant limits. Then the left Kan extension $F^*\colon P(\mathbb{A}) \to P(\mathbb{B})$ also preserves these limits. This is critical, and for the separate classes of limits needs to be proved on a case by case basis. The case when S is the 2-monad for a terminal object is simple. If F preserves the terminal, then so does $F^*\,\mathrm{y}_{\mathbb{A}}$ (being naturally isomorphic to F). But the Yoneda $\mathrm{y}_{\mathbb{A}}\colon \mathbb{A} \to P(\mathbb{A})$ preserves the terminal object, and hence so does F^*. The case when S is the 2-monad for finite products needs a little more work. Suppose that F and hence $F^*\,\mathrm{y}_{\mathbb{A}}$ preserves finite products. As the Yoneda $\mathrm{y}_{\mathbb{A}}\colon \mathbb{A} \to P(\mathbb{A})$ preserves finite products, F^* preserves finite products of representables. But the objects of $P(\mathbb{A})$ are colimits of representables. Since F^* and products by objects (are left adjoints and so) preserve colimits, it follows that F^* preserves finite products. Finally, the case when S is the monad for finite limits is similar, though in this case the result is standard. If $\mathbb{A}$ has finite limits and $F\colon \mathbb{A} \to P(\mathbb{B})$ preserves finite limits and hence is flat [43, §VII.10, Corollary 3], i.e. F^* preserves finite limits.

Thus, in each case F^* is a pseudomorphism of strict S-algebras; and as we observed above, any 2-cell between pseudomorphisms will be an algebra 2-cell. Hence, the left Kan extension gives us a functor

$$S\text{-Alg}_{\mathbf{CAT}}[\mathbb{A}, P(\mathbb{B})] \xrightarrow{(-)^*} S\text{-Alg}_{\mathbf{CAT}}[P(\mathbb{A}), P(\mathbb{B})]\,.$$

Now we exploit the fact that the relative pseudomonad for presheaves is lax idempotent (see Example 5.6). So we have an adjunction

$$\begin{tikzcd} \mathbf{CAT}[\mathbb{A}, P(\mathbb{B})] \arrow[r, shift left=1.5, "(-)^*"] \arrow[r, phantom, "\perp"] & \mathbf{CAT}[P(\mathbb{A}), P(\mathbb{B})]\,. \arrow[l, shift left=1.5, "(-)\,\mathrm{y}_{\mathbb{A}}"] \end{tikzcd} \tag{7.1}$$

We observed that S-Alg$[\mathbb{A}, P(\mathbb{B})]$ and S-Alg$[P(\mathbb{A}), P(\mathbb{B})]$ are full subcategories of $\mathbf{CAT}[\mathbb{A}, P(\mathbb{B})]$ and $\mathbf{CAT}[P(\mathbb{A}), P(\mathbb{B})]$, and it is clear from the above discussion that this adjunction restricts to an adjunction

$$\begin{tikzcd} S\text{-Alg}_{\mathbf{CAT}}[\mathbb{A}, P(\mathbb{B})] \arrow[r, shift left=1.5, "(-)^*"] \arrow[r, phantom, "\perp"] & S\text{-Alg}_{\mathbf{CAT}}[P(\mathbb{A}), P(\mathbb{B})]\,. \arrow[l, shift left=1.5, "(-)\,\mathrm{y}_{\mathbb{A}}"] \end{tikzcd}$$

Thus, in view of Proposition 5.5, we have established the claim. ☐

Remark 7.2. With the experience of these examples of liftings, it is easy to give examples of 2-monads which do not lift.

- Consider the 2-monad for a category with zero object (i.e. object which is both terminal and initial). No presheaf category (over **Set**) has a zero object. So the 2-monad cannot lift. The same applies to the monad for direct sums or biproducts (in the terminology of [42]).
- Consider the 2-monad for a category with initial object. Given a category $\mathbb{A}$ with initial object, while the presheaf category $P(\mathbb{A})$ does indeed have an initial object, the Yoneda embedding does not preserve it. Hence the 2-monad cannot lift.
- Consider the 2-monad for a category with equalisers. Given a category $\mathbb{A}$ with equalisers, the presheaf category $P(\mathbb{A})$ also has equalisers, and the Yoneda embedding $\mathrm{y}_{\mathbb{A}} \colon \mathbb{A} \to P(\mathbb{A})$ preserves them. But now suppose that $\mathbb{A}$ has equalisers and that $F \colon \mathbb{A} \to \mathbf{Set}$ preserves them. It does not follow that $F^* \colon P(\mathbb{A}) \to \mathbf{Set}$ preserves equalisers. For a counterexample one can obviously just take $\mathbb{A}$ to be the fork i.e. the generic equaliser. Then for example take $F \colon \mathbb{A} \to \mathbf{Set}$ mapping the parallel pair to the identity and twist on 2 with equaliser 0. Because of this failure it follows that the 2-monad cannot lift.

Next, we consider 2-monads associated with various notions of monoidal category. To start with, we consider 2-monads which are flexible in the sense of [9, 10] and we have again a situation of the form as in (6.1).

Theorem 7.3. *Let $S \colon \mathbf{CAT} \to \mathbf{CAT}$ be the 2-monad for monoidal categories, or symmetric monoidal categories, or monoidal categories in which the unit is a terminal object, or symmetric monoidal categories in which the unit is a terminal object. The relative pseudomonad of presheaves $P \colon \mathbf{Cat} \to \mathbf{CAT}$ has a lifting to strict S-algebras,*

$$\bar{P} \colon S\text{-Alg}_{\mathbf{Cat}} \to S\text{-Alg}_{\mathbf{CAT}}\,.$$

Proof. The base case is that of a monoidal category. We discuss that case and derive the others. We use the analysis in [30] of the univeral property of Day's convolution tensor product [19]. We write **Mon** (respectively, **MON**) for the 2-category of small (respectively, locally small) monoidal categories, strong monoidal functors and monoidal natural transformations. For cocomplete categories $\mathbb{A}$, $\mathbb{B}$, a functor $F \colon \mathbb{A} \times \mathbb{B} \to \mathbb{C}$ is *separately cocontinuous* if for every

$a \in \mathbb{A}$, $b \in \mathbb{B}$ both $F(a,-)\colon \mathbb{B} \to \mathbb{C}$ and $F(-,b)\colon \mathbb{A} \to \mathbb{C}$ are cocontinous. We write $\mathbf{COC}[\mathbb{A},\mathbb{B};\mathbb{C}]$ for the category of such functors and natural transformations between them. A cocomplete category $\mathbb{A}$ equipped with a monoidal structure is *monoidally cocomplete* if the tensor product is separately cocontinuous. We then have a straightforward 2-category **MONCOC** of monoidally cocomplete locally small categories, strong monoidal functors, and monoidal transformations. Note that **MONCOC** is a sub-2-category of **COC**, but obviously not locally full.

In [19] Brian Day showed how for any small monoidal category $\mathbb{A}$, the category $P(\mathbb{A})$ of presheaves on $\mathbb{A}$ can be equipped with a canonical monoidal structure, often called the convolution tensor product, which makes $\mathbb{A}$ into a monoidally cocomplete category, defined by letting

$$(F_1 \hat{\otimes} F_2)(a) =_{\mathrm{def}} \int^{a_1,a_2 \in \mathbb{A}} F_1(a_1) \times F_2(a_2) \times \mathbb{A}[a, a_1 \otimes a_2]$$

for $F_1, F_2 \in P(\mathbb{A})$ and $a \in \mathbb{A}$. Furthermore, the Yoneda $\mathrm{y}_{\mathbb{A}}\colon \mathbb{A} \to P(\mathbb{A})$ has then the structure of a strong monoidal functor. We have the adjoint equivalence obtained in [30]

$$\begin{tikzcd} \mathbf{MON}[\mathbb{A},\mathbb{B}] \arrow[r, shift left, "(-)^*"] \arrow[r, phantom, "\perp"] & \mathbf{MONCOC}[P(\mathbb{A}),\mathbb{B}]\,, \arrow[l, shift left, "(-)\,\mathrm{y}_{\mathbb{A}}"] \end{tikzcd} \tag{7.2}$$

as required. In particular, for any pair of monoidal categories $\mathbb{A}$, $\mathbb{B}$ we have an adjoint equivalence

$$\begin{tikzcd} \mathbf{MON}[\mathbb{A},P(\mathbb{B})] \arrow[r, shift left, "(-)^*"] \arrow[r, phantom, "\perp"] & \mathbf{MONCOC}[P(\mathbb{A}),P(\mathbb{B})]\,. \arrow[l, shift left, "(-)\,\mathrm{y}_{\mathbb{A}}"] \end{tikzcd}$$

In our terminology, the adjoint equivalences in (7.2) amounts to saying that we have a relative pseudoadjunction

$$\begin{tikzcd} & \mathbf{MONCOC} \arrow[d] \\ \mathbf{Mon} \arrow[ur, "P"] \arrow[r] & \mathbf{MON}\,. \end{tikzcd}$$

This provides exactly a lifting of the relative pseudomonad $P\colon \mathbf{Cat} \to \mathbf{CAT}$ to a relative pseudomonad $\bar{P}\colon S\text{-Alg}_{\mathbf{Cat}} \to S\text{-Alg}_{\mathbf{CAT}}$. All these considerations extend to symmetric monoidal categories, again by the results in [19, 30]. For the 2-monads for monoidal categories with the condition that the unit is terminal, the lift follows from the above, observing that the unit of the convolution monoidal structure is the Yoneda embedding of the unit on the base category and so it remains a terminal object. $\square$

Our final group of examples of a lifting involve 2-monads on **CAT** which are not flexible. In this case, we have a lifting to pseudoalgebras, in the sense of Definition 6.2.

Theorem 7.4. *Let $S\colon \mathbf{Cat} \to \mathbf{Cat}$ be the 2-monad for either strict monoidal categories, or symmetric strict monoidal categories, or strict monoidal category in which the unit is terminal, or symmetric strict monoidal categories, or symmetric strict monoidal categories in which the unit is terminal. Then the relative pseudomonad $P\colon \mathbf{Cat} \to \mathbf{CAT}$ has a lifting to pseudo-S-algebras,*

$$\bar{P}\colon S\text{-Alg}_{\mathbf{Cat}} \to \text{Ps-}S\text{-Alg}_{\mathbf{CAT}}\,.$$

Proof. There is a direct and an indirect approach to this. Directly, one follows through the arguments of the previous section making the necessary adjustments. Indirectly, observe that in each case the 2-monad S' is the monad whose strict algebras are categories with unbiased structure as in the list in Theorem 7.3. Now S-Alg is a full sub-2-category of $S'\text{-Alg} \cong \text{Ps-}S\text{-Alg}$.

So the lifting of the relative pseudomonad of presheaves $P\colon \mathbf{Cat} \to \mathbf{CAT}$ to $\bar{P}\colon S'\text{-Alg}(\mathbf{Cat}) \to S'\text{-Alg}(\mathbf{CAT})$ restricts to $S\text{-Alg}(\mathbf{Cat}) \to S'\text{-Alg}(\mathbf{Cat})$. □

Corollary 7.5. *All the 2-monads on* **Cat** *listed in Theorems 7.1, 7.3 and 7.4 admit an extension to pseudomonads on* **Prof**.

Proof. Immediate consequence of Theorem 6.3 and Theorems 7.1, 7.3 and 7.4. □

For each of the monads $S\colon \mathbf{Cat} \to \mathbf{Cat}$ above, one can consider the Kleisli bicategory associated to the pseudomonad $\tilde{S}\colon \mathbf{Prof} \to \mathbf{Prof}$ determined by Corollary 7.5. The composition functors of these Kleisli bicategories can be understood as generalizations of various kinds of substitution monoidal structures [35, 58], among those giving rise to the notions of a many-sorted Lawvere theory and of a coloured operad. We conclude the paper by illustrating this idea in the case of coloured operads.

Example 7.6. As an illustration of the theory developed here, we revisit the construction of the bicategory of generalized species of structures of [22] and relate more precisely its composition with the substitution monoidal structure for coloured operads [4]. For this, let us begin by recalling the definition of the 2-monad $S\colon \mathbf{Cat} \to \mathbf{Cat}$ for symmetric strict monoidal categories. Let $\mathbb{X} \in \mathbf{Cat}$. For $n \in \mathbb{N}$, define $S_n(\mathbb{X})$ to be the category having as objects n-tuples $\bar{x} = (x_1, \ldots, x_n)$ of objects $x_i \in \mathbb{X}$ and as morphisms $(\sigma, \bar{f})\colon \bar{x} \to \bar{x}'$ pairs consisting of a permutation $\sigma \in \Sigma_n$ and an n-tuple of morphisms $f_i\colon x_i \to x'_{\sigma(i)}$. We then let

$$S(\mathbb{X}) =_{\mathrm{def}} \bigsqcup_{n \in \mathbb{N}} S_n(\mathbb{X})\,.$$

The category $S(\mathbb{X})$ is equipped with a strict symmetric monoidal structure: the tensor product, written $\bar{x} \oplus \bar{x}'$, is given by concatenation of sequences, and the unit, written u, is given by the empty sequence; and the symmetry is given by a permutation of identity maps. This definition can be extended easily to obtain a 2-functor $S\colon \mathbf{Cat} \to \mathbf{Cat}$. The multiplication of the monad is given by taking a sequence of sequences and forgetting the bracketing, while the unit has components $e_{\mathbb{X}}\colon \mathbb{X} \to S(\mathbb{X})$ mapping $x \in \mathbb{X}$ to the singleton sequence $(x) \in S(\mathbb{X})$. By the theory developed above, and Corollary 7.5 in particular, we obtain a pseudomonad

$$\tilde{S}\colon \mathbf{Prof} \to \mathbf{Prof}\,. \tag{7.3}$$

For our purposes, it is convenient to describe explicitly the relative pseudomonad associated to $\tilde{S}$. Its action on objects is the function mapping $\mathbb{X} \in \mathbf{Cat}$ to $S(\mathbb{X}) \in \mathbf{Cat}$. The component of the unit for $\mathbb{X} \in \mathbf{Cat}$ is the profunctor $\tilde{e}_{\mathbb{X}}\colon \mathbb{X} \to S(\mathbb{X})$ corresponding to the functor

$$\mathbb{X} \xrightarrow{e_{\mathbb{X}}} S(\mathbb{X}) \xrightarrow{\mathrm{y}_{S(\mathbb{X})}} PS(\mathbb{X})\,,$$

The extension functors of the relative pseudomonad have the form

$$(-)^\sharp\colon \mathbf{Prof}[\mathbb{X}, S(\mathbb{Y})] \to \mathbf{Prof}[S(\mathbb{X}), S(\mathbb{Y})]\,,$$

where $\mathbb{X}, \mathbb{Y}$ are small categories. For a functor $F\colon \mathbb{X} \to PS(\mathbb{Y})$, we can define the functor $F^\sharp\colon S(\mathbb{X}) \to PS(\mathbb{Y})$ recalling that, since $S(\mathbb{Y})$ has a symmetric strict monoidal structure, $PS(\mathbb{Y})$ has a unbiased (in the sense of [47]) symmetric monoidal structure. Hence, by the universal property of $S(\mathbb{X})$, we have an essentially unique $F^\sharp$ fitting into a diagram of the form

$$\begin{tikzcd} \mathbb{X} \arrow[r, "e_{\mathbb{X}}"] \arrow[dr, bend right, "F"'] & S(\mathbb{X}) \arrow[d, "F^\sharp"] \\ & PS(\mathbb{Y})\,. \end{tikzcd}$$

For brevity, we omit the description of the invertible natural transformations

$$\eta_F \colon F \Rightarrow F \circ \tilde{e}_{\mathbb{X}}\,, \qquad \mu_{F,G} \colon (G^\sharp \circ F)^\sharp \Rightarrow G^\sharp \circ F^\sharp\,, \qquad \kappa_{\mathbb{X}} \colon (\tilde{e}_{\mathbb{X}})^\sharp \Rightarrow \mathrm{Id}_{\mathbb{X}}\,.$$

The Kleisli bicategory of $\tilde{S}$ is the bicategory S-**Prof** of S-profunctors defined in [24], which has small categories as objects and hom-categories defined by

$$S\text{-}\mathbf{Prof}[\mathbb{X}, \mathbb{Y}] = \mathbf{Prof}[\mathbb{X}, S(\mathbb{Y})] = \mathbf{CAT}[S(\mathbb{Y})^{\mathrm{op}} \times \mathbb{X}, \mathbf{Set}]\,.$$

Indeed, one can readily check that composition and identity morphisms of S-**Prof**, as defined in [24], coincide with those given by instantiating the general definition of a Kleisli bicategory. Following [24], we write **CatSym** for the bicategory of categorical symmetric sequences, which is defined as the opposite of S-**Prof**. Thus, the objects of **CatSym** are small categories and its hom-categories are given by

$$\mathbf{CatSym}[\mathbb{X}, \mathbb{Y}] =_{\mathrm{def}} S\text{-}\mathbf{Prof}[\mathbb{Y}, \mathbb{X}] = \mathbf{CAT}[S(\mathbb{X})^{\mathrm{op}} \times \mathbb{Y}, \mathbf{Set}]\,.$$

Given categorical symmetric sequences $F \colon \mathbb{X} \to \mathbb{Y}$ and $G \colon \mathbb{Y} \to \mathbb{Z}$, i.e. functors $F \colon S(\mathbb{X})^{\mathrm{op}} \times \mathbb{Y} \to \mathbf{Set}$ and $G \colon S(\mathbb{Y})^{\mathrm{op}} \times \mathbb{Z} \to \mathbf{Set}$, their composite $G \circ F \colon \mathbb{X} \to \mathbb{Z}$ in **CatSym** is given by considering F and G as S-profunctors in the opposite direction, $G \colon \mathbb{Z} \to \mathbb{Y}$, $F \colon \mathbb{Y} \to \mathbb{X}$, taking their composition in S-**Prof** using the definition of composition in a Kleisli bicategory, and then regarding the result as a categorical symmetric sequence from $\mathbb{X}$ to $\mathbb{Z}$. Explicitly, we obtain that

$$\begin{aligned}(G \circ F)(\bar{x}; z) =_{\mathrm{def}} \bigsqcup_{m \in \mathbb{N}} \int^{(y_1, \ldots, y_m) \in S_m(\mathbb{Y})} & G[y_1, \ldots, y_m; z] \times \\ \int^{\bar{x}_1 \in S(\mathbb{X})} \cdots \int^{\bar{x}_m \in S(\mathbb{X})} & F[\bar{x}_1; y_1] \times \ldots \times F[x_m; y_m] \times S(\mathbb{X})[\bar{x}, \bar{x}_1 \oplus \ldots \oplus \bar{x}_m]\end{aligned} \tag{7.4}$$

Remarkaly, this formula yields the definition of the substitution monoidal structure for coloured operads given in [3] by considering the special case where $\mathbb{X}$ and $\mathbb{Y}$ are discrete and coincide with a fixed set C, the set of colours of the coloured operads being considered.

The bicategory **CatSym** can also be seen to be equivalent to the bicategory of generalized species of structures **Esp** defined in [22]. To see this, let us recall the definition of **Esp**. For this, observe that the duality pseudofunctor $(-)^\perp \colon \mathbf{Prof} \to \mathbf{Prof}$ defined by $\mathbb{X}^\perp =_{\mathrm{def}} \mathbb{X}^{\mathrm{op}}$ allows us to turn this pseudomonad in (7.3) into a pseudocomonad. The bicategory **Esp** is then defined as the coKleisli bicategory of this pseudocomonad. More explicitly, its objects are small categories and its hom-categories are given by

$$\mathbf{Esp}[\mathbb{X}, \mathbb{Y}] = \mathbf{Prof}[S(\mathbb{X}), \mathbb{Y}] = \mathbf{CAT}[\mathbb{Y}^{\mathrm{op}} \times S(\mathbb{X}), \mathbf{Set}]\,.$$

The bicategory **Esp** is then equivalent to **CatSym**, via the pseudofunctor that sends $\mathbb{X}$ to $\mathbb{X}^{\mathrm{op}}$. Indeed,

$$\mathbf{CatSym}[\mathbb{X}, \mathbb{Y}] = \mathbf{CAT}[S(\mathbb{X}^{\mathrm{op}}) \times \mathbb{Y}, \mathbf{Set}] \cong [\mathbb{Y} \times S(\mathbb{X})^{\mathrm{op}}, \mathbf{Set}] = \mathbf{Esp}[\mathbb{X}^{\mathrm{op}}, \mathbb{Y}^{\mathrm{op}}]\,.$$

Furthermore, the definition of the composition of generalized species of structures defined via co-Kleisli composition (see [22] for details) can easily be seen to correspond to the composition operation of categorical symmetric sequences in (7.4).

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Computer Laboratory, University of Cambridge
E-mail address: Marcelo.Fiore@cl.cam.ac.uk

School of Mathematics, University of Leeds
E-mail address: n.gambino@leeds.ac.uk

DPMMS, University of Cambridge
E-mail address: m.hyland@dpmms.cam.ac.uk

Computer Laboratory, University of Cambridge
E-mail address: Glynn.Winskel@cl.cam.ac.uk