On the expressivity of symmetry in event structures*

Sam Staton and Glynn Winskel
Computer Laboratory, University of Cambridge

***Abstract*—This paper establishes a bridge between presheaf models for concurrency and the more operationally-informative world of event structures. It concentrates on a particular presheaf category, consisting of presheaves over finite partial orders of events; such presheaves form a model of nondeterministic processes in which the computation paths have the shape of partial orders. It is shown how with the introduction of symmetry event structures represent *all* presheaves over finite partial orders. This is in contrast with plain event structures which only represent certain *separated* presheaves. Specifically a coreflection from the category of presheaves to the category of event structures with symmetry is exhibited. It is shown how the coreflection can be cut down to an equivalence between the presheaf category and the subcategory of *graded* event structures with symmetry. Event structures with *strong* symmetries are shown to represent precisely all the separated presheaves. The broader context and specific applications to the unfolding of higher-dimensional automata and Petri nets, and weak bisimulation on event structures are sketched.**

I. Background

There is a renewed interest in causal models such as event structures and Petri nets across a variety of areas: distributed computation; security; systems biology; model checking; semantics of types, proofs and processes.

There is a call for a mathematically flexible theory to support semantics in terms of causal models. Here the rather concrete nature of causal models is a mixed blessing. On the one hand, their concreteness brings them close to an operational realisation. On the other, it can hamper their mathematical versatility. Constructions that cry out to be universal constructions in a category often are not because the associated diagrams do not strictly commute. The situation is ameliorated to a surprising extent through the introduction of a formal theory of symmetry, so that constructions become universal but 'up to symmetry'.

Behavioural symmetry is an important and natural feature within many processes. Often we wish to consider two computations of a process as essentially similar, for example but for the contingencies of naming. At the same time the introduction of a formal treatment of symmetry to causal models has been shown to boost their expressivity considerably, both in terms of the maps and universal constructions that causal models can support, and the forms of processes and types that they can provide semantics for [12], [13], [14].

* Dedicated to the memory of Robin Milner.

This paper provides technical results establishing a bridge between the mathematically rich but sometimes operationally colourless world of presheaf models for concurrency and the more operationally-informative world of causal models.

It concentrates on a particular presheaf category, consisting of presheaves over finite partial orders of events. Such presheaves form a model of nondeterministic processes in which the computation paths have the shape of partial orders, and, for instance, the category of event structures embeds fully and faithfully in the presheaf category. It is shown how, with the introduction of symmetry, event structures represent *all* presheaves over finite partial orders. This is in stark contrast with plain event structures which only represent certain *separated* presheaves. Specifically a coreflection from the category of presheaves to the category of event structures with symmetry is exhibited. It is shown how the coreflection can be cut down to an equivalence between the presheaf category and the subcategory of *graded* event structures with symmetry. Event structures with *strong* symmetries are shown to represent precisely all the separated presheaves.

Two new conditions on event structures play an important technical role. The first is that of *consistent-countability*, a relaxation of the restriction of countability so that countability need only apply to the sets of events that can occur in a possible history. Several universal constructions 'up to symmetry' depend crucially on this condition, as does the reflection from separated presheaves to event structures with strong symmetries described here. The second condition is the assumption of a *grading* on an event structure, which emerged in characterising those event structures obtained from presheaves in the coreflection mentioned above. Intuitively a grading describes a well-founded generation process to build up an event structure. Not surprisingly we have found grading a powerful tool in defining operations on event structures, and speculate that it can have applications in other areas such as probabilistic event structures, where showing the existence of extra local structure to support probability distributions ('branching cells' [1] and 'finitary tests' [10]) is sometimes problematic. Graded event structures are automatically consistent-countable and, curiously, can always be described in terms of a binary consistency (or conflict) relation.

The particular work described here has potential applications to the unfolding of higher-dimensional automata and

Petri nets, and weak bisimulation on event structures. We are especially keen to explore the latter in future work. But we see the main importance of this work as its being part of a broader programme [14] to develop an intensional semantics of computation, where causal models appear in describing the intensional manner of computation. (Another strand of this programme is to explain and extend game semantics using spans of event structures with symmetry.) A presheaf captures the different ways of computation through its different elements. Those ways are often described concretely by configurations of an event structure. However, as the work here shows, traditional event structures can be overly discriminating. As measured against a presheaf semantics, the ways of computation are more accurately expressed as configurations equivalent up to symmetry.

II. Event structures and presheaves

We briefly introduce and motivate the two main models we shall relate.

A. Event structures

Event structures represent a process, or system, as a set of event occurrences with relations to express how events causally depend on others, or exclude other events from occurring. In one of their simpler forms they consist of a set of events on which there is a consistency relation, expressing when events can occur together in a history, and a partial order of causal dependency—writing $e' \leq e$ if the occurrence of e depends on the previous occurrence of e'. Formally, an *event structure* comprises $(E, \mathrm{Con}, \leq)$, consisting of a set E of *events* which are partially ordered by $\leq$, the *causal dependency relation*, and a *consistency relation* Con consisting of set of finite subsets of E, which satisfy

$$\{e' \mid e' \leq e\} \text{ is finite for all } e \in E,$$
$$\{e\} \in \mathrm{Con} \text{ for all } e \in E,$$
$$\text{if } y \subseteq x \in \mathrm{Con} \text{ then } y \in \mathrm{Con},$$
$$\text{if } x \in \mathrm{Con} \text{ and } e \leq e' \in x \text{ then } x \cup \{e\} \in \mathrm{Con}.$$

The *configurations*, $\mathcal{C}(E)$, of an event structure E consist of those subsets $x \subseteq E$ which are

- *Consistent:* $\forall X \subseteq x.\ X$ is finite $\implies X \in \mathrm{Con}$, and
- *Down-closed:* $\forall e, e'.\ e' \leq e \in x \implies e' \in x$.

We write $\mathcal{C}^\circ(E)$ for its *finite* configurations.

The configurations of an event structure are ordered by inclusion, where $x \subseteq x'$, i.e. x is a sub-configuration of x', means that x is a sub-history of x'. Note that an individual configuration inherits an order of causal dependency on its events from the event structure, so that the history of a process is captured through a partial order of events. For an event e the set $[e] = \{e' \in E \mid e' \leq e\}$ is a configuration describing the whole causal history of the event e.

As affirmation of its intended meaning, notice that a finite subset of events is in the consistency relation iff it occurs as a subset of a (finite) configuration. Similarly, two events are in the causal dependency relation, $e' \leq e$, iff any (finite) configuration containing e also contains e'.

Prime algebraic domains. It is well-known that the partial order of configurations of an event structure ordered by inclusion forms a *finitary prime algebraic domain* (a Scott domain in which the complete primes form a subbasis) satisfying the further property that every finite element dominates only finitely many elements; moreover, precisely all such domains can be obtained up to isomorphism from event structures [7], [15]. A finitary prime algebraic domain determines an event structure: its events are the complete primes w.r.t. the domain order, causal dependency is got as a restriction of the domain order and consistency is given by compatibility. Sometimes we will find it useful to describe an event structure via a prime algebraic domain which determines it.

Rigid maps. A rigid map of event structures $(E, \mathrm{Con}, \leq) \to (E', \mathrm{Con}', \leq')$ is a function on events $f \colon E \to E'$ that is

- *Configuration-preserving:* if $x \in \mathcal{C}^\circ(E)$ then $f(x) \in \mathcal{C}^\circ(E')$;
- *Locally injective:* for $x \in \mathcal{C}^\circ(E)$, if $e_1, e_2 \in x$ and $f(e_1) = f(e_2) \in E'$, then $e_1 = e_2$;
- *Monotone:* if $e_1 \leq e_2$ then $f(e_1) \leq' f(e_2)$.

Such a rigid map can alternatively be described as being a function on events $f \colon E \to E'$ which is configuration-preserving and locally injective and moreover satisfies

$$\forall x \in \mathcal{C}^\circ(E).\ \forall y' \in \mathcal{C}^\circ(E').\ y' \subseteq fx \implies \exists x' \in \mathcal{C}^\circ(E).\ x' \subseteq x \ \&\ fx' = y\,.$$

The configuration x' is necessarily unique by the local injectivity of f.

Rigid maps are the primary class of map that we will consider in this paper. They are emerging as *the* fundamental maps of event structures (in the sense that other maps can be defined from them as Kleisli maps w.r.t. monads up to symmetry). We write $\mathcal{E}$ for the category of event structures and rigid maps.

In defining symmetries in event structures we use open maps [6]. Open maps are a generalisation of functional bisimulations, known from transition systems. They are specified as those maps in a category which satisfy a path-lifting property w.r.t. a chosen subcategory of paths (or path shapes). Here we take the subcategory of paths $\mathbf{P}$ to be the full subcategory of finite elementary event structures, i.e., finite event structures in which the set of all events is consistent. Thus paths have the form of finite, partially ordered sets on which we inherit rigid maps. Open maps

$f : E \to E'$ satisfy a *path-lifting property*:

$$\begin{tikzcd} P \arrow[r] \arrow[d] & E \arrow[d, "f"] \\ Q \arrow[r] \arrow[ur, dashed] & E' \end{tikzcd}$$

Any commuting square with $P, Q \in \mathbf{P}$ factors into two commuting triangles as shown. Bisimulation is then expressed as a span of open maps.

Open maps of event structures can be characterised as rigid maps $f \colon E \to E'$ for which

$$\begin{aligned} &\forall x \in \mathcal{C}^\circ(E).\ \forall y' \in \mathcal{C}^\circ(E').\ f(x) \subseteq y' \\ &\qquad \implies \exists x' \in \mathcal{C}^\circ(E).\ x \subseteq x'\ \&\ f(x') = y' \,. \end{aligned}$$

B. Presheaves

The category $\mathbf{P}$ comprises paths, or path shapes, in the form of finite partial orders of events; its maps describe how one path can be extended to another. The presheaf category $[\mathbf{P}^{\mathrm{op}}, \mathbf{Set}]$ is the free colimit completion of $\mathbf{P}$. Correspondingly, a presheaf, that is to say a functor $A \colon \mathbf{P}^{\mathrm{op}} \to \mathbf{Set}$, describes a gluing together of a collection of partial-order paths, with the set $A(P)$ describing the contributions of paths of shape P. We can read a presheaf A as describing a nondeterministic process in which $A(P)$ describes the set of all computation paths of shape P that the process can perform.

It is reasonable to assume that the process has exactly one state associated with the empty path. For this reason, we will be primarily interested in those presheaves that are rooted, in the sense that $A(\emptyset)$ has exactly one element. The category of rooted presheaves over $\mathbf{P}$ is equivalent to the category of presheaves $[\mathbf{P}_+^{\mathrm{op}}, \mathbf{Set}]$, where $\mathbf{P}_+$ is the category of non-empty paths.

An event structure is another way to describe a nondeterministic process having computation paths with shapes in $\mathbf{P}$. As this would lead one to expect, there is an embedding of the category of event structures $\mathcal{E}$ in the presheaf category $[\mathbf{P}_+^{\mathrm{op}}, \mathbf{Set}]$.

C. Event structures as presheaves

The inclusion $I : \mathbf{P}_+ \hookrightarrow \mathcal{E}$ induces a functor,

$$\begin{aligned} \mathit{Nerve} : \mathcal{E} &\to [\mathbf{P}_+^{\mathrm{op}}, \mathbf{Set}] \\ E &\mapsto \mathcal{E}(I(-), E)\,, \end{aligned}$$

which is full and faithful into presheaves over $\mathbf{P}_+$.

But not all presheaves are represented by event structures. Those presheaves A that are representable can be characterised as *separated* (see Section VI for a definition) and satisfying a *mono* condition saying that all maps from representables to A are mono [11]. Event structures give an operational reading of the presheaves they represent. Viewed in this light those presheaves without an event-structure representation lack an operational explanation.

This lack of expressivity of traditional event structures means that essential constructions cannot always be carried out within event structures, or for that matter within other causal models. There are two immediate examples, involving constructions on higher-dimensional automata and for weak bisimulation on causal models. The unfolding of higher-dimensional automata is naturally described as a presheaf over $\mathbf{P}_+$, in general not representable by an event structure. Milner's definition of weak bisimulation is based on a construction on transition systems saturating a transition system with all invisible τ-transitions it can support; weak bisimulation between transition systems is then defined as strong bisimulation between their saturations. Generalisations of this saturation construction to event structures do not in general yield event structures, but rather presheaves over $\mathbf{P}_+$ outside those representable by event structures. Both these anomalous constructions are described in more detail later, in Section VII.

What is missing from event structures to provide the necessary increase in expressivity? As we will see, an answer is supplied through a formal treatment of symmetry in event structures.

III. Event structures with symmetry

Informally a behavioural symmetry in a process should express when one computation path in the process is similar to another. In formalising this idea there is a general scheme we can follow based on relations in categories. In a symmetry, similar paths should have similar pasts and futures, and this will lead us to base the relations on open maps. In short, a symmetry in a process will be expressed as a bisimulation equivalence.

Relations in categories. Event structures with symmetry are instances of the following general concepts. Let X be an object of a category $\mathcal{A}$. Recall that a relation on X is an object R of $\mathcal{A}$ together with two morphisms $R \rightrightarrows X$—so forming a span—that are jointly monic. Morphisms between relations $R \rightrightarrows X$ and $S \rightrightarrows Y$ are pairs of morphisms $R \to S$ and $X \to Y$ which commute with the projections—by joint monicity of $S \rightrightarrows Y$ the morphism $R \to S$ has to be unique. If $\mathcal{A}$ has products, then a relation can be equivalently given by a monomorphism $R \rightarrowtail X \times X$. If $\mathcal{A}$ has pullbacks, we can formulate diagrammatically the requirement that R be an equivalence relation (e.g. [5, A1.3.6]).

If $\mathcal{A}$ is equipped with a class of open maps, then we say that a relation S on X is a *symmetry* in X if it is an equivalence relation and the projections $R \rightrightarrows X$ are both open; this amounts to $R \rightrightarrows X$ being a bisimulation equivalence.

The transition from event structures to event structures with symmetry resembles the exact-completion, which has been used to construct the effective topos from a category of assemblies [3], [8].

Event structures with symmetry. Rather than continuing the development at the abstract level, we can take advantage of the particular model of event structures to give explicit descriptions of equivalence relations and symmetries there.

We begin by noting that the category of event structures and rigid maps has binary products. We build the product $E \times E'$ of two event structures E and E' out of a prime algebraic domain, defined as follows. The underlying set of the domain consists of all order isomorphisms

$$\varphi : x \cong x'$$

between configurations x of E and x' of E'. The order on configurations is that inherited from their ambient event structures. We can equip the pairs of events in the isomorphism φ with the order induced by that on the configurations—the isomorphism ensures x and x' agree—and thus regard φ as an elementary event structure. We order two isomorphisms by letting $\varphi \sqsubseteq \varphi'$ if $\varphi \subseteq \varphi'$ and the inclusion induces a rigid map. The resulting partial order $\sqsubseteq$ on isomorphisms is a finitary prime algebraic domain; its finite elements are precisely the finite isomorphisms φ and its complete primes are the isomorphisms with a top element. As described earlier, the prime algebraic domain gives rise to an event structure with the complete primes as events. This is our definition of the product $E \times E'$.

An equivalence relation on an event structure $(E, \mathrm{Con}, \leq)$ is a family $\mathbb{R}$ of isomorphisms $\theta \colon x \cong y$ between configurations in $\mathcal{C}^\circ(E)$, such that

- **(R)** for $\theta \colon x \cong y$ in $\mathbb{R}$, whenever $x' \subseteq x$ with $x' \in \mathcal{C}^\circ(E)$, then there is a (necessarily unique) $y' \in \mathcal{C}^\circ(E)$ with $y' \subseteq y$ such that the restriction of θ to $\theta' \colon x' \cong y'$ is in $\mathbb{R}$.
- **(E)** the identities $\mathrm{id}_x \colon x \cong x$ are in $\mathbb{R}$ for all $x \in \mathcal{C}^\circ(E)$; if $\theta \colon x \cong y$ is in $\mathbb{R}$ then so is the inverse $\theta^{-1} \colon y \cong x$; and if $\theta \colon x \cong y$ and $\psi \colon y \cong z$ are in $\mathbb{R}$, then so is their composition $\psi \circ \theta \colon x \cong z$.

An event structure with symmetry is an event structure with an equivalence relation $\mathbb{S}$ that satisfies the following axiom:

- **(O)** for $\theta \colon x \cong y$ in $\mathbb{S}$, whenever $x \subseteq x'$ for $x' \in \mathcal{C}^\circ(E)$, then there is an extension of θ to $\theta' \colon x' \cong y'$ for some (not necessarily unique) $y' \in \mathcal{C}^\circ(E)$ with $y \subseteq y'$.

We will also be interested in those relations $R \rightarrowtail (E \times E)$ that are strong monomorphisms. Concretely, an equivalence relation $\mathbb{R}$ on an event structure is strong if it satisfies the following axiom:

- **(S)** an isomorphism $\theta \colon x \cong y$ between configurations is in $\mathbb{R}$ whenever, for every event $e \in x$, the restricted isomorphism $\theta|_{[e]} \colon [e] \cong [\theta(e)]$ is in $\mathbb{R}$.

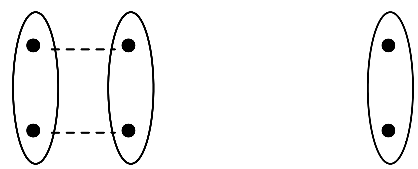
Any ordinary event structure can be considered to have the (strong) symmetry that only contains the identity isomorphisms.

A map between event structures with equivalence relations, $f \colon (E, \mathbb{R}) \to (E', \mathbb{R}')$, is a map of event structures, $f \colon E \to E'$, such that moreover whenever an isomorphism $\theta \colon x \cong y$ is in $\mathbb{R}$, then $f\theta \colon f\,x \cong f\,y$ is in $\mathbb{R}'$ (where $f\theta = \{(f(e), f(e')) \mid (e, e') \in \theta\}$).

We write $\mathcal{SE}$ for the category of event structures with symmetry and maps between them, and we write $\mathcal{SSE}$ for the full subcategory where the symmetry relations are required to be strong. We write $\mathcal{RE}$ for the category of event structures with equivalence relations, and maps between them.

Symmetries induce a notion of equivalence on morphisms, and in fact the categories $\mathcal{SE}$ and $\mathcal{SSE}$ are enriched in the category of sets with equivalence relations. Two maps $f, g \colon (E, \mathbb{R}) \to (E', \mathbb{R}')$ between event structures with equivalence relations are equivalent, written $f \sim g$, if for all $x \in \mathcal{C}^\circ(E)$, the bijection $\{(f(e), g(e)) \mid e \in x\}$ is in $\mathbb{R}'$. We regard two event structures as equivalent, writing $E \simeq E'$, if there are $f : E \to E'$ and $g : E' \to E$ such that $f \circ g \sim \mathrm{id}_{E'}$ and $g \circ f \sim \mathrm{id}_E$.

Example 1. The following illustration shows two event structures that are equivalent but not isomorphic. The dots indicate events, the ovals indicate the largest configurations. The event structures have discrete causal dependency. The left-hand event structure is considered with a strong symmetry relation, indicated by the dotted lines.



Remark. Any category enriched in the category of sets with equivalence relations can be understood as an ordinary category by quotienting the hom-sets. In some circumstances it is instructive to know whether diagrams commute on the nose or only up to symmetry. We will be quite casual about this in this extended abstract, and extend standard categorical terminology such as 'adjunction' and 'reflection' to categories enriched in equivalence relations when the terminology applies to their quotients.

Consistent-countable event structures. An event structure $(E, \mathrm{Con}, \leq)$ is *consistent-countable* if there exists a function $f \colon E \to \omega$ into the natural numbers that is locally injective, i.e., if $\{e_1, e_2\} \in \mathrm{Con}$ and $f(e_1) = f(e_2)$ then $e_1 = e_2$. We write $\mathcal{E}_\omega$ for the category of consistent-countable event structures, and $\mathcal{SE}_\omega$ for the category of consistent-countable event structures with symmetry.

Proposition 2. *Let E and E' be event structures, and suppose that E is consistent-countable. Then the product event structure $E \times E'$ is also consistent-countable.*

Proof. There is a function $f \colon E \to \omega$ that is locally injective, and the projection $E \times E' \to E$ is also locally injective; hence the composite $E \times E' \to E \to \omega$ is locally injective. ■

The category of consistent countable event structures with symmetry has a terminal object, $(\mathrm{T}, \leq_{\mathrm{T}}, \mathrm{Con}_{\mathrm{T}}, \mathbb{S}_{\mathrm{T}})$, up to symmetry. As earlier with the product, we can build the event structure $(\mathrm{T}, \leq_{\mathrm{T}}, \mathrm{Con}_{\mathrm{T}})$ out of the complete primes of a prime algebraic domain. The underlying set of the domain consists of all partial orders P on subsets of the natural numbers. We order two such partial orders by letting $P \sqsubseteq P'$ if $P \subseteq P'$ and the inclusion induces a rigid map. The resulting partial order $\sqsubseteq$ over these partial orders is a finitary prime algebraic domain with finite elements the finite partial orders and complete primes the partial orders with a top element. The prime algebraic domain determines an event structure with the complete primes as events. This we take as our definition of the event structure T. The configurations of T have the form

$$x_P = \{[e]_P \mid e \in P\}$$

where $(P, \leq_P)$ is partial order on a subset of natural numbers and $[e]_P = \{e' \in P \mid e' \leq_P e\}$. Its symmetry comprises all isomorphisms $\bar{\psi} : x_P \cong x_Q$ where

$$\bar{\psi} =_{\mathrm{def}} \{([e]_P, [e']_Q) \mid (e, e') \in \psi\}$$

where $\psi : P \cong Q$ is an order isomorphism between finite partial orders on subsets of natural numbers.

There are generally many maps from an event structure to T, and indeed T has many endomorphisms. However, these maps are all $\sim$-equivalent, which is why we say that T is terminal 'up to symmetry'.

The proof of the theorem below uses the construction of product with the terminal object to cut down any event structure, regardless of size, to a consistent-countable one.

Theorem 3. *The category $\mathcal{SE}_\omega$ of consistent-countable event structures with symmetry is a coreflective subcategory (up to symmetry) of the category $\mathcal{SE}$ of all event structures with symmetry.*

Terminology. When we say that $\mathcal{SE}_\omega$ is a coreflective subcategory of $\mathcal{SE}$ up to symmetry, we mean that there is a functor $R\colon \mathcal{SE} \to \mathcal{SE}_\omega$ and a bijection of quotiented hom-sets:

$$\mathcal{SE}(E, E')/_\sim \;\cong\; \mathcal{SE}_\omega(E, R(E'))/_\sim$$

natural in $E \in \mathcal{SE}_\omega$, $E' \in \mathcal{SE}$.

Proof of Theorem 3. The right adjoint to $\mathcal{SE}_\omega \hookrightarrow \mathcal{SE}$ maps an event structure with symmetry E in $\mathcal{SE}$ to the product $E \times \mathrm{T}$ with the terminal object of $\mathcal{SE}_\omega$. ∎

Example 4. The category $\mathcal{E}_\omega$ of consistent-countable event structures is *not* a coreflective subcategory of the category $\mathcal{E}$ of all event structures. A simple example of an event structure that is not consistent-countable is the first uncountable ordinal ω_1, considered as an event structure with discrete partial order and with all finite subsets consistent. We can consider any ordinal as an event structure in a similar way. Notice that ω_1 is a colimit in $\mathcal{E}$ of the chain of all countable ordinals considered as event structures and ordered by inclusion. Suppose that there is a coreflection $R\colon \mathcal{E} \to \mathcal{E}_\omega$, and we will derive a contradiction. By the universality of the counit of this coreflection, $\varepsilon_{\omega_1}\colon R(\omega_1) \to \omega_1$, the consistent-countable event structure $R(\omega_1)$ forms a co-cone over the chain of all countable ordinals. But ω_1 is colimiting for this cone, and so there is a map of event structures $\omega_1 \to R(\omega_1)$. We conclude that the composite $(\omega_1 \to R(\omega_1) \to \omega)$ must be an injection. This is absurd, and so there can be no such coreflection.

IV. A COREFLECTION

For technical reasons, we will now take the category of paths $\mathbf{P}_+$ be a *small* subcategory that is equivalent to the full subcategory of event structures $\mathcal{E}$ comprising finite non-empty elementary event structures. (This does not disturb any of the previous development.)

Consider the category $[\mathbf{P}_+^{\mathrm{op}}, \mathbf{Set}]$ of presheaves over $\mathbf{P}_+$ and natural transformations between them. We define the *nerve* of an event structure with an equivalence relation $(E, \mathbb{R})$ to be the presheaf in $[\mathbf{P}_+^{\mathrm{op}}, \mathbf{Set}]$ given by $P \mapsto \mathcal{RE}(P, (E, \mathbb{R}))/_\sim$. The nerve construction extends naturally to a functor $\mathit{Nerve}\colon \mathcal{RE} \to [\mathbf{P}_+^{\mathrm{op}}, \mathbf{Set}]$.

The central result of this section is the following:

Theorem 5. *The nerve functor $\mathcal{SE}_\omega \to [\mathbf{P}_+^{\mathrm{op}}, \mathbf{Set}]$ has a left adjoint which is full and faithful (thus forming a coreflection).*

To prove this theorem, it is helpful to consider the category $\mathcal{RE}_\omega$ of consistent-countable event structures with equivalence relations. The category $\mathcal{SE}_\omega$ of consistent-countable event structures with symmetry is a full subcategory of this. We will derive a coreflection as a composite, $[\mathbf{P}_+^{\mathrm{op}}, \mathbf{Set}] \rightleftharpoons \mathcal{RE}_\omega \rightleftharpoons \mathcal{SE}_\omega$.

Theorem 6. *The embedding $\mathcal{SE} \hookrightarrow \mathcal{RE}$ has a left adjoint $O\colon \mathcal{RE} \to \mathcal{SE}$ up to symmetry. The functor O preserves countable consistency. For an event structure with an equivalence relation, $(E, \mathbb{R})$, we have a natural isomorphism $\mathit{Nerve}(E, \mathbb{R}) \cong \mathit{Nerve}(O(E, \mathbb{R}))$.*

Proof. The reflection O takes an event structure $(E, \mathrm{Con}, \leq)$ with an equivalence relation $\mathbb{R}$ to the event structure that is inductively defined as follows. We have four inductively defined sets: $O(E)$, $O(\mathrm{Con})$, $O(\leq)$ and $O(\mathbb{R})$. In what follows, we write x_θ and y_θ for the domain and codomain of an isomorphism θ. We write $x \longrightarrow\!\!\!\!\!\subset x \cup \{e\}$ to mean that $x \cup \{e\}$ is a configuration of $O(E)$ covering the configuration x. The sets $(O(E), O(\mathrm{Con}), O(\leq), O(\mathbb{R}))$ are the least satisfying the following rules, and such that $O(\mathbb{R})$

satisfies axioms **(R)** and **(E)**.

$$\frac{e \in E}{e \in O(E)} \qquad \frac{\theta \in O(\mathbb{R}) \quad x_\theta \multimap\!\!\!\!\!\subset x_\theta \cup \{e\}}{\mathsf{ev}(e,\theta) \in O(E)}$$

$$\frac{X \in \mathrm{Con}}{X \in O(\mathrm{Con})} \qquad \frac{\theta \in O(\mathbb{R}) \quad x_\theta \multimap\!\!\!\!\!\subset x_\theta \cup \{e\} \quad X \subseteq x_\theta}{(\theta(X) \cup \{\mathsf{ev}(e,\theta)\}) \in O(\mathrm{Con})}$$

$$\frac{e' \leq e}{e' \; O(\leq) \; e} \qquad \frac{\theta \in O(\mathbb{R}) \quad x_\theta \multimap\!\!\!\!\!\subset x_\theta \cup \{e\} \quad e' \; O(\leq) \; e}{\theta(e') \; O(\leq) \; \mathsf{ev}(e,\theta)}$$

$$\frac{\theta \in \mathbb{R}}{\theta \in O(\mathbb{R})} \qquad \frac{\theta \in O(\mathbb{R}) \quad x_\theta \multimap\!\!\!\!\!\subset x_\theta \cup \{e\}}{\theta \cup \{(e, \mathsf{ev}(e,\theta))\} \in O(\mathbb{R})}$$

To see that O is left adjoint to the embedding of $\mathcal{SE}$ into $\mathcal{RE}$, consider an event structure with symmetry, $(E, \mathrm{Con}, \leq, \mathbb{S})$. The counit $\varepsilon\colon O(E) \to E$ is defined by induction on $O(E)$, as follows. First, for $e \in E$, we let $\varepsilon(e) = e$. Second, for $\theta \in O(\mathbb{R})$ and $x_\theta \multimap\!\!\!\!\!\subset x_\theta \cup \{e\}$, we define $\varepsilon(\mathsf{ev}(e,\theta))$ as follows. We assume that ε is defined on $x_\theta \cup \{e\}$ and y_θ, and that $\varepsilon(\theta) = \{(\varepsilon(e_1), \varepsilon(e_2)) \mid (e_1, e_2) \in \theta\}$ is in $\mathbb{R}$. By axiom **(O)**, there must be an event $e' \in E$ such that the relation $\varepsilon(\theta) \cup \{(\varepsilon(e), e')\}$ is an isomorphism in $\mathbb{R}$. We define $\varepsilon(\mathsf{ev}(e,\theta))$ to be this event, e'.

We have chosen a particular event e', and so this construction does not uniquely determine $\varepsilon\colon O(E) \to E$, but it is unique up to symmetry.

If E is consistent-countable, then the function $E \to \omega$ is extended to a locally injective function $O(E) \to \omega$, defined by recursion.

We conclude that $\mathit{Nerve}(E) \cong \mathit{Nerve}(O(E))$ by induction on the definition of $O(E)$. First, we observe that for every configuration $x \in \mathcal{C}^\circ(O(E))$ there is $y \in \mathcal{C}^\circ(E)$ and $\theta\colon x \cong y$ in $O(\mathbb{R})$. Secondly, we observe that for $x, y \in \mathcal{C}^\circ(E)$, if $\theta\colon x \cong y$ is in $O(\mathbb{R})$ then θ is already in $\mathbb{R}$. In checking this, one must take care over the transitivity of $O(\mathbb{R})$. ∎

Theorem 7. *The nerve functor $\mathcal{RE}_\omega \to [\mathbf{P}_+^{\mathrm{op}}, \mathbf{Set}]$ has a left adjoint R which is full and faithful.*

Proof. The left adjoint to $\mathit{Nerve}\colon \mathcal{RE}_\omega \to [\mathbf{P}_+^{\mathrm{op}}, \mathbf{Set}]$ takes a presheaf $A\colon \mathbf{P}_+^{\mathrm{op}} \to \mathbf{Set}$ and returns the event structure with equivalence relation $(E, \mathrm{Con}, \leq, \mathbb{R})$ whose events are triples (P, a, e), where $P \in \mathbf{P}_+$, $a \in A(P)$, and $e \in P$. The partial order is given by saying that $(P, a, e) \leq (P', a', e')$ if $P = P'$, $a = a'$, and $e \leq_P e'$. The configurations $\mathcal{C}^\circ(E)$ are those sets of triples having common first and second components. Thus a configuration is determined by a triple (P, a, x) of a finite non-empty poset P, an element $a \in A(P)$, and a down-set x of P.

We consider the least family $\mathbb{R}$ closed under **(E)** and **(R)** and such that for every map $j\colon Q \to P$ in $\mathbf{P}_+$, every element $a \in A(P)$, and every down-closed subset x of Q, the isomorphism $\theta\colon (P, a, j(x)) \cong (Q, A(j)(a), x)$ is in $\mathbb{R}$.

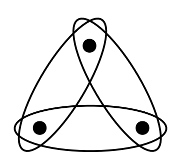
This event structure is consistent countable. For every finite partial order P, we pick an injection $f_P\colon P \rightarrowtail \omega$, and we thus define a locally injective function $\bar{f}\colon E \to \omega$ by $\bar{f}(P, a, e) = f_P(e)$.

This construction extends straightforwardly to a full and faithful functor $R\colon [\mathbf{P}_+^{\mathrm{op}}, \mathbf{Set}] \to \mathcal{RE}_\omega$. The counit $\varepsilon_E\colon R(\mathit{Nerve}(E)) \to E$ is given as follows. For each $P \in \mathbf{P}_+$ and every $\sim$-equivalence class $[f]_\sim\colon P \to E$ of morphisms we pick a representative, f. Now, for $e \in P$, we let $\varepsilon_E(P, [f]_\sim, e) = f(e)$. It doesn't matter which representative of the equivalence class $[f]_\sim$ is chosen. ∎

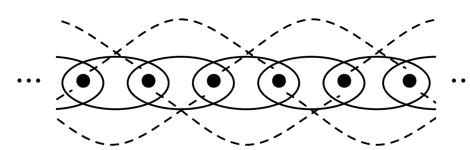
Theorem 5 follows from Theorems 6 and 7. The composite $O \circ R\colon [\mathbf{P}_+^{\mathrm{op}}, \mathbf{Set}] \to \mathcal{SE}_\omega$ is left adjoint to the nerve functor $\mathcal{SE}_\omega \to [\mathbf{P}_+^{\mathrm{op}}, \mathbf{Set}]$. The left adjoint is full and faithful because R is full and faithful and O preserves nerves.

Example 8. The nerve functor $\mathcal{SE}_\omega \to [\mathbf{P}_+^{\mathrm{op}}, \mathbf{Set}]$ is not an equivalence of categories, because it is not full. To demonstrate this, we consider two non-equivalent event structures with symmetry that have the same nerve.

- First, consider the event structure $(\mathrm{E3}, \mathrm{Con}_{\mathrm{E3}}, \leq_{\mathrm{E3}})$. There are exactly three events in E3; the partial order $\leq_{\mathrm{E3}}$ is discrete; and the consistent sets are those sets with two or fewer elements. In the following illustration, the dots represent events, and the ovals indicate the only non-trivial consistent sets.



- Second, consider the event structure $(\mathbb{Z}3, \mathrm{Con}_{\mathbb{Z}3}, \leq_{\mathbb{Z}3}, \mathbb{S}_{\mathbb{Z}3})$. The events are all integers; the partial order is discrete; and the consistent sets are generated by saying that pairs $\{i, i+1\}$ are consistent, for every integer i. The isomorphism family $\mathbb{S}_{\mathbb{Z}3}$ is generated by the isomorphisms $\{(i, i+3), (i+1, i+4)\} : \{i, i+1\} \cong \{i+3, i+4\}$ for all integers i. Notice that this is a strong symmetry. In the following illustration, dotted lines indicate events that are related by the symmetry, and ovals indicate the only non-trivial consistent sets.



There are various maps of event structures $(\mathbb{Z}3, \mathbb{S}_{\mathbb{Z}3}) \to \mathrm{E3}$, which roll up $\mathbb{Z}3$ into E3. For example, one map takes an event i to an event $(i \bmod 3)$. However there is no map of event structures $\mathrm{E3} \to (\mathbb{Z}3, \mathbb{S}_{\mathbb{Z}3})$, and yet the nerves of these event structures are isomorphic. Write 1 for a partial order with one element, and 2 for a discrete partial order with two elements.

- The hom-sets $\mathcal{SE}(1, \mathrm{E3})$ and $\mathcal{SE}(1, \mathbb{Z}3)$ both have three elements, up to symmetry;

- The hom-sets $\mathscr{SE}(2, \mathrm{E3})$ and $\mathscr{SE}(2, \mathbb{Z}3)$ both have six elements, up to symmetry;
- When P is not isomorphic to 1 or 2, the hom-sets $\mathscr{SE}(P, \mathrm{E3})$ and $\mathscr{SE}(P, \mathbb{Z}3)$ are both empty.

V. An equivalence of categories

We characterise the images of the left adjoint to the nerve functor in the coreflection $[\mathbf{P}_+^{\mathrm{op}}, \mathbf{Set}] \rightleftharpoons \mathscr{SE}_\omega$, and thus show how to cut the coreflection down to an equivalence.

Definition 9. *An event structure $(E, \mathrm{Con}, \leq)$ is* graded *if there is a well-founded relation $\prec$ on events E called the* grading *relation (its reflexive closure is written $\preceq$) such that*

(i) $\preceq^{-1}\{e\} =_{\mathrm{def}} \{e' \in E \mid e' \preceq e\}$ *is a finite configuration for all events e,*
(ii) *the restriction $\preceq \cap (x \times x)$ is a total order, for all configurations x.*

Notice that the simple event structures in Example 1 are graded. The event structure $\mathbb{Z}3$ (Example 8) is another example of a graded event structure. One grading is

$$\cdots \succ -2 \succ -1 \succ 0 \prec 1 \prec 2 \prec \cdots$$

As this makes clear, a grading need not be unique; any integer could be chosen as the minimum event, instead of 0.

Proposition 10. *The image of any presheaf under the left adjoint of the adjunction $[\mathbf{P}_+^{\mathrm{op}}, \mathbf{Set}] \rightleftharpoons \mathscr{SE}_\omega$ is graded.*

Proof. (Idea) We refer to the construction O in Theorem 6. The image of a presheaf A is an event structure in the form of a sum of finite partial orders indexed by elements of A, endowed with an equivalence relation, on which the inductive construction O then acts, generating new events. Once each finite partial order is equipped with a chosen linearisation of its events, this initial grading is extended to the new events, setting $e' \prec \mathsf{ev}(e, \theta)$ when e' is in the range y_θ of the isomorphism θ to be extended. ∎

Graded event structures are automatically consistent-countable. Curiously, graded event structures, and so those event structures with symmetry obtained from a presheaf, can be described by a binary consistency (or conflict) relation:

Proposition 11. *If $\prec$ is a grading of an event structure $(E, \mathrm{Con}, \leq)$, then $\prec^*$ is a forest, E is consistent-countable and its consistency relation satisfies*

$$X \in \mathrm{Con} \iff \forall e_1, e_2 \in X.\ \{e_1, e_2\} \in \mathrm{Con}. \qquad (\dagger)$$

Proof. We show first that $\prec^*$ is a forest. As $\prec$ is well-founded and $\prec^{-1}\{e\}$ is finite it follows by König's lemma that $\prec^{*-1}\{e\}$ is finite for all events e. Hence to show $\prec^*$ is a forest it suffices to show the property

$$e' \prec^+ e \ \&\ e'' \prec^+ e \implies e' \prec^* e'' \text{ or } e'' \prec^* e'$$

for all events e, e', e''. Suppose this were to fail, then w.l.o.g. it would fail for chains

$$e' = e'_1 \prec \cdots e'_m \prec e \quad \text{ and } \quad e'' = e''_1 \prec \cdots e''_n \prec e\,,$$

of minimal combined length $m+n$. By minimality $e'_m \neq e''_n$. But $\prec^{-1}\{e\}$ is a finite configuration within which either $e'_m \prec e''_n$ or $e''_n \prec e'_m$. Either way we obtain chains of smaller combined length violating the property—a contradiction. Hence $\prec^*$ forms a forest.

This implies the consistent-countability of E. Map each event to its height in the forest. Distinct events with the same height cannot be in the same configuration and so must be inconsistent.

To show (†), consider a finite set X such that for all $e_1, e_2 \in X$, $\{e_1, e_2\} \in \mathrm{Con}$. This means that for all e_1, e_2 in X, either $e_1 \preceq e_2$ or $e_2 \preceq e_1$. It follows that $\preceq$ provides a total order on X. (To see that $\preceq$ is transitive, consider events $e_1 \preceq e_2 \preceq e_3$ in X; now either $e_1 \preceq e_3$ or $e_3 \preceq e_1$; in the latter case $e_1 = e_2 = e_3$ since $\prec$ is well founded.) Let e be the top element of X with regard to the total order $\preceq$. We know that $X \subseteq \preceq^{-1}\{e\}$, and $\preceq^{-1}\{e\}$ is a configuration, so X is consistent. ∎

For instance, the event structure ω_1 in Example 4 is not consistent-countable, while the event structure E3 in Example 8 cannot be described by a binary conflict relation. These event structures are not graded.

Theorem 12. *Let $\mathscr{SG}$ be the subcategory of $\mathscr{SE}_\omega$ where the objects are graded event structures with symmetry. The restriction of the nerve functor to $\mathscr{SG} \to [\mathbf{P}_+^{\mathrm{op}}, \mathbf{Set}]$ yields an equivalence of categories.*

Proof. The restricted nerve functor is essentially surjective by the coreflection (Thm. 6) and Proposition 10. It is faithful because the nerve functor $\mathscr{SE} \to [\mathbf{P}_+^{\mathrm{op}}, \mathbf{Set}]$ is faithful (up to symmetry), essentially from the definition of $\sim$.

It remains for us to explain why the nerve functor $\mathscr{SG} \to [\mathbf{P}_+^{\mathrm{op}}, \mathbf{Set}]$ is full. Let $(E, \mathbb{S})$ be a graded event structure with symmetry, and let $(E', \mathbb{S}')$ be any event structure with symmetry. We will show that for every natural transformation $\phi \colon \mathit{Nerve}(E, \mathbb{S}) \to \mathit{Nerve}(E', \mathbb{S}')$ there is an event structure map $f \colon (E, \mathbb{S}) \to (E', \mathbb{S}')$ such that $\phi = \mathit{Nerve}(f)$. Given ϕ, we define $f \colon (E, \mathbb{S}) \to (E', \mathbb{S}')$ by well-founded induction on $\prec^+$, for a grading $\prec$ on E.

For $e \in E$, we write E_e for the sub-event-structure of E whose events are $\{e' \mid e' \prec^* e\}$. We will define a family of maps $f_e \colon E_e \to E'$ that satisfy the following properties:

$$\begin{aligned} & e' \prec e \implies f_{e'} = f_e|_{E_{e'}} \\ \& \ \ & \forall P \in \mathbf{P}_+.\ \forall j \colon P \to E_e.\ \phi_P([j]_\sim) = [f_e \circ j]_\sim \end{aligned} \qquad (\ddagger)$$

As induction hypothesis, we suppose that $f_{e'} \colon E_{e'} \to E'$ satisfying (‡) is defined for all $e' \prec^+ e$. We define f_e satisfying (‡) as follows.

First, we note that there are no events in E_e causally dependent on e. We consider the set $\bar{E}_e = E_e \setminus \{e\}$ which is a sub-event-structure of E_e. From the fact that $\prec^*$ is a forest, there is $e_{\max} \prec e$ such that $\bar{E}_e \subseteq {\prec^*}^{-1}\{e_{\max}\}$. From the induction hypothesis, (‡) holds of $e_{\max}$. The corresponding facts restrict to $\bar{E}_e$, and we obtain a map $\bar{f}_e \colon \bar{E}_e \to (E', \mathbb{S}')$ such that

$$\begin{aligned} & e' \prec e \implies f_{e'} = \bar{f}_e|_{E_{e'}} \\ \& \ & \forall P \in \mathbf{P}_+.\ \forall j \colon P \to \bar{E}_e.\ \phi_P([j]_\sim) = [\bar{f}_e \circ j]_\sim . \end{aligned}$$

Now consider the configurations $\preceq^{-1}\{e\}$ and $\prec^{-1}\{e\}$, as posets. We will assume that $\preceq^{-1}\{e\}$ and $\prec^{-1}\{e\}$ are objects of $\mathbf{P}_+$. (If they are not in $\mathbf{P}_+$, we simply pick isomorphic posets that are.) Consider the following commuting diagram of rigid inclusion maps:

$$\begin{tikzcd} \prec^{-1}\{e\} \arrow[r, "j"] \arrow[d, "k'"'] & \preceq^{-1}\{e\} \arrow[d, "k"] \\ \bar{E}_e \arrow[r, "j'"] & E_e \end{tikzcd}$$

From the naturality of ϕ and $\phi_{\prec^{-1}\{e\}}([j' \circ k']_\sim) = [\bar{f}_e \circ k']_\sim$, we can obtain $g \colon \preceq^{-1}\{e\} \to E'$ such that $\phi_{\preceq^{-1}\{e\}}([k]_\sim) = [g]_\sim$ and $g \circ j \sim \bar{f}_e \circ k'$. From the latter, there must be an isomorphism $\theta \colon g{\prec^{-1}}\{e\} \cong \bar{f}_e{\prec^{-1}}\{e\}$ in $\mathbb{S}'$. But $\prec^{-1}\{e\} \longrightarrow\!\!\!\!\subset \preceq^{-1}\{e\}$, so $g{\prec^{-1}}\{e\} \longrightarrow\!\!\!\!\subset g{\preceq^{-1}}\{e\}$. Hence by Axiom **(O)**, there is an event $e'' \in E'$ such that θ extends to an isomorphism

$$\theta' \colon g{\preceq^{-1}}\{e\} \cong (\bar{f}_e{\prec^{-1}}\{e\}) \cup \{e''\}$$

in $\mathbb{S}'$. Define $f_e(e) = e''$, for some choice of e''.

It follows from the induction hypothesis and the naturality of ϕ that $f_e \colon E_e \to E'$ is an event structure map satisfying (‡). Let $\mathbb{R}_e$ be the restriction of the equivalence relation $\mathbb{S}$ from E to E_e. It follows from (‡) that f_e respects relations to become a map $(E_e, \mathbb{R}_e) \to (E', \mathbb{S}')$.

Finally, we define $f \colon E \to E'$ by $f(e) = f_e(e)$. By construction, f preserves symmetry and $\mathit{Nerve}(f) = \phi$. ∎

VI. Separated presheaves

The requirement **(S)** of a symmetry to be strong is mathematically natural, and we now investigate connections between event structures with strong symmetries and presheaves that are separated. In addition to the coreflection of Section IV, there is also a reflection of categories.

We say that a presheaf A in $[\mathbf{P}_+^{\mathrm{op}}, \mathbf{Set}]$ is *separated* if for all $P \in \mathbf{P}_+$, and $a, a' \in A(P)$, we have the following property:

- if, for all e in P, we have $a|_{[e]} = a'|_{[e]}$, then $a = a'$.

Here, we are writing $a|_{[e]}$ for the element $A(j)(a)$, where $j \colon [e] \to P$ is the embedding. We write $\mathbf{Sep}(\mathbf{P}_+)$ for the category of separated presheaves. (The separatedness condition is for the Grothendieck topology on $\mathbf{P}_+$ whose covering sieves are those that are jointly surjective.)

Proposition 13. *The nerve of an event structure with equivalence is separated if and only if the equivalence relation is strong.*

Proof. Consider an event structure with an equivalence relation, $(E, \mathbb{R})$. Consider a path $P \in \mathbf{P}_+$, and two maps $f, g \colon P \to E$. Suppose that, for all $e \in P$, we have $f|_{[e]} \sim g|_{[e]}$, i.e., the isomorphism $\{f(e'), g(e') \mid e' \le e\}$ is in $\mathbb{R}$. The nerve $\mathit{Nerve}_{(E,\mathbb{R})}$ is separated at elements f and g, i.e. $f \sim g$, if and only if the the isomorphism $\{f(e), g(e) \mid e \in P\}$ is in $\mathbb{R}$, which is axiom **(S)**. ∎

Theorem 14.

1) *The nerve functor $\mathcal{SSE}_\omega \to \mathbf{Sep}(\mathbf{P}_+)$, from consistent-countable event structures with strong symmetry, has a left adjoint that is full and faithful.*
2) *Let $\mathcal{SSG}$ be the category of graded event structures with strong symmetry. The restriction of the nerve functor to $\mathcal{SSG} \to \mathbf{Sep}(\mathbf{P}_+)$ yields an equivalence of categories.*

Proof. We deal with item 1. This is a corollary of Theorem 5, which says that the nerve functor $\mathcal{SE}_\omega \to [\mathbf{P}_+^{\mathrm{op}}, \mathbf{Set}]$ has a full and faithful left adjoint $(O \circ R)$. This left adjoint takes separated presheaves to event structures with strong symmetry. To see this, consider a separated presheaf A, and consider the event structure with symmetry $O(R(A))$ given by the left adjoint. The nerve of this event structure must be isomorphic to A, since $O \circ R$ is full and faithful. Thus the nerve is separated, and by Proposition 13, $O(R(A))$ has strong symmetry. This proves item 1. Item 2 is a corollary of Theorem 12 by a similar argument. ∎

Example 15. An example of an event structure with symmetry but without strong symmetry: the event structure consists of two ‘green’ events and two ‘red’ events with discrete causal dependency relation. All sets of at most two events are consistent. The isomorphism family describing its symmetry consists of the bijection $\emptyset \cong \emptyset$, all bijections between singletons, and all bijections $\{e_1, e_2\} \cong \{e'_1, e'_2\}$ between pairs of distinct events *provided* e_1, e_2 have the same colour iff e'_1, e'_2 have the same colour. This family of isomorphisms satisfies axioms **(R)**, **(E)** and **(O)**, but not **(S)**: it is not strong. Consider, for instance, a bijection

$$\{e_1, e_2\} \cong \{e'_1, e'_2\}$$

with e_1 red and e_2 green, while both e'_1 and e'_2 are red; the bijection is not in the isomorphism family although all its restrictions to bijections between singletons are. Consequently its nerve cannot be a separated presheaf.

In addition to the coreflection of Theorem 14, we introduce a reflection.

Theorem 16. *The nerve functor $\mathcal{SSE}_\omega \to \mathbf{Sep}(\mathbf{P}_+)$ has a right adjoint that is full and faithful.*

Proof. We describe a right adjoint to the nerve functor $\mathcal{SSE}_\omega \to \mathbf{Sep}(\mathbf{P}_+)$. We begin by defining a *protoconfiguration* to be a finite poset whose elements are natural numbers; a *protoevent* is a protoconfiguration that has a top element. Thus a protoevent is an event of the event structure T introduced in Section III. Without loss of generality, we assume that $\mathbf{P}_+$ contains all the protoconfigurations. We define an event structure with symmetry $(E, \mathrm{Con}, \leq, \mathbb{S})$ as follows. The events are pairs (P, a) where P is a protoevent and a is in $A(P)$. The partial order is such that $(P, a) \leq (P', a')$ whenever P is a down-closed subset of P' and $a = a'|_P$. The configurations $\mathcal{C}^\circ(E)$ are determined as follows. For every non-empty protoconfiguration P and every $a \in A(P)$, the set $x_{(P,a)} = \{([e], a|_{[e]}) \mid e \in P\}$ is a configuration in $\mathcal{C}^\circ(E)$.

The symmetry $\mathbb{S}$ is defined to contain all the isomorphisms that arise as follows. Let P and P' be protoconfigurations, and consider $a \in A(P)$. Every isomorphism $\theta\colon P' \cong P$ induces an isomorphism $x_{(P', A(\theta)(a))} \cong x_{(P,a)}$, which is in $\mathbb{S}$.

This event structure $(E, \mathrm{Con}, \leq, \mathbb{S})$ is consistent-countable. Pick an enumeration of the finite sets of natural numbers. We use this to define a function $E \to \omega$ which is injective on configurations: we map an event (P, a) to the natural number indexing the set of natural numbers underlying the protoevent P.

A universal natural isomorphism $\varepsilon_A\colon \mathit{Nerve}(E) \cong A$ is defined as follows. Consider P in $\mathbf{P}_+$, and $f\colon P \to E$. Note that the image of f is a configuration of E, which is determined by a protoconfiguration Q and $a \in A(Q)$. Moreover, the function f defines an isomorphism $\bar{f}\colon P \to Q$. We let $\varepsilon_{A,P}(f) = A\bar{f}(a)$. ∎

VII. Applications

We conclude with pointers to immediate future work, sketching two areas which lead to constructions in presheaves not previously representable by event structures. The constructions are now representable within event structures *with symmetry*, though it remains to carry out the constructions directly there.

Unfolding higher-dimensional automata

Higher-dimensional automata (hda's, [9]) are most concisely described as cubical sets, i.e. as presheaves over $\mathbf{C}$, a category of cube shapes. The objects of $\mathbf{C}$ are natural numbers n, thought of as n-dimensional cubes (or hypercubes). Each dimension has a direction. A morphism $m \to n$ is a *face map*, an embedding of the m-dimensional cube as a face of the n-dimensional cube, preserving the direction of each dimension. (Here, in $\mathbf{C}$, there are no degeneracy maps between cubes, which we assume are not oriented.)

We can identify an n-dimensional cube with the configurations of an elementary event structure comprising the discrete poset of n (concurrent) events. A face map $m \to n$ determines a function between the configurations of m and n. But these functions need not come from rigid maps—face maps need not fix the initial empty configuration. However, by modifying the maps of $\mathbf{P}$ to allow the initial configuration to shift, we can obtain a new category $\mathbf{A}$, which contains both the category $\mathbf{C}$ of cube shapes and the path category $\mathbf{P}$. Precisely, the objects of $\mathbf{A}$ are finite partially ordered sets with maps $P \to Q$ given as pairs (j, y) where $j\colon P \to Q$ is a monotone injective function, and y is a configuration of Q, disjoint from the image of f, such that

$$\forall x \in \mathcal{C}^\circ(P).\ y \cup (j\,x) \in \mathcal{C}^\circ(Q)\ ;$$

two maps, $(j, y)\colon P \to Q$ and $(k, z)\colon Q \to R$, compose as $(k \circ j, z \cup (k\,y))\colon P \to R$.

Note that a function $j\colon P \to Q$ is a rigid map iff $(j, \emptyset)$ is a map $P \to Q$ in $\mathbf{A}$, and so we have an identity-on-objects functor $J\colon \mathbf{P} \to \mathbf{A}$. When P and Q are finite discrete posets, the maps $P \to Q$ in $\mathbf{A}$ are exactly the face maps. We thus obtain a full and faithful functor $K\colon \mathbf{C} \to \mathbf{A}$.

Now we can construct a functor $H\colon \mathbf{P} \to [\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]$; it takes P in $\mathbf{P}$ to the presheaf $\mathbf{A}(K(\_), J(P))$. Taking its left Kan extension over the Yoneda embedding of $\mathbf{P}$ in $[\mathbf{P}^{\mathrm{op}}, \mathbf{Set}]$ we obtain a functor

$$H_!\colon [\mathbf{P}^{\mathrm{op}}, \mathbf{Set}] \to [\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]\ .$$

For general reasons the functor $H_!$ has a right adjoint H^*:

$$[\mathbf{P}^{\mathrm{op}}, \mathbf{Set}] \rightleftharpoons [\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]$$

H^* takes an hda Y to the presheaf $[\mathbf{C}^{\mathrm{op}}, \mathbf{Set}](H(\_), Y)$ in $[\mathbf{P}^{\mathrm{op}}, \mathbf{Set}]$. The presheaf $H^*(Y)$ is not rooted in general; its elements at $\emptyset$ corresponds to the set of points (0-dimensional cubes) in Y. But $H^*(Y)$ does decompose into a sum of rooted presheaves; each choice of element at $\emptyset$ determines a rooted component—the behaviour, described as a presheaf over $\mathbf{P}$, for this choice of initial state in Y.

Via the equivalence of Section V, hda's with a choice of initial state, unfold to presheaves representable by event structures with symmetry. Restricting to such hda's which are separated, now w.r.t. a basis of jointly surjective maps in $\mathbf{C}$, will ensure that they are sent to separated presheaves over $\mathbf{P}$ and so are representable by event structures with strong symmetry. General Petri nets give rise to separated hda's (for example, with the 'self-concurrent individual token interpretation' of [9]).

Weak bisimulation on event structures

There is a problem in giving an account of weak bisimulation in causal models, and in event structures in particular.

A systematic way to define weak bisimulation on presheaf models was described in [4]. It was developed with respect to any hiding functor $h : \mathbf{P} \to \mathbf{Q}$. Here we think of $\mathbf{P}$ as a category of computation paths with explicit invisible actions and $\mathbf{Q}$ as paths without; for instance $\mathbf{P}$ might be strings of actions with τ while $\mathbf{Q}$ is strings of just visible actions; alternatively $\mathbf{P}$ might be finite partial orders of events including explicit internal events while $\mathbf{Q}$ as partial orders with all events visible.

A treatment of weak bisimulation depends on an operation of hiding which makes certain events of a process invisible. For example, Milner's method of defining weak bisimulation involves hiding the τ-actions a process can do by allowing arbitrarily many τ-actions to participate in any transition between states—a sort of hiding by obfuscation.

The general treatment of weak bisimulation in presheaf models goes via an intermediate construction of hiding on processes regarded as bundles. A presheaf over $\mathbf{P}$ can be viewed as a discrete fibration and so as an object in $\mathbf{Cat}/\mathbf{P}$, a bundle over $\mathbf{P}$ in $\mathbf{Cat}$. Such bundles can be regarded as generalised transition systems. We can express the operation of hiding directly on bundles. Via composition with the hiding functor h a bundle over $\mathbf{P}$ becomes a bundle over $\mathbf{Q}$. The operation of taking a presheaf over $\mathbf{P}$ to a bundle over $\mathbf{Q}$ has a right adjoint. This adjunction induces a 'hiding' monad T_h on presheaves over $\mathbf{P}$. We can now express when two presheaves A, B over $\mathbf{P}$ are weakly bisimilar, viz. if $T_h(A)$ and $T_h(B)$ are open-map bisimilar as presheaves over $\mathbf{P}$.

The major difficulty in making this account of weak bisimulation work entirely within the model of event structures has been that the monad T_h does not preserve the property of being representable by an event structure (some discussion on this can be found in [2]). Now, through the representation results here, this obstacle is removed. The way is open to explore accounts of weak bisimulation directly on (labelled) event structures with symmetry.

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