Borel Determinacy of Concurrent Games

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Abstract. Just as traditional games can be represented by trees, so concurrent games can be represented by event structures. We show the determinacy of such concurrent games with Borel sets of configurations as winning conditions, provided they are race-free and bounded-concurrent. Both properties are shown necessary. The determinacy proof proceeds via a reduction to the determinacy of tree games, and the determinacy of these in turn reduces to the determinacy of Gale-Stewart games.

1 Introduction

In logic the study of determinacy in games (the existence of a winning strategy or counter-strategy) dates back, at least, to Zermelo's work [12] on finite games which showed that all perfect-information finite games are determined. Since then, more complex games and determinacy results have been studied, *e.g.* for games with plays of infinite length. A research line that began in the 1950s with the seminal work of Gale and Stewart [5] on open games culminated with the work of Martin [6] who showed that two-player zero-sum sequential games with perfect information in which the winning conditions were Borel are determined.

In computer science determinacy results have most often been used rather than investigated. Frequently decision and verification problems are represented by games with winning conditions where winning strategies encode solutions to the problems being represented by the games. The determinacy of games ensures that in all cases there is a solution to the decision or verification problem under consideration, so is a computationally desirable property.

A common feature of the games mentioned above is that they are generally represented as trees. As a consequence, the plays of such games form total orders—the branches. The games we consider in this paper are not restricted to games represented by trees. Instead, they are played on games represented by event structures. Event structures [9] are the *concurrency* analogue of trees. Just as transitions systems unfold to trees, so Petri nets and asynchronous transition systems unfold to event structures. Plays are now partial orders of moves.

The concurrent games we consider are an extension of those introduced in [10]. Games there can be thought of as highly-interactive, distributed games between Player (thought of as a team of players) and Opponent (a team of opponents). The games model, as first introduced in [10], was extended with winning conditions in [3]. There a determinacy result was given for well-founded games (*i.e.* where only *finite* plays are possible) provided they are *race-free*, *i.e.* neither player could interfere with the moves available to the other—a property satisfied by all best-known games on trees/graphs, both sequential and concurrent.

Here we extend the main result of [3] by providing a much more general determinacy theorem. We consider concurrent games in which plays may be *infinite* and where the winning set of configurations forms a Borel set.

In particular we show that such games are determined provided that they are race-free and satisfy a structural condition we call *bounded concurrency.* Bounded concurrency expresses that no move of one of the players can be concurrent with infinitely many moves of the other—a condition trivially satisfied when *e.g.* all plays are finite, the games are sequential, or the games have rounds where simple choices are made (usual in traditional concurrent games). Bounded concurrency and race-freedom hold implicitly in games as traditionally defined.

We also show in what sense both bounded concurrency and race-freedom are necessary for Borel determinacy. Our determinacy proof follows by a reduction to the determinacy of Borel games, shown by Martin [6].

Related Work Determinacy problems have been studied for more than a century: for finite games [12]; open games [5]; Borel games [6]; or Blackwell games [7], to mention a few particularly relevant to concurrency and computer science. Whereas the determinacy theorem in [3] is a concurrent generalisation of Zermelo's determinacy theorem for finite games, the determinacy theorem in this paper generalises the Borel determinacy theorem for infinite games from trees to event structures, so from total orders to partial orders of moves.

The results here apply to zero-sum concurrent games with perfect information. The games here require additional structure in order to model imperfect information [4] or stochastic features, so the determinacy result here does not apply directly to Blackwell games [7], the imperfect-information concurrent games played on graphs in [2] or the nonzero-sum concurrent games of [1].

Structure of the Paper In Section 2 we present concurrent games represented as event structures. Section 3 introduces tree and Gale-Stewart games as variants of concurrent games. In Section 4 race-freedom and bounded concurrency are studied. Section 5 contains the determinacy theorem, preceding the conclusion.

2 Concurrent Games on Event Structures

An *event structure* comprises $(E, \leq, \mathrm{Con})$, consisting of a set E, of *events* which are partially ordered by $\leq$, the *causal dependency relation*, and a nonempty *consistency relation* Con consisting of finite subsets of E, which satisfy axioms:

$$\begin{array}{l}\{e' \mid e' \leq e\} \text{ is finite for all } e \in E,\\ \{e\} \in \mathrm{Con} \text{ for all } e \in E,\\ Y \subseteq X \in \mathrm{Con} \implies Y \in \mathrm{Con}, \quad \text{and}\\ X \in \mathrm{Con} \;\&\; e \leq e' \in X \implies X \cup \{e\} \in \mathrm{Con}.\end{array}$$

The *configurations* of E consist of those subsets $x \subseteq E$ which are

Consistent: $\forall X \subseteq x.\ X$ is finite $\Rightarrow X \in \mathrm{Con}$, and
Down-closed: $\forall e, e'.\ e' \leq e \in x \implies e' \in x$.

We write $\mathcal{C}^\infty(E)$ for the set of configurations of E and $\mathcal{C}(E)$ for the finite configurations. Two events e_1, e_2 which are both consistent and incomparable with respect to causal dependency in an event structure are regarded as *concurrent*, written $e_1\, co\, e_2$. In games the relation of *immediate* dependency $e \rightarrowtriangle e'$, meaning e and e' are distinct with $e \leq e'$ and no event in between plays an important role. For $X \subseteq E$ we write $[X]$ for $\{e \in E \mid \exists e' \in X.\ e \leq e'\}$, the down-closure of X; note if $X \in \mathrm{Con}$ then $[X] \in \mathrm{Con}$. We use $x \longrightarrow\!\!\!\!\subset y$ to mean y covers x in $\mathcal{C}^\infty(E)$, *i.e.*, $x \subset y$ with nothing in between, and $x \stackrel{e}{\longrightarrow\!\!\!\!\subset} y$ to mean $x \cup \{e\} = y$ for $x, y \in \mathcal{C}^\infty(E)$ and event $e \notin x$. We use $x \stackrel{e}{\longrightarrow\!\!\!\!\subset}$, expressing that event e is enabled at configuration x, when $x \stackrel{e}{\longrightarrow\!\!\!\!\subset} y$ for some configuration y.

Let E and E' be event structures. A *map* of event structures is a partial function on events $f : E \to E'$ such that for all $x \in \mathcal{C}(E)$ its direct image $fx \in \mathcal{C}(E')$ and if $e_1, e_2 \in x$ and $f(e_1) = f(e_2)$ (with both defined) then $e_1 = e_2$. The map expresses how the occurrence of an event e in E induces the coincident occurrence of the event $f(e)$ in E' whenever it is defined. Maps of event structures compose as partial functions, with identity maps given by identity functions. We say that the map is *total* if the function f is total. Say a total map of event structures is *rigid* when it preserves causal dependency.

The category of event structures is rich in useful constructions on processes. In particular, *pullbacks* are used to define the composition of *strategies*, while *restriction* (a form of equalizer) and the *defined part* of maps will be used in defining strategies. Any map of event structures $f : E \to E'$, which may be a partially defined on events, has a *defined part* the total map $f_0 : E_0 \to E'$, in which the event structure E_0 has events those of E at which f is defined, with causal dependency and consistency inherited from E, and where f_0 is simply f restricted to its domain of definition. Given an event structure E and a subset $R \subseteq E$ of its events, the *restriction* $E \upharpoonright R$ is the event structure comprising events $\{e \in E \mid [e] \subseteq R\}$ with causal dependency and consistency inherited from E; we sometimes write $E \setminus S$ for $E \upharpoonright (E \setminus S)$, where $S \subseteq E$.

Event Structures with Polarity Both a game and a strategy in a game are represented with event structures with polarity, comprising an event structure E together with a polarity function $pol : E \to \{+, -\}$ ascribing a polarity $+$ (Player) or $-$ (Opponent) to its events; the events correspond to moves. Maps of event structures with polarity, are maps of event structures which preserve polarities. An event structure with polarity E is *deterministic* iff

$$\forall X \subseteq_{\mathrm{fin}} E.\ Neg[X] \in \mathrm{Con}_E \implies X \in \mathrm{Con}_E\,,$$

where $Neg[X] =_{\mathrm{def}} \{e' \in E \mid pol(e') = - \ \&\ \exists e \in X.\ e' \leq e\}$. We write $Pos[X]$ if $pol(e') = +$. The *dual*, $E^\perp$, of an event structure with polarity E comprises the same underlying event structure E but with a reversal of polarities.

Given two sets of events x and y, we write $x \subset^+ y$ to express that $x \subset y$ and $pol(y \setminus x) = \{+\}$; similarly, we write $x \subset^- y$ iff $x \subset y$ and $pol(y \setminus x) = \{-\}$.

Games and Strategies Let A be an event structure with polarity—a game; its events stand for the possible moves of Player and Opponent and its causal dependency and consistency relations the constraints imposed by the game.

A *strategy (for Player)* in A is a total map $\sigma : S \to A$ from an event structure with polarity S, which is both *receptive* and *innocent*. Receptivity ensures an openness to all possible moves of Opponent. Innocence, on the other hand, restricts the behaviour of Player; Player may only introduce new relations of immediate causality of the form $\ominus \rightarrowtriangle \oplus$ beyond those imposed by the game.

Receptivity: A map σ is *receptive* iff
$\sigma x \overset{a}{\longrightarrow\!\!\!\subset}$ & $pol_A(a) = - \Rightarrow \exists! s \in S.\ x \overset{s}{\longrightarrow\!\!\!\subset}$ & $\sigma(s) = a$.
Innocence: A map σ is *innocent* iff
$s \rightarrowtriangle s'$ & $(pol(s) = +$ or $pol(s') = -)$ then $\sigma(s) \rightarrowtriangle \sigma(s')$.
Say a strategy $\sigma : S \to A$ is *deterministic* if S is deterministic.

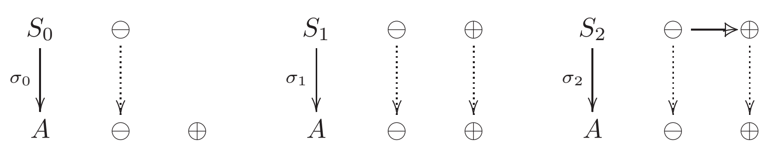
Composing Strategies Suppose that $\sigma : S \to A$ is a strategy in a game A. A counter-strategy is a strategy of Opponent, so a strategy $\tau : T \to A^{\perp}$ in the dual game. The effect of playing-off a strategy σ against a counter-strategy τ is described via a pullback. Ignoring polarities, we have total maps of event structures $\sigma : S \to A$ and $\tau : T \to A$. Form their pullback,

$$\begin{array}{ccc} P & \xrightarrow{\Pi_2} & T \\ {\scriptstyle \Pi_1}\downarrow & \lrcorner & \downarrow{\scriptstyle \tau} \\ S & \xrightarrow[\sigma]{} & A\,. \end{array}$$

The event structure P describes the play resulting from playing-off σ against τ. Because σ or τ may be nondeterministic there can be more than one maximal configuration z in $\mathcal{C}^{\infty}(P)$. A maximal z images to a configuration $\sigma \Pi_1 z = \tau \Pi_2 z$ in $\mathcal{C}^{\infty}(A)$. Define the set of *results* of playing-off σ against τ to be

$$\langle \sigma, \tau \rangle =_{\text{def}} \{\sigma \Pi_1 z \mid z \text{ is maximal in } \mathcal{C}^{\infty}(P)\}\,.$$

Example 1. Let $\sigma_i : S_i \to A$ be a strategy in $A = \oplus\ co\ \ominus$



There are three analogous counter-strategies $\tau_j : T_j \to A^{\perp}$, $j = 0, 1, 2$, for Opponent. The results of playing each σ_i against each τ_j are:

$$\langle \sigma_i, \tau_j \rangle = \begin{cases} \{\emptyset\} & \text{if } i \in \{0,2\} \text{ \& } j \in \{0,2\}, \\ \{\{\oplus\}\} & \text{if } i = 1 \text{ \& } j = 0, \\ \{\{\ominus\}\} & \text{if } i = 0 \text{ \& } j = 1, \\ \{\{\oplus, \ominus\}\} & \text{if } i = 1 \text{ \& } j = 1. \end{cases}$$

Note that Player (or Opponent) can try to force *some* play to happen sequentially by adding causal dependencies, *e.g.* when using strategy σ_2 (or τ_2). This situation may lead to a deadlock as with σ_2 played-off against τ_2 when both players are waiting for their opponent to play first. □

Determinacy and Winning Conditions A *game with winning conditions* [3] comprises $G = (A, W)$ where A is an event structure with polarity and the set $W \subseteq \mathcal{C}^\infty(A)$ consists of the *winning configurations (for Player)*. Define the *losing conditions (for Player)* to be $L = \mathcal{C}^\infty(A) \setminus W$. The dual $G^\perp$ of a game with winning conditions $G = (A, W)$ is defined to be $G^\perp = (A^\perp, L)$, a game where the roles of Player and Opponent are reversed, as are correspondingly the roles of winning and losing conditions.

A strategy in G is a strategy in A. A strategy in G is regarded as *winning* if it always prescribes moves for Player to end up in a winning configuration, no matter what the activity or inactivity of Opponent. Formally, a strategy $\sigma : S \to A$ in G is *winning (for Player)* if $\sigma x \in W$ for all $\oplus$-maximal configurations $x \in \mathcal{C}^\infty(S)$—a configuration x is $\oplus$-maximal if whenever $x \stackrel{s}{\longrightarrow\!\!\subset}$ then the event s has $-$ve polarity. Equivalently, a strategy σ for Player is winning if when played against any counter-strategy τ of Opponent, the final result is a win for Player; precisely, it can be shown [3] that a strategy σ is a winning for Player iff all the results $\langle \sigma, \tau \rangle$ lie within W, for any counter-strategy τ of Opponent. Sometimes we say a strategy σ *dominates* a counter-strategy τ (and *vice versa*) when $\langle \sigma, \tau \rangle \subseteq W$ (respectively, $\langle \sigma, \tau \rangle \subseteq L$). A game with winning conditions is *determined* when either Player or Opponent has a winning strategy in the game.

Example 2. Consider the game A with two inconsistent events $\oplus$ and $\ominus$ with the obvious polarities and winning conditions $W = \{\{\oplus\}\}$. The game (A, W) is not determined: no strategy of either player dominates all counter-strategies of the other player. Any strategy $\sigma : S \to A$ cannot be winning as it must by receptivity have $\ominus \mapsto \ominus$, so a $\oplus$-maximal configuration of S with image $\{\ominus\} \notin W$. By a symmetric argument no counter-strategy for Opponent can be winning. □

3 Tree Games and Gale–Stewart Games

We introduce tree games as a special case of concurrent games, traditional Gale–Stewart games as a variant, and show how to reduce the determinacy of tree games to that of Gale–Stewart games. Via Martin's theorem for the determinacy of Gale–Stewart games with Borel winning conditions we show that tree games with Borel winning conditions are determined.

3.1 Tree Games

Definition 3. Say E, an event structure with polarity, is *tree-like* iff it has empty concurrency relation (so $\leq_E$ forms a forest), all events enabled by the initial configuration $\emptyset$ have the same polarity, and is such that polarities alternate along branches, *i.e.* if $e \rightarrowtriangle e'$ then $pol_E(e) \neq pol_E(e')$. A *tree game* is (E, W), a concurrent game with winning conditions in which E is tree-like. □

Proposition 4. *Let E be a tree-like event structure with polarity. Then, its finite configurations $\mathcal{C}(E)$ form a tree w.r.t. $\subseteq$. Its root is the empty configuration $\emptyset$. Its (maximal) branches may be finite or infinite; finite sub-branches correspond to finite configurations of E; infinite branches correspond to infinite configurations of E. Its arcs, associated with $x \overset{e}{\longrightarrow\!\!\subset} x'$, are in 1-1 correspondence with events $e \in E$. The events e associated with initial arcs $\emptyset \overset{e}{\longrightarrow\!\!\subset} x$ all have the same polarity. In a branch*

$$\emptyset \overset{e_1}{\longrightarrow\!\!\subset} x_1 \overset{e_2}{\longrightarrow\!\!\subset} x_2 \overset{e_3}{\longrightarrow\!\!\subset} \cdots \overset{e_i}{\longrightarrow\!\!\subset} x_i \overset{e_{i+1}}{\longrightarrow\!\!\subset} \cdots$$

the polarities of the events $e_1, e_2, \ldots, e_i, \ldots$ alternate.

Proposition 4 gives the precise sense in which the terms 'arc,' 'sub-branch' and 'branch' are synonyms for the terms 'events,' 'configurations' and 'maximal configurations' when an event structure with polarity is tree-like. Notice that for a non-empty tree-like event structure with polarity, all the events that can occur initially share the same polarity. We say a non-empty tree game (E, W) has polarity $+$ or $-$ depending on whether its initial events are +ve (positive) or −ve (negative). We adopt the convention that the empty game $(\emptyset, \emptyset)$ has polarity $+$, and the empty game $(\emptyset, \{\emptyset\})$ has polarity $-$.

Proposition 5. *Let $f : S \to A$ be a total map of event structures with polarity and let A be tree-like. Then, it follows that S is also tree-like and that the map f is innocent. The map f is a strategy if and only if it is receptive.*

3.2 Gale–Stewart Games

Gale–Stewart games are a variant of tree games in which all maximal configurations of the tree game are infinite, and more importantly where Player and Opponent *must* play to a maximal, infinite configuration. Note that this is in general not the case for concurrent games where neither player is forced to play.

Definition 6. A *Gale–Stewart* game (G, V) comprises

- G, a tree-like event structure with polarity for which all maximal configurations are infinite, and
- V, a subset of infinite configurations—the *winning* configurations for Player.

A *winning strategy* in (G, V) is $\sigma : S \to G$, a deterministic strategy such that $\sigma x \in V$ for all maximal (and hence necessarily infinite) x in $\mathcal{C}^\infty(S)$. □

This is not the way a Gale–Stewart game and a winning strategy in a Gale–Stewart game are traditionally defined. However, because σ is deterministic it is injective as a map on configurations, so corresponds to the subfamily of configurations $T = \{\sigma x \mid x \in \mathcal{C}^\infty(S)\}$ of $\mathcal{C}^\infty(G)$. The family of configurations T forms a subtree of the tree of configurations of G. Its properties, given below, reconcile our definition based on event structures with the traditional one.

Proposition 7. *A winning strategy in a Gale–Stewart game* (G, V) *is a non-empty subset* $T \subseteq \mathcal{C}^\infty(G)$ *such that*

(i) $\forall x, y \in \mathcal{C}^\infty(G).\ \ y \subseteq x \in T \implies y \in T$,

(ii) $\forall x, y \in \mathcal{C}(G).\ \ x \in T\ \&\ x \stackrel{-}{\longrightarrow\!\!\subset} y \implies y \in T$,

(iii) $\forall x, y_1, y_2 \in T.\ x \stackrel{+}{\longrightarrow\!\!\subset} y_1\ \&\ x \stackrel{+}{\longrightarrow\!\!\subset} y_2 \implies y_1 = y_2$, *and*

(iv) all $\subseteq$*-maximal members of* T *are infinite and in* V.

A Gale–Stewart game (G, V) has a *dual* game $(G, V)^* =_{\text{def}} (G^\perp, V^*)$, where V^* is the set of all maximal configurations in $\mathcal{C}^\infty(G) \setminus V$. A winning strategy for Opponent in (G, V) is a winning strategy (for Player) in the dual game $(G, V)^*$.

For any event structure A there is a topology on $\mathcal{C}^\infty(A)$ given by the Scott open subsets [8]. The $\subseteq$-maximal configurations in $\mathcal{C}^\infty(A)$ inherit a sub-topology from that on $\mathcal{C}^\infty(A)$. The Borel subsets of a topological space comprise the sigma-algebra generated by the open subsets, *i.e.* the Borel sets are constructed by closing the open subsets under countable union, countable intersection and complement. Martin proved in [6] that Gale–Stewart games (G, V), with V Borel, are determined.

3.3 Determinacy of Tree Games

The determinacy of tree games with Borel winning conditions is shown by a reduction to the determinacy of Gale–Stewart games. Let (E, W) be a tree game. We construct a Gale–Stewart game $\text{GS}(E, W) = (G, V)$ and a partial map *proj* : $G \to E$. The events of G are built as sequences of events in E together with two new symbols δ^- and δ^+ decreed to have polarity $-$ and $+$, respectively; the symbols δ^- and δ^+ represent delay moves by Opponent and Player.

An event of G is a non-empty finite sequence $[e_1, \cdots, e_k]$ of symbols from the set $E \cup \{\delta^-, \delta^+\}$ where: e_1 has the same polarity as (E, W); polarities alternate along the sequence; and for all subsequences $[e_1, \cdots, e_i]$, with $i \leq k$, $\{e_1, \cdots, e_i\} \cap E \in \mathcal{C}(E)$. Causal dependency is given by $[e_1, \cdots, e_k] \leq_G [e_1, \cdots, e_k, e_{k+1}]$ and consistency by compatibility w.r.t. $\leq_G$. Events $[e_1, \cdots, e_k]$ of G have the same polarity as their last entry e_k. Note that G is tree-like and that all maximal configurations are infinite (because of delay moves).

The map *proj* : $G \to E$ takes an event $[e_1, \cdots, e_k]$ of G to e_k if $e_k \in E$, and is undefined otherwise. The set V consists of all infinite, maximal configurations for which *proj* $x \in W$. We have built a Gale–Stewart game $\text{GS}(E, W) = (G, V)$. Note, as *proj* is Scott-continuous on configurations, if W is Borel then so is V. The construction respects the duality on games: $\text{GS}((E, W)^\perp) = (\text{GS}(E, W))^*$.

Lemma 8. *Suppose* σ *is a winning strategy for* $\text{GS}(E, W)$. *Then proj* $\circ\, \sigma$ *has defined part* σ_0, *a winning strategy for* (E, W).

Dually, a winning counter-strategy in $\text{GS}(E, W)$ yields a winning counter-strategy in (E, W). Hence by Martin's Borel-determinacy theorem [6]:

Theorem 9. *Tree games with Borel winning conditions are determined.*

4 Race-freedom and Bounded-concurrency

Not all games are determined, *cf.* Example 2. However, a determinacy theorem holds for well-founded games (games where all configurations are finite) which satisfy a property called *race-freedom* (from [3]): in a game A, for $x \in \mathcal{C}(A)$,

$$x \overset{a}{\longrightarrow\!\!\subset} \;\&\; x \overset{a'}{\longrightarrow\!\!\subset} \;\&\; pol(a) \neq pol(a') \implies x \cup \{a, a'\} \in \mathcal{C}(A)\,. \qquad (\mathbf{Race-free})$$

Note that the game in Example 2 is not *race-free*, but well-founded; tree games are race-free, but not necessarily well-founded. It may be easy to believe that a nondeterministic winning strategy always has a winning deterministic sub-strategy. This is not so and determinacy does not hold even for well-founded race-free games if we restrict to deterministic strategies, *cf.* [3]. Other observations made in [3]: being race-free is necessary for determinacy, in the sense that without it there are winning conditions for which a well-founded game is not determined; race-freedom is not sufficient to ensure determinacy when infinite behaviour is allowed, *i.e.* when A is not well-founded, as is illustrated in the following example.

Example 10. Let A be the event structure with polarity consisting of one positive event $\oplus$ which is concurrent with an infinite chain of alternating negative and positive events, *i.e.* for each i we have both $\oplus\ co\ \oplus_i$ and $\oplus\ co\ \ominus_i$, $i \in \mathbb{N}$,

$$A = \quad \oplus \quad\quad \ominus_1 \rightarrowtriangle \oplus_1 \rightarrowtriangle \ominus_2 \rightarrowtriangle \oplus_2 \rightarrowtriangle \cdots$$

and Borel winning conditions (for Player) given by

$$W = \{\emptyset, \{\ominus_1, \oplus_1\}, ..., \{\ominus_1, \oplus_1, ..., \ominus_i, \oplus_i\}, ..., A\}.$$

So, Player wins if (i) no event is played, or (ii) the event $\oplus$ is not played and the play is finite and finishes in some $\oplus_i$, or (iii) all of the events in A are played. Otherwise, Opponent wins.

Player does not have a winning strategy because Opponent has an infinite family of *spoiler* strategies, not all be dominated by a single strategy of Player. The inclusion maps $\tau_\infty : T_\infty \to A^\perp$ and $\tau_i : T_i \to A^\perp$, $i \in \mathbb{N}$, are strategies for Opponent where $T_\infty^\perp =_{\mathrm{def}} A$ and $T_i^\perp =_{\mathrm{def}} A \setminus \{e' \in A \mid \ominus_i \leq e'\}$, for $i \in \mathbb{N}$.

Any strategy for Player that plays $\oplus$ is dominated by some strategy τ_i for Opponent; likewise, any strategy for Player that does not play $\oplus$ and plays only finitely many positive events $\oplus_i$ is also dominated by some strategy τ_i for Opponent. Moreover, a strategy for Player that does not play $\oplus$ and plays all of the events $\oplus_i$ in A is dominated by τ_∞. So, Player does not have a winning strategy in this game. Similarly, Opponent does not have a winning strategy in A because Player has two strategies that cannot be both dominated by any strategy for Opponent. Let $\sigma_{\overline{\oplus}} : S_{\overline{\oplus}} \to A$ and $\sigma_\oplus : S_\oplus \to A$ be strategies for Player such that $S_{\overline{\oplus}} =_{\mathrm{def}} A \setminus \{\oplus\}$ and $S_\oplus =_{\mathrm{def}} A$.

On the one hand, any strategy for Opponent that plays only finitely many (possibly zero) negative events $\ominus_i$ is dominated by $\sigma_{\overline{\oplus}}$; on the other, any strategy for Opponent that plays all of the negative events $\ominus_i$ in A is dominated by $\sigma_\oplus$. Thus neither player has a winning strategy in this game! □

In the above example, to win Player should only make the move $\oplus$ when Opponent has played a specified infinite number of moves. We can banish such difficulties by insisting that in a game A no event is concurrent with infinitely many events of the opposite polarity. This property is called *bounded-concurrency*:

$$\forall y \in \mathcal{C}^\infty(A).\ \forall a \in y.\ \{a' \in y \mid a\ co\ a'\ \&\ pol(a) \neq pol(a')\} \text{ is finite.} \qquad (\mathbf{Bounded-concurrent})$$

Bounded concurrency is in fact a *necessary* structural condition for determinacy with respect to Borel winning conditions.

Notation *For configurations y, y' of A, we shall write $max_+(y', y)$ iff y' is $\oplus$-maximal in y, i.e. $y' \stackrel{e}{\longrightarrow\!\!\subset}\ \&\ pol(e) = + \implies e \notin y$; similarly, $\overline{max}_+(y', y)$ iff y' is not $\oplus$-maximal in y. We use max_- analogously when $pol(e) = -$.* □

We show that if a *countable* race-free A is not bounded-concurrent, then there is Borel W so that the game (A, W) is not determined. Bounded-concurrency is thus shown necessary: Since A is not bounded-concurrent, there is $y \in \mathcal{C}^\infty(A)$ and $e \in y$ such that e is concurrent with infinitely many events of opposite polarity in y. W.l.o.g. assume that $pol(e) = +$, that $y_e =_{\text{def}} y \setminus \{e\}$ is a configuration and that $y = [e] \cup [\{a \in y \mid pol_A(a) = -\}]$. The following rules determine whether $y' \in \mathcal{C}^\infty(A)$ is in W or L:

1. $y' \supseteq y \implies y' \in W$;
2. $y' \subset y\ \&\ e \in y' \implies y' \in L$;
3. $y' \subset y\ \&\ e \notin y'\ \&\ max_+(y', y_e)\ \&\ \overline{max}_-(y', y_e) \implies y' \in W$;
4. $y' \subset y\ \&\ e \notin y'\ \&\ \overline{max}_+(y', y_e)$ or $max_-(y', y_e) \implies y' \in L$;
5. $y' \not\supseteq y\ \&\ (y' \cap y) \subset^- y' \implies y' \in W$;
6. $y' \not\supseteq y\ \&\ (y' \cap y) \subset^+ y' \implies y' \in L$;
7. otherwise assign y' (arbitrarily) to W.

No y' is assigned as winning for both Player and Opponent: the implications' antecedents are exhaustive and pairwise mutually exclusive.[3] Informally, rules 3 and 4 ensure that to win both players' strategies must progress towards y; rules 5 and 6 that to win no player can deviate from y; rules 1 and 2 that for Player to win they should make move e iff Opponent plays all their moves in y.

Lemma 11. *For A and W as above, W is a Borel subset of $\mathcal{C}^\infty(A)$ and the game (A, W) is not determined.*

Proof. (Sketch) Countability of A ensures that W defined using the scheme above is Borel. We first show: (i) if $\sigma : S \to A$ is a winning strategy for Player then y is σ-reachable, *i.e.*, there is $x \in \mathcal{C}^\infty(S)$ such that $\sigma x = y$—equivalently, there is τ such that $y \subseteq^+ y'$ for some $y' \in \langle \sigma, \tau \rangle$. And, (ii) if τ is a winning strategy for Opponent then y is τ-reachable.

Define the (deterministic) strategies $\tau_\infty : T_\infty \to A^\perp$ for Opponent and $\sigma_{\overline{\oplus}} : S_{\overline{\oplus}} \to A$ for Player as the following inclusion maps:

[3] The set W in Example 10 is an instance of this scheme—use rules 1 and 3.

$$\tau_\infty : A^\perp \upharpoonright (\{a \in A \mid a \in y \text{ or } pol_A(a) = +\}) \hookrightarrow A^\perp,$$
$$\sigma_{\overline{\oplus}} : A \upharpoonright (\{a \in A \mid a \in y_e \text{ or } pol_A(a) = -\}) \hookrightarrow A.$$

For (i) suppose $\sigma : S \to A$ is a winning strategy for Player. Let $y' \in \langle \sigma, \tau_\infty \rangle$. Thus $y' \in W$. Since $Neg[(\tau_\infty T_\infty)^\perp] \subseteq y$ then $(y' \cap y) \subset^- y'$ does not hold (to discard rule 5); and, because $\{-\} \not\subset pol_A(y' \setminus y)$ one can discard rule 7 too. Moreover, since $max_-(y', y_e)$ holds then $max_+(y', y_e)$ & $\overline{max}_-(y', y_e)$ does not hold (to discard 3). Then, necessarily $y' \supseteq y$ (by rule 1). However, because of the definition of τ_∞ this implies $y' \supseteq^+ y$ and that y is σ-reachable.

For (ii) suppose $\tau : T \to A^\perp$ is a winning strategy for Opponent. It is sufficient to show that y_e is τ-reachable as then y will also be τ-reachable by receptivity. Let $y' \in \langle \sigma_{\overline{\oplus}}, \tau \rangle$. Thus $y' \in L$. As $Pos[\sigma_{\overline{\oplus}} S_{\overline{\oplus}}] \subseteq y_e$ then $(y' \cap y) \subset^+ y'$ does not hold (to discard rule 6). Since there is no $s_e \in S_{\overline{\oplus}}$ such that $\sigma_{\overline{\oplus}}(s_e) = e$ then the antecedent of rule 2, *i.e.*, $y' \subset y$ & $e \in y'$, does not hold (to discard rule 2). And since $max_+(y', y_e)$ holds for all $y' \in \langle \sigma_{\overline{\oplus}}, \tau \rangle$ then, because $y' \in L$, we have that $max_-(y', y_e)$ holds too (by rule 4). Hence, y_e is τ-reachable.

To conclude we show there is no winning strategy for either player. If σ is a winning strategy for Player then by (i) there is $x \in \mathcal{C}^\infty(S)$ such that $\sigma x = y$; in particular there is $s_e \in x$ such that $\sigma(s_e) = e$. Define the inclusion map $\tau_{\text{fin}} : A^\perp \upharpoonright (\{a \in A^\perp \mid a \in \sigma[s_e]_S \text{ or } pol_A(a) = +\} \hookrightarrow A^\perp$ as a spoiler strategy. Then τ_{fin} is a strategy for Opponent for which there is $y' \in \langle \sigma, \tau_{\text{fin}} \rangle$ with $e \in y'$ and where y' only contains finitely many $-$ve events. Either $y' \subset y$ whence $y' \in L$ by (2), or $y' \not\subset y$ whereupon $(y' \cap y) \subset^+ y'$ so $y' \in L$ by (6). Hence as τ_{fin} is a strategy for Opponent not dominated by σ the latter cannot be winning.

If τ is a winning strategy for Opponent then y is τ-reachable. Define a spoiler strategy as the inclusion map $\sigma_\oplus : A \upharpoonright (\{a \in A \mid a \in y \text{ or } pol_A(a) = -\} \hookrightarrow A$. Then $\sigma_\oplus$ is a strategy for which there is $y' \in \langle \sigma_\oplus, \tau \rangle$ with $y' \supseteq y$. By (1), $y' \in W$, so $\sigma_\oplus$ is not dominated by τ, which then cannot be a winning strategy either. □

5 From Concurrent to Tree Games

We now construct a tree game $\mathrm{TG}(A, W)$ from a concurrent game (A, W). We can think of the events of $\mathrm{TG}(A, W)$ as corresponding to (non-empty) *rounds* of $-$ve (negative) or $+$ve (positive) events in the original concurrent game (A, W). When (A, W) is race-free and bounded-concurrent, a winning strategy for $\mathrm{TG}(A, W)$ will induce a winning strategy for (A, W). In this way we reduce determinacy of concurrent games to determinacy of tree games.

5.1 The Tree Game of a Concurrent Game

Let (A, W) be a concurrent game; from the game (A, W) we construct a tree game $\mathrm{TG}(A, W) = (TA, TW)$. The construction of TA depends on whether $\emptyset \in W$. When $\emptyset \in W$, define an alternating sequence of (A, W) to be a sequence

$$\emptyset \subset^- x_1 \subset^+ x_2 \subset^- \cdots \subset^+ x_{2i} \subset^- x_{2i+1} \subset^+ x_{2i+2} \subset^- \cdots$$

of configurations in $\mathcal{C}^\infty(A)$—the sequence need not be maximal. Define the $-$ve events of $\mathrm{TG}(A, W)$ to be $[\emptyset, x_1, x_2, \ldots, x_{2k-2}, x_{2k-1}]$, *i.e.* finite alternating sequences of the form $\emptyset \subset^- x_1 \subset^+ x_2 \subset^- \cdots \subset^+ x_{2k-2} \subset^- x_{2k-1}$, and let the $+$ve events to be $[\emptyset, x_1, x_2, \ldots, x_{2k-1}, x_{2k}]$, *i.e.* finite alternating sequences of the form $\emptyset \subset^- x_1 \subset^+ x_2 \subset^- \cdots \subset^- x_{2k-1} \subset^+ x_{2k}$, where $k \geq 1$. The causal dependency relation on TA is given by the relation of initial sub-sequence, with a finite subset of events being consistent if and only if the events are all initial sub-sequences of a common alternating sequence.

It is easy to see that a configuration of TA corresponds to an alternating sequence, the $-$ve events of TA matching arcs $x_{2k-2} \subset^- x_{2k-1}$ and the $+$ve events arcs $x_{2k-1} \subset^+ x_{2k}$. As such, we say a configuration $y \in \mathcal{C}^\infty(TA)$ is winning, and in TW, if and only if y corresponds to an alternating sequence of the form $\emptyset \ \cdots \subset^+ x_i \subset^- x_{i+1} \subset^+ \cdots$ for which $\bigcup_i x_i \in W$.

When $\emptyset \in L$, we define an alternating sequence of (A, W) as a sequence

$$\emptyset \subset^+ x_1 \subset^- x_2 \subset^+ \cdots \subset^- x_{2i} \subset^+ x_{2i+1} \subset^- x_{2i+2} \subset^+ \cdots$$

of configurations in $\mathcal{C}^\infty(A)$. In this case, the $-$ve events of $\mathrm{TG}(A, W)$ are finite alternating sequences ending in x_{2k}, while the $+$ve events end in x_{2k-1}, for $k \geq 1$. The remaining parts of the definition proceed analogously.

We have constructed a tree game $\mathrm{TG}(A, W)$ from (A, W). The construction respects duality on games: $\mathrm{TG}((A, W)^\perp) = (\mathrm{TG}(A, W))^\perp$.

Proposition 12. *Suppose (A, W) is a bounded-concurrent game. Every maximal alternating sequence has one of two forms,*

(i) finite:

$$\emptyset \ \cdots \subset^+ x_i \subset^- x_{i+1} \subset^+ \cdots x_k \ ,$$

where x_i is finite for all $0 < i < k$ (where possibly x_k is infinite), or

(ii) infinite:

$$\emptyset \ \cdots \subset^+ x_i \subset^- x_{i+1} \subset^+ \cdots \ ,$$

where each x_i is finite.

Example 13. Let (A, W) be the concurrent game with A as in Example 1 and $W = \{\emptyset, \{\oplus, \ominus\}\}$. Player has an obvious winning strategy: await Opponent's move and then make their move. Because $\emptyset \in W$, its tree game is

$$e_1 = [\emptyset, \{\ominus\}] \longrightarrow e_2 = [\emptyset, \{\ominus\}, \{\ominus, \oplus\}]$$

In the tree game the empty and maximal branches are winning. Its Gale–Stewart game has events which correspond to the non-empty subsequences of

$$(\delta^- \delta^+)^* e_1 (\delta^+ \delta^-)^* e_2 (\delta^- \delta^+)^*$$

and branches which comprise consecutive sequences of such. An infinite branch is winning if it only has delay events or contains e_1 and e_2. Player has a winning strategy: delay while Opponent delays and play e_2 when Opponent plays e_1. $\square$

5.2 Concurrent Strategies from Tree-Strategies

Now assume that the game (A, W) is race-free and bounded-concurrent. Suppose that $str : T \to TA$ is a (winning) strategy in the tree game $TG(A, W)$. Note that T is necessarily tree-like. We will show how to construct $\sigma_0 : S \to A$, a (winning) strategy in the original concurrent game (A, W). We construct S indirectly from $\mathcal{Q}$, a prime-algebraic domain [8,11], built as follows. For technical reasons, when defining $\mathcal{Q}$ it is convenient to assume that $A \cap (A \times T) = \emptyset$. Via *str* a sub-branch $\vec{t} = (t_1, \cdots, t_i, \cdots)$ of T determines a *tagged alternating sequence*

$$\emptyset \quad \cdots \quad \overset{t_{i-1}}{\subset^-} x_{i-1} \overset{t_i}{\subset^+} x_i \overset{t_{i+1}}{\subset^-} \cdots$$

where $str(t_i) = [\emptyset, \ldots, x_{i-1}, x_i]$. (the arc t_i is associated with a round extending x_{i-1} to x_i in the original game.) Define $q(\vec{t})$ to be the partial order with events

$$\bigcup\{(x_i \setminus x_{i-1}) \mid t_i \text{ is a } -\text{ve arc of } \vec{t}\} \ \cup \ \bigcup\{(x_i \setminus x_{i-1}) \times \{t_i\} \mid t_i \text{ is a } +\text{ve arc of } \vec{t}\}$$

—so a copy of the events $\bigcup_i x_i$ but with +ve events tagged by the +ve arc of T at which they occur—with order a copy of that $\bigcup_i x_i$ inherits from A with additional causal dependencies from (with x_{i-1}^- the set of −ve events in x_{i-1})

$$x_{i-1}^- \ \times \ ((x_i \setminus x_{i-1}) \times \{t_i\})$$

—making the +ve events occur after the −ve events which precede them in the alternating sequence.

Define the partial order $\mathcal{Q}$ as follows. Its elements are posets q, not necessarily finite, where for some sub-branch $(t_1, t_2, \cdots, t_i, \cdots)$ of T there is a *rigid inclusion* $q \hookrightarrow q(t_1, t_2, \cdots, t_i, \cdots)$, *i.e.* if $q(\vec{t}) \in \mathcal{Q}$ and $q \hookrightarrow q(\vec{t})$ is a rigid inclusion (regarded as a map of event structures) then $q \in \mathcal{Q}$. The order on $\mathcal{Q}$ is that of rigid inclusion. Define the function $\sigma : \mathcal{Q} \to \mathcal{C}^\infty(A)$ by taking

$$\sigma q = \{a \in A \mid a \text{ is } -\text{ve } \& \ a \in q\} \ \cup \ \{a \in A \mid \exists t \in T.\ a \text{ is } +\text{ve } \& \ (a, t) \in q\}$$

for $q \in \mathcal{Q}$. We check $\sigma q \in \mathcal{C}^\infty(A)$. Clearly, we have that $\sigma q(\vec{t}) = \bigcup_{i \in I} x_i$ where $\emptyset \ \cdots \ \overset{t_{i-1}}{\subset^-} x_{i-1} \overset{t_i}{\subset^+} x_i \overset{t_{i+1}}{\subset^-} \cdots$ is the tagged alternating sequence determined by $\vec{t} =_{\text{def}} (t_1, \cdots, t_i, \cdots)$. Any q for which there is a *rigid inclusion* $q \hookrightarrow q(\vec{t})$, *i.e.* which preserves causal dependency, is sent to a sub-configuration of $\bigcup_i x_i$.

Proposition 14. *Let* $str : T \to TA$ *be a strategy in the tree game* $TG(A, W)$ *and let* $(t_1, \cdots, t_i, \cdots)$ *be a sub-branch of* T*, so corresponding to some configuration* $\{t_1, \cdots, t_i, \cdots\} \in \mathcal{C}^\infty(T)$*. Then,*

$$str\{t_1, \cdots, t_i, \cdots\} \in TW \iff \sigma q(t_1, \cdots, t_i, \cdots) \in W\,.$$

The following proposition justifies writing $\subseteq$ for the order of $\mathcal{Q}$.

Proposition 15. *For all* $q, q' \in \mathcal{Q}$*, whenever there is an inclusion of the events of* q *in the events of* q' *there is a rigid inclusion* $q \hookrightarrow q'$*.*

The next lemma is crucial and depends critically on (A, W) being *race-free* and *bounded-concurrent*.

Lemma 16. *The order* $(\mathcal{Q}, \subseteq)$ *is a prime algebraic domain in which the primes are precisely those (necessarily finite) partial orders in* $\mathcal{Q}$ *with a top element.*

Proof. (Sketch) Any compatible finite subset X of $\mathcal{Q}$ has a least upper bound: if all the members of X include rigidly in a common q then taking the union of their images in q, with order inherited from q, provides their least upper bound. Provided $\mathcal{Q}$ has least upper bounds of directed subsets it will then be consistently complete with the additional property that every $q \in \mathcal{Q}$ is the least upper bound of the primes below it—this will make $\mathcal{Q}$ a prime algebraic domain. It then remains to show that $\mathcal{Q}$ has least upper bounds of directed sets.

Let S be a directed subset of $\mathcal{Q}$. The +ve events of orders $q \in S$ are tagged by +ve arcs of T. As S is directed the +ve tags which appear throughout all $q \in S$ must determine a common sub-branch of T, *viz.*, $\vec{t} =_{\mathrm{def}} (t_1, t_2, \cdots, t_i, \cdots)$. Every +ve arc of the sub-branch appears in some $q \in S$ and all −ve arcs are present only by virtue of preceding a +ve arc. Forming the partial order $\bigcup S$ comprising the union of the events of all $q \in S$ with order the restriction of that on $q(\vec{t})$ we obtain a rigid inclusion $\bigcup S \hookrightarrow q(\vec{t})$ and so a least upper bound of S in $\mathcal{Q}$—from which prime algebraicity follows. □

Prime algebraic domains determine event structures in a simple way [8, 11]: define S to be the event structure with polarity, with events the primes of $\mathcal{Q}$; causal dependency the restriction of the order on $\mathcal{Q}$; with a finite subset of events consistent if they include rigidly in a common element of $\mathcal{Q}$. The polarity of events of S is the polarity in A of its top element (the event is a prime in $\mathcal{Q}$).

Define $\sigma_0 : S \to A$ to be the function which takes a prime with top element an untagged event $a \in A$ to a and top element a tagged event (a, t) to a.

Lemma 17. *The function which takes* $q \in \mathcal{Q}$ *to the set of primes below* q *in* $\mathcal{Q}$ *gives an order isomorphism* $\mathcal{Q} \cong \mathcal{C}^\infty(S)$. *The function* $\sigma_0 : S \to A$ *is a strategy for which the following commutes:*

$$\begin{array}{ccc} \mathcal{Q} & \cong & \mathcal{C}^\infty(S) \\ {\scriptstyle\sigma}\downarrow & \swarrow{\scriptstyle\sigma_0} & \\ \mathcal{C}^\infty(A) & & \end{array}$$

We obtain a winning strategy in a concurrent game from a winning strategy in its tree game:

Theorem 18. *Suppose that* $str : T \to TA$ *is a winning strategy in the tree game* $\mathrm{TG}(A, W)$. *Then* $\sigma_0 : S \to A$ *is a winning strategy in* (A, W).

Proof. (Sketch) For σ_0 to be a winning strategy we require that $\sigma_0 x \in W$ for every $\oplus$-maximal $x \in \mathcal{C}^\infty(S)$. Via the order isomorphism $\mathcal{Q} \cong \mathcal{C}^\infty(S)$ (Lemma 17) we can carry out the proof in $\mathcal{Q}$ rather than $\mathcal{C}^\infty(S)$. For any q which is $\oplus$-maximal in $\mathcal{Q}$ (*i.e.* whenever $q \subseteq^+ q'$ in $\mathcal{Q}$ then $q = q'$) we require that $\sigma q \in W$.

Letting q be $\oplus$-maximal in $\mathcal{Q}$, because there is a rigid inclusion $q \hookrightarrow q(\vec{t})$ for some sub-branch $\vec{t} = (t_1, \cdots, t_i, \cdots)$ of T, we can show that $q = q(\vec{u})$ for some $\oplus$-maximal branch $\vec{u}$ of T. This implies that its image $str\{\vec{u}\}$ is in TW, as str is a winning strategy in $\mathrm{TG}(A, W)$. By Proposition 14, we have that $str\{\vec{u}\} \in TW \iff \sigma q(\vec{u}) \in W$. Hence, $\sigma q \in W$, as required. □

Corollary 19. *Let* (A, W) *be a race-free, bounded-concurrent game. If the tree game* $\mathrm{TG}(A, W)$ *has a winning strategy, then* (A, W) *has a winning strategy.*

As TG respects duality, a winning counter-strategy for $\mathrm{TG}(A, W)$ determines a winning counter-strategy for (A, W). Corollary 19 and Theorem 9 guarantee winning strategies in (A, W) from winning strategies in $\mathrm{GS}(\mathrm{TG}(A, W))$. We can now establish a *concurrent* analogue of Martin's determinacy theorem [6].

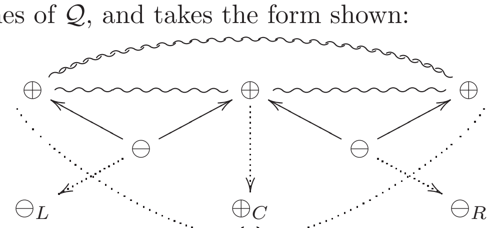
Theorem 20 (Concurrent Borel determinacy). *Any race-free, bounded-concurrent game* (A, W)*, in which* W *is a Borel subset of* $\mathcal{C}^\infty(A)$*, is determined.*

We illustrate the construction of Theorem 18, how a winning strategy for a concurrent game is built from that of its tree game.

Example 21. Let (A, W) be a concurrent game where A is $\ominus_L\, co \oplus_C co \ominus_R$ and the set $\{\emptyset, \{\ominus_L, \oplus_C\}, \{\ominus_R, \oplus_C\}, \{\ominus_L, \ominus_R, \oplus_C\}\}$ is W, that is, Player's winning conditions in A. Player has a winning strategy. The maximal alternating sequences upon which the tree game $\mathrm{TG}(A, W)$ is constructed are:

1. $t^1_{max} = \emptyset \subset^- \{\ominus_L\} \subset^+ \{\ominus_L, \oplus_C\} \subset^- \{\ominus_L, \oplus_C, \ominus_R\}$,
2. $t^2_{max} = \emptyset \subset^- \{\ominus_R\} \subset^+ \{\ominus_R, \oplus_C\} \subset^- \{\ominus_R, \oplus_C, \ominus_L\}$,
3. $t^3_{max} = \emptyset \subset^- \{\ominus_L, \ominus_R\} \subset^+ \{\ominus_L, \ominus_R, \oplus_C\}$.

Its winning configurations correspond to those sub-branches terminating in W. It has a winning strategy str is given by the identity function on $\mathrm{TG}(A, W)$. We construct a winning strategy via a prime algebraic domain $\mathcal{Q}$ which has elements partial orders built out of tagged alternating sequences determined by str. In this example str is deterministic so the tagging plays no essential role and we can build the partial orders in $\mathcal{Q}$ from the alternating sequences above. The three maximal alternating sequences above are associated with the following partial orders on the three events $\{\ominus_L, \oplus_C, \ominus_R\}$: the first with just $\ominus_L \rightarrowtriangle \oplus_C$; the second with $\ominus_R \rightarrowtriangle \oplus_C$; and the last with both $\ominus_L \rightarrowtriangle \oplus_C$ and $\ominus_R \rightarrowtriangle \oplus_C$. There are other partial orders in $\mathcal{Q}$ associated with sub-branches. The event structure S of the winning non-deterministic concurrent strategy $\sigma_0 : S \to A$ is built from the complete primes of $\mathcal{Q}$, and takes the form shown:



Wiggly lines denote conflict and the dotted arrows the map σ_0. □

6 Concluding Remarks

Event structures have a central status within models for concurrency, both 'interleaving' and 'partial-order' based, and are formally related to other models by adjunctions. One can expect this central status to be inherited within games. Indeed, working with such a detailed model exposes new structure and new subtleties, which readily appear when studying determinacy issues.

For instance, in traditional 'interleaving' games on graphs or trees, both race-freedom and bounded-concurrency hold implicitly. At each vertex every player makes a simple choice *independently* of the others, implying race-freedom. Strategies are generally defined as functions from partial plays to partial plays via *rounds* which ensures bounded-concurrency. Round-free asynchrony, not studied before, makes the determinacy problem considerably harder.

Our determinacy result is, in a sense, the *strongest* one can hope to obtain (with respect to the descriptive complexity of the winning sets) for concurrent games on event structures—and hence on partial orders—since any generalisation of the winning sets would require an extension of the determinacy theorem by Martin [6]—well known to be at the limits of traditional set theory.

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