Prime Algebraicity

Glynn Winskel, University of Cambridge Computer Laboratory, England

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Abstract

A prime algebraic lattice can be characterised as isomorphic to the downwards-closed subsets, ordered by inclusion, of its complete primes. It is easily seen that the downwards-closed subsets of a partial order form a completely distributive algebraic lattice when ordered by inclusion. The converse also holds; any completely distributive algebraic lattice is isomorphic to such a set of downwards-closed subsets of a partial order. The partial order can be recovered from the lattice as the order of the lattice restricted to its complete primes. Consequently prime algebraic lattices are precisely the completely distributive algebraic lattices. The result extends to Scott domains. Several consequences are explored briefly: the representation of Berry's dI-domains by event structures; a simplified form of information systems for completely distributive Scott domains; and a simple domain theory for concurrency.

Introduction

It is 30 years since Mogens Nielsen, Gordon Plotkin and I introduced prime algebraic lattices, and the more general prime algebraic domains, as an intermediary in relating Petri nets and Scott domains [19]. The recognition that prime algebraic lattices were well-known in another guise, that of completely distributive algebraic lattices, came a little later, partly while I was a postdoc visiting Mogens in Aarhus, with the final pieces falling into place early after my move to CMU in 1982. The first part of this article is essentially based on a CMU research report [25] from my time there.[1]

Since their introduction prime algebraic domains have come to play a significant role in several other areas and have broader relevance today. I would particularly like to draw attention to their part in: stable domain theory; a recent domain theory for concurrency; and a potential pedagogic role through providing a simplified form of information system with which to represent domains. These uses are summarised in section 3.

1 Prime algebraic lattices

1.1 Basic definitions

The following definitions are well-known, see *e.g.* [11, 8, 14].

For a partial order $\mathbb{L} = (L, \sqsubseteq)$, the *covering relation* $-\!\!\!<$ is defined by

$$x -\!\!\!< y \iff x \sqsubseteq y \ \&\ x \neq y \ \&\ (\forall z.\ x \sqsubseteq z \sqsubseteq y \Rightarrow x = z \text{ or } z = y)$$

for $x, y \in L$.

Recall a *directed set* of a partial order $(L, \sqsubseteq)$ is a non-null subset $S \subseteq L$ such that $\forall s, t \in S \exists u \in S.\ s \sqsubseteq u \ \&\ t \sqsubseteq u$.

A *complete lattice* is a partial order $\mathbb{L} = (L, \sqsubseteq)$ which has joins $\bigsqcup X$ and meets $\bigsqcap X$ of arbitrary subsets X of L. We write $x \sqcup y$ for $\bigsqcup \{x, y\}$, and $x \sqcap y$ for $\bigsqcap \{x, y\}$.

An *isolated* (*i.e. finite* or *compact*) element of a complete lattice $\mathbb{L} = (L, \sqsubseteq)$ is an element $x \in L$ such that for any directed subset $S \subseteq L$ when $x \sqsubseteq \bigsqcup S$ there is $s \in S$ such that $x \sqsubseteq s$. (In a computational framework the isolated elements are that information which a computation can realise—use or produce—in finite time—see [21].)

When there are enough isolated elements to form a basis, a complete lattice is said to be *algebraic i.e.* an *algebraic lattice* is a complete lattice $\mathbb{L} = (L, \sqsubseteq)$ for which $x = \bigsqcup \{e \sqsubseteq x \mid e \textit{ is isolated}\}$ for all $x \in L$.

Let $\mathbb{L} = (L, \sqsubseteq)$ be a complete lattice. We are interested in these distributivity laws:

$$\bigsqcap_{i \in I} \bigsqcup_{j \in J(i)} x_{i,j} = \bigsqcup_{f \in K} \bigsqcap_{i \in I} x_{i,f(i)} \tag{1}$$

[1] I am grateful to a reviewer for pointing out that the results of the CMU report are implicit in the earlier works of R-E. Hoffmann [13] and of J.D. Lawson [16], the latter of which has come to be known as the "Lawson duality for continuous posets".

where K is the set of functions $f : I \to \bigcup_{i \in I} J(i)$ such that $f(i) \in J(i)$; when $\mathbb{L}$ satisfies (1) it is said to be *completely distributive*.

$$(\bigsqcup X) \sqcap y = \bigsqcup\{x \sqcap y \mid x \in X\} \tag{2}$$

where $X \subseteq L$ and $y \in L$; condition (2) is associated with $\mathbb{L}$ being a *complete Heyting algebra*.

$$(\bigsqcap X) \sqcup y = \bigsqcap\{x \sqcup y \mid x \in X\} \tag{3}$$

where $X \subseteq L$ and $y \in L$.

$$(x \sqcup y) \sqcap z = (x \sqcap z) \sqcup (y \sqcap z) \tag{4}$$

where $x, y, z \in L$. This finite distributive law is equivalent to its dual in a complete lattice—see [11]. Note that simple arguments by induction show that (4) implies finite versions of (1)—in which the indexing sets are restricted to be finite—and (2) and (3)—in which the set X is restricted to be finite.

Clearly if a complete lattice is completely distributive, *i.e.* satisfies (1), then it also satisfies (2), (3) and (4).

The following definitions are perhaps less standard. Given a partial order $\mathbb{P}$, we shall order the set of downwards-closed subsets of $\mathbb{P}$ by inclusion. The points of $\mathbb{P}$ can be recovered as the complete primes in this order of downwards-closed subsets. (At the time of [19] we were thinking of structures like $\mathbb{P}$, and accompanying structures of downwards-closed subsets, as associated with sets of events ordered by a causal dependency relation—see [19, 23, 24, 7]. Their application is broader nowadays.)

Definition: Let $\mathbb{P} = (P, \leq)$ be a partial order. A subset X of P is *downwards-closed* iff

$$p' \leq p \in X \Rightarrow p' \in X$$

for $p, p' \in P$.

Let X be a subset of P. Define the *downwards-closure* of X to be

$$\downarrow X =_{def} \{p' \in P \mid \exists p \in X.\ p' \leq p\}.$$

By convention we write $\downarrow p$ for $\downarrow\{p\} = \{p' \in P \mid p' \leq p\}$ when $p \in P$.

Definition: Let $\mathbb{L} = (L, \sqsubseteq)$ be a complete lattice. A *complete prime* of $\mathbb{L}$ is an element $p \in L$ such that

$$p \sqsubseteq \bigsqcup X \Rightarrow \exists x \in X.\ p \sqsubseteq x\,.$$

The lattice $\mathbb{L}$ is *prime algebraic* iff $x = \bigsqcup\{p \sqsubseteq x \mid p \textit{ is a complete prime}\}$, for all $x \in L$.

The definition of prime algebraic was introduced in [19]. However, it turns out that the concept was already familiar in another guise; for complete lattices it is equivalent to algebraicity with complete distributivity.

1.2 Characterisations

Firstly we recall a theorem from [19]. A prime algebraic complete lattice can always be represented, to within isomorphism, as the lattice, ordered by inclusion, of the downwards-closed subsets of its complete primes.

Theorem 1 *(i) Let $\mathbb{P} = (P, \leq)$ be a partial order. Its downwards-closed subsets ordered by inclusion, $(\mathcal{L}(\mathbb{P}), \subseteq)$, form a prime algebraic complete lattice; the complete primes of $(\mathcal{L}(\mathbb{P}), \subseteq)$ have the form $\downarrow p$ for $p \in P$. The partial order $\mathbb{P}$ is isomorphic to $(\{\downarrow p \mid p \in P\}, \subseteq)$, the restriction of the ordering on downwards-closed subsets to the complete primes, with isomorphism given by the map $p \mapsto \downarrow p$, for $p \in P$.*

(ii) Let $\mathbb{L} = (L, \sqsubseteq)$ be a prime algebraic complete lattice. Let $\mathbb{P} = (P, \leq)$ be the partial order consisting of the complete primes of $\mathbb{L}$ ordered by the restriction $\leq = \sqsubseteq \upharpoonright P$ of $\sqsubseteq$ to P. Then $\theta : (\mathcal{L}(\mathbb{P}), \subseteq) \cong \mathbb{L}$ where $\theta(X) = \bigsqcup X$ for $X \in \mathcal{L}(\mathbb{P})$, with inverse ϕ given by $\phi(x) = \{p \in P \mid p \sqsubseteq x\}$ for $x \in L$.

Proof: (i) Let $\mathbb{P} = (P, \leq)$ be a partial order. It is easy to see that $\mathcal{L}(\mathbb{P})$ is a complete lattice in which joins are unions and meets are intersections.

Suppose x is a complete prime of $(\mathcal{L}(\mathbb{P}), \subseteq)$. Then obviously

$$x = \bigcup \{\downarrow p \mid p \in x\}$$

which implies $x = \downarrow p$ for some $p \in P$. To see the converse, consider an element of the form $\downarrow p$, for $p \in P$. If $\downarrow p \subseteq \bigcup X$ for $X \subseteq \mathcal{L}(\mathbb{P})$ then $p \in x$ for some $x \in X$. But x is downwards-closed so $\downarrow p \subseteq x$. Thus $\downarrow p$ is a complete prime.

It is easy to see that the map $p \mapsto \downarrow p$, for $p \in P$, is an order isomorphism between $\mathbb{P}$ and $(\{\downarrow p \mid p \in P\}, \subseteq)$.

(ii) Let $\mathbb{L} = (L, \sqsubseteq)$ be a prime algebraic complete lattice. Let $\mathbb{P} = (P, \leq)$ be the complete primes of $\mathbb{L}$ ordered by the restriction of $\sqsubseteq$.

Obviously the maps θ and ϕ are monotonic *i.e.* order preserving. We show they are mutual inverses and so give the required isomorphism.

Firstly we show $\theta \circ \phi = 1$. Thus we require $x = \bigsqcup\{p \in P \mid p \sqsubseteq x\}$ for all $x \in L$. But this is just the condition of prime algebraicity.

Now we show $\phi \circ \theta = 1$. Let $X \in \mathcal{L}(P, \leq)$. We require $X = \phi \circ \theta(X)$ *i.e.* $X = \{p \in P \mid p \sqsubseteq \bigsqcup X\}$. Clearly $X \subseteq \{p \in P \mid p \sqsubseteq \bigsqcup X\}$. Conversely if $p \sqsubseteq \bigsqcup X$, where p is a complete prime, then certainly $p \sqsubseteq q$ for some $q \in X$. However X is downwards-closed so $p \in X$, showing the converse inclusion.

Thus we have established the required isomorphism. $\square$

Corollary 2 *A prime algebraic complete lattice is completely distributive (and so satisfies the distributive laws (2), (3) and (4), as well as (1)).*

Proof: The distributive laws clearly hold for downwards-closed subsets ordered by inclusion and these represent all the prime algebraic complete lattices to within isomorphism. $\square$

The next step is to show the prime algebraic complete lattices are the completely distributive algebraic lattices. A key idea is that algebraicity implies a form of discreteness; any distinct comparable pair of elements of an algebraic lattice are separated by a covering interval. The proof uses Zorn's lemma.

Lemma 3 *Let $\mathbb{L} = (L, \sqsubseteq)$ be an algebraic lattice. Then*

$$\forall x, y \in \mathbb{L}.\ x \sqsubseteq y \ \&\ x \neq y \Rightarrow \exists z, z' \in \mathbb{L}.\ x \sqsubseteq z \multimap z' \sqsubseteq y.$$

Proof: Suppose x, y are distinct elements of $\mathbb{L}$ such that $x \sqsubseteq y$. Because $\mathbb{L}$ is algebraic there is an isolated element b such that $b \not\sqsubseteq x$ & $b \sqsubseteq y$. By Zorn's lemma there is a maximal chain C of elements above x and strictly below $x \sqcup b$. If $\bigsqcup C = x \sqcup b$, then as C is directed we would have $c \sqsupseteq b$, so $c = x \sqcup b$, for some $c \in C$—a contradiction. Hence $\bigsqcup C \sqsubset x \sqcup b$, and by the maximality of C we must have $\bigsqcup C \multimap x \sqcup b$ yielding $x \sqsubseteq \bigsqcup C \multimap x \sqcup b \sqsubseteq y$. $\square$

In proving the next theorem we use such coverings to construct complete primes of a lattice. The distributive laws (2) and (3)—implied of course by (1)—make it possible to find $\sqsubseteq$– minimum coverings which correspond to complete primes. Algebraicity ensures there are enough covering intervals, and so complete primes, for the lattice to be prime algebraic.

Theorem 4 *Let $\mathbb{L}$ be a complete lattice. Then $\mathbb{L}$ is prime algebraic iff it is algebraic and satisfies the distributive laws (2) and (3).*

Proof: "*only if*": Let $\mathbb{L}$ be a prime algebraic complete lattice. Let $\mathbb{P}$ be the ordering of $\mathbb{L}$ restricted to its complete primes. By the previous theorem we know $\mathbb{L} \cong \mathcal{L}(\mathbb{P})$ so it is sufficient to prove properties for $\mathcal{L}(\mathbb{P})$. We have already seen the distributivity laws follow from the corresponding laws for sets.

The isolated elements of $\mathcal{L}(\mathbb{P})$ are easily shown to be precisely the downwards-closures of finite subsets of P. Suppose $x \in \mathcal{L}(\mathbb{P})$ is isolated. Obviously $x = \bigcup\{\downarrow X \mid X \subseteq_{\text{fin}} x\}$. But the set $\{\downarrow X \mid X \subseteq_{\text{fin}} x\}$ is clearly directed so, because x is isolated, $x = \downarrow X$ for some finite set $X \subseteq P$. Conversely, it is clear that an element of the form $\downarrow X$, for a finite $X \subseteq P$, is necessarily isolated; if $\downarrow X \subseteq \bigcup S$ for a directed subset S of $\mathcal{L}(\mathbb{P})$ then X, and so $\downarrow X$, is included in the union of a finite subset of S, and so in an element of S. Clearly now every element of $\mathcal{L}(\mathbb{P})$ is the least upper bound of the isolated elements below it, making $\mathcal{L}(\mathbb{P})$ algebraic.

Thus $\mathbb{L}$ is an algebraic lattice satisfying the distributive laws (1), (2), (3) and (4).

"*if*": Let $\mathbb{L} = (L, \sqsubseteq)$ be an algebraic lattice satisfying the distributive laws (2) and (3).

Let $x \multimap\!\!\subset x'$ in $\mathbb{L}$. Define

$$pr[x, x'] = \bigsqcap\{y \in L \mid x' \leq x \sqcup y\}.$$

We show $p = \text{pr}[x, x']$ is a complete prime of $\mathbb{L}$. Note first that

$$x \sqcup p = \bigsqcap\{x \sqcup y \mid x' \sqsubseteq x \sqcup y\} = x'$$

by distributive law (3). Now suppose $p \sqsubseteq \bigsqcup Z$ for some $Z \subseteq L$. Then

$$p = (\bigsqcup Z) \sqcap p = \bigsqcup\{z \sqcap p \mid z \in Z\}$$

by the distributive law (2). Write $Z' = \{z \sqcap p \mid z \in Z\}$, so $p = \bigsqcup Z'$. Then

$$x' = x \sqcup p = x \sqcup (\bigsqcup Z') = \bigsqcup\{x \sqcup z' \mid z' \in Z'\}.$$

Clearly $x \sqsubseteq x \sqcup z' \sqsubseteq x'$ for all $z \in Z'$. As $x \multimap\!\!\subset x'$ we must have $x' = x \sqcup z'$ for some $z' \in Z'$; otherwise $x = x \sqcup z'$ for all $z' \in Z'$ giving the contradiction $x = \bigsqcup\{x \sqcup z' \mid z' \in Z'\} = x'$. But then $p \sqsubseteq z'$ from the definition of p. However $z' = z \sqcap p$ for some $z \in Z$. Therefore $p \sqsubseteq z$ for some $z \in Z$. Thus p is a complete prime of $\mathbb{L}$.

That $\mathbb{L}$ is prime algebraic follows provided for $z \in L$, we have

$$z = \bigsqcup\{pr[x, x'] \mid x \longrightarrow\!\!\!\!\subset x' \sqsubseteq z\}.$$

Let $z \in L$. Write

$$w = \bigsqcup\{pr[x, x'] \mid x \longrightarrow\!\!\!\!\subset x' \sqsubseteq z\}.$$

Clearly $w \sqsubseteq z$. Suppose $w \neq z$. Then, by the lemma,

$$w \sqsubseteq x \longrightarrow\!\!\!\!\subset x' \sqsubseteq z$$

for some $x, x' \in L$. Write $p = pr[x, x']$. Then $p \sqsubseteq w$ making $x \sqcup p = x$, a contradiction as $x \sqcup p = x'$. Thus each element of $\mathbb{L}$ is the least upper bound of the complete primes below it, as required.

Thus we have established the required equivalence between prime algebraic complete lattices and algebraic lattices satisfying (2) and (3). □

Corollary 5 *Let $\mathbb{L}$ be a complete lattice. The following are equivalent:*
(i) *$\mathbb{L}$ is prime algebraic,*
(ii) *$\mathbb{L}$ is isomorphic to $(\mathcal{L}(\mathbb{P}), \subseteq)$ for some partial order $\mathbb{P}$,*
(iii) *$\mathbb{L}$ is algebraic and completely distributive,*
(iv) *$\mathbb{L}$ is algebraic and satisfies the distributive laws (2) and (3).*

Proof: Combining previous results. □

1.3 The finitary case

In the special case when the algebraic lattice satisfies a finiteness restriction we can obtain a similar representation of algebraic lattices mentioning just the finite distributive law (4). The finiteness restriction says every isolated element dominates only a finite number of elements. The corresponding axiom has been called *axiom F*, sometimes *axiom I*, in [15, 4, 23]. (This restriction arises naturally for computations. When a partial order models the events and causal dependency relation of a computation it is generally true that an event is causally dependent on only a finite set of events. The associated downwards-closed subsets then satisfy the finiteness restriction.)

Definition: An algebraic lattice $\mathbb{L} = (L, \sqsubseteq)$ is said to satisfy *axiom F* when $\{y \in L \mid y \sqsubseteq x\}$ is finite for all isolated elements $x \in L$.

Theorem 6 *Let $\mathbb{L}$ be an algebraic lattice which satifies axiom F. Then $\mathbb{L}$ is prime algebraic iff $\mathbb{L}$ satisfies the finite distributive law (4).*

Proof: The "only if" part follows from theorem 4. The converse, "if" part, follows from theorem 4 provided we can show that, in the presence of axiom F, the finite distributive law (4) implies the infinite distributive laws (2) and (3).

Let $\mathbb{L}$ be an an algebraic lattice which satifies axiom F and the finite distributive law (4).

We show that $\mathbb{L}$ satisfies the infinite distributive law (2). Let $X \subseteq L$ and $y \in L$. Clearly $\bigsqcup\{x \sqcap y \mid x \in X\} \sqsubseteq (\bigsqcup X) \sqcap y$. To show the converse inequality, suppose b is isolated and $b \sqsubseteq (\bigsqcup X) \sqcap y$. Then as $b \sqsubseteq \bigsqcup X$ and b is isolated, for some finite $X' \subseteq_{\text{fin}} X$ we have $b \sqsubseteq \bigsqcup X'$. Thus

$$\begin{aligned} b \sqsubseteq (\textstyle\bigsqcup X) \sqcap y \quad &\Rightarrow \quad b \sqsubseteq (\textstyle\bigsqcup X') \sqcap y \\ &\Rightarrow \quad b \sqsubseteq \textstyle\bigsqcup\{x \sqcap y \mid x \in X'\} \text{ (by the finite distributive law (4))} \\ &\Rightarrow \quad b \sqsubseteq \textstyle\bigsqcup\{x \sqcap y \mid x \in X\}. \end{aligned}$$

Therefore, because $\mathbb{L}$ is algebraic, we have the converse inequality. Combining the inequalities we obtain (2), $\bigsqcup\{x \sqcap y \mid x \in X\} = (\bigsqcup X) \sqcap y$.

Now we show $\mathbb{L}$ satisfies the infinite distributive law (3). Let $X \subseteq L$ and $y \in L$. Clearly

$$(\textstyle\bigsqcap X) \sqcup y \sqsubseteq \textstyle\bigsqcap \{x \sqcup y \mid x \in X\}.$$

We require the converse inequality. Suppose b is isolated and

$$b \sqsubseteq \textstyle\bigsqcap \{x \sqcup y \mid x \in X\}.$$

Then

$$\begin{aligned} b = (\textstyle\bigsqcap \{x \sqcup y \mid x \in X\}) \sqcap b &= \textstyle\bigsqcap \{(x \sqcup y) \sqcap b \mid x \in X\} \\ &= \textstyle\bigsqcap \{(x \sqcap b) \sqcup (y \sqcap b) \mid x \in X\}. \end{aligned}$$

Now b dominates only a finite number of elements. Thus there is some finite subset $X' \subseteq_{\text{fin}} X$ for which

$$\{x \sqcap b \mid x \in X'\} = \{x \sqcap b \mid x \in X\}.$$

So in addition,

$$b = \textstyle\bigsqcap \{(x \sqcap b) \sqcup (y \sqcap b) \mid x \in X\} = \textstyle\bigsqcap \{(x \sqcap b) \sqcup (y \sqcap b) \mid x \in X'\}.$$

Now by the finite distributive law (4),

$$\begin{aligned} b =& (\bigsqcap \{x \sqcap b \mid x \in X'\}) \sqcup (y \sqcap b) \\ =& (\bigsqcap \{x \sqcap b \mid x \in X\}) \sqcup (y \sqcap b) \\ =& (\bigsqcap X \sqcap b) \sqcup (y \sqcap b) \sqsubseteq (\bigsqcap X) \sqcup y. \end{aligned}$$

By algebraicity we obtain

$$\bigsqcap \{x \sqcup y \mid x \in X\} \sqsubseteq (\bigsqcap X) \sqcup y.$$

Combining the inequalities we obtain (3). □

2 Prime algebraic domains

2.1 Basic Definitions

Let $\mathbb{D} = (D, \sqsubseteq)$ be a partial order.

Say a set $X \subseteq D$ is *compatible* iff it has an upper bound. Say a set $X \subseteq D$ is *finitely compatible* iff if every finite $Y \subseteq X$ is compatible. In particular, a directed set is finitely compatible.

Say $\mathbb{D}$ is *consistently complete* iff every finitely compatible subset $X \subseteq D$ has a least upper bound $\bigsqcup X$. Note a consistent complete partial order has a least element, *viz.* $\bot = \bigsqcup \emptyset$, though it may not have a greatest. Note any consistent complete partial order has greatest lower bounds of any nonempty subsets: for X nonempty, $\bigsqcap X = \bigsqcup \{d \mid \forall x \in X.\ d \sqsubseteq x\}$.

As before, when $\mathbb{D} = (D, \sqsubseteq)$ is consistently complete, an *isolated* element is an element $x \in D$ such that for any directed subset $S \subseteq D$ when $x \sqsubseteq \bigsqcup S$ there is $s \in S$ such that $x \sqsubseteq s$.

$\mathbb{D}$ is a *Scott domain* (or simply a *domain*) iff it is consistently complete and algebraic in the sense that $x = \bigsqcup \{e \sqsubseteq x \mid e \text{ is isolated}\}$ for all $x \in D$.

In the case where $\mathbb{D} = (D, \sqsubseteq)$ is a consistently complete partial order, not all subsets have joins so we need to modify the statement of the distributivity laws:

$$\bigsqcap_{i \in I} \bigsqcup_{j \in J(i)} x_{i,j} = \bigsqcup_{f \in K} \bigsqcap_{i \in I} x_{i,f(i)} \qquad (1')$$

where K is the set of functions $f : I \to \bigcup_{i \in I} J(i)$ such that $f(i) \in J(i)$, provided the sets $\{x_{i,j} \mid j \in J(i)\}$ are compatible for all $i \in I$; when $\mathbb{D}$ satisfies

(1′) it is said to be *completely distributive*.

$$(\bigsqcup X) \sqcap y = \bigsqcup\{x \sqcap y \mid x \in X\} \tag{2′}$$

provided X is a compatible subset of D and $y \in D$.

$$(\bigsqcap X) \sqcup y = \bigsqcap\{x \sqcup y \mid x \in X\} \tag{3′}$$

where $X \subseteq D$ and $y \in D$, provided $\{x, y\}$ is compatible for all $x \in X$.

$$(x \sqcup y) \sqcap z = (x \sqcap z) \sqcup (y \sqcap z) \tag{4′}$$

where $x, y, z \in D$, provided $\{x, y\}$ is compatible. Now, this finite distributive law implies its 'dual' while the converse implication does not hold.

Let $\mathbb{D} = (D, \sqsubseteq)$ be a consistently complete partial order. A *complete prime* of $\mathbb{D}$ is an element $p \in D$ such that for any compatible $X \subseteq D$

$$p \sqsubseteq \bigsqcup X \Rightarrow \exists x \in X.\ p \sqsubseteq x\,.$$

The partial order $\mathbb{D}$ is a *prime algebraic domain* iff

$$x = \bigsqcup\{p \sqsubseteq x \mid p \textit{ is a complete prime}\}\,,$$

for all $x \in D$.

Complete primes now inherit a consistency relation as well as a partial order from their ambient prime algebraic domain. This suggests a representation of prime algebraic domains in terms of partial orders with a consistency relation. It is useful to relax the definition to a *preorder* with consistency.

2.1.1 Preorders with consistency

A *preorder with consistency* comprises $\mathbb{P} = (P, \lesssim_{\mathbb{P}}, \mathrm{C}on_{\mathbb{P}})$ where $(P, \lesssim_{\mathbb{P}})$ is a preorder and $\mathrm{C}on_{\mathbb{P}}$ is a family of finite subsets of P such that

$$\begin{array}{l} \{p\} \in \mathrm{C}on_{\mathbb{P}}\,, \\ Y \subseteq X \in \mathrm{C}on_{\mathbb{P}} \Rightarrow Y \in \mathrm{C}on_{\mathbb{P}}\,, \quad \text{and} \\ X \in \mathrm{C}on_{\mathbb{P}} \;\&\; p' \lesssim_{\mathbb{P}} p \in X \Rightarrow X \cup \{p'\} \in \mathrm{C}on_{\mathbb{P}} \end{array}$$

for all $p \in P$ and finite $X, Y \subseteq P$. We say the preorder with consistency is *finitary* iff each set $\{p' \in P \mid p' \lesssim_{\mathbb{P}} p\}$ is finite for $p \in P$. A *partial order with consistency* meets the additional requirement that the preorder is a partial order. We can identify a preorder (partial order) with the preorder (partial order) with consistency comprising all finite subsets.

Let $\mathbb{P} = (P, \lesssim_{\mathbb{P}}, \mathrm{Con}_{\mathbb{P}})$ be a preorder with consistency. The preorder determines an equivalence relation $\simeq_{\mathbb{P}}$ on P, given by $p \simeq_{\mathbb{P}} p'$ iff $p \lesssim_{\mathbb{P}} p'$ and $p' \lesssim_{\mathbb{P}} p$. Quotiented by $\simeq_{\mathbb{P}}$, the preorder with consistency $\mathbb{P}$ yields a partial order with consistency $\mathbb{P}/{\simeq_{\mathbb{P}}} = (P/{\simeq_{\mathbb{P}}}, \lesssim_{\mathbb{P}}/{\simeq_{\mathbb{P}}}, \mathrm{Con})$, where

$$Y \in \mathrm{Con} \ \text{ iff } \ \exists X \in \mathrm{Con}_{\mathbb{P}}.\ Y = X/{\simeq_{\mathbb{P}}}\,.$$

An *equivalence* of preorders with consistency $\mathbb{P} = (P, \lesssim_{\mathbb{P}}, \mathrm{Con}_{\mathbb{P}})$ and $\mathbb{Q} = (Q, \lesssim_{\mathbb{Q}}, \mathrm{Con}_{\mathbb{Q}})$ is a function $f : P \to Q$ such that

$$\begin{array}{ll} \forall p, p' \in P.\ \ p \lesssim_{\mathbb{P}} p' \iff f(p) \lesssim_{\mathbb{Q}} f(p')\,, & \\ \forall X \subseteq P.\ \ X \in \mathrm{Con}_{\mathbb{P}} \iff fX \in \mathrm{Con}_{\mathbb{Q}}\,, & \text{and} \\ \forall q \in Q \exists p \in P.\ f(p) \simeq_{\mathbb{Q}} q\,. & \end{array}$$

When such a map exists we say $\mathbb{P}$ is *equivalent* to $\mathbb{Q}$. If $\mathbb{P}$ is equivalent to $\mathbb{Q}$, then $\mathbb{Q}$ is equivalent to $\mathbb{P}$, via any function $g : Q \to P$ for which $g(q) = p$ for some choice of p with $f(p) \simeq_{\mathbb{Q}} q$. A preorder with consistency is equivalent to its quotient, a partial order with consistency.

Let $\mathbb{P} = (P, \lesssim_{\mathbb{P}}, \mathrm{Con}_{\mathbb{P}})$ be a preorder with consistency. As before we say $x \subseteq A$ is downwards-closed iff

$$p' \lesssim_{\mathbb{P}} p \in x \ \Rightarrow \ p' \in x$$

for all $p, p' \in P$. In addition, we say x is *consistent* iff for all finite subsets $X \subseteq x$ we have $X \in \mathrm{Con}_{\mathbb{P}}$. Write $\mathcal{L}(\mathbb{P})$ for the collection of all downwards-closed, consistent sets of $\mathbb{P}$. As earlier, we adopt the notation $\downarrow X$ for the *downwards-closure* w.r.t. $\lesssim_{\mathbb{P}}$ of $X \subseteq A$ and $\downarrow p$ for the downwards-closure of a singleton $\{p\}$ in P. A simple argument shows that if $X \in \mathrm{Con}_{\mathbb{P}}$ then $\downarrow X$ is consistent, so $\downarrow X \in \mathcal{L}(\mathbb{P})$.

Equivalent preorders with consistency represent isomorphic prime algebraic domains (the converse follows from theorem 8 below):

Proposition 7 *Let $\mathbb{P} = (P, \lesssim_{\mathbb{P}}, \mathrm{Con}_{\mathbb{P}})$ and $\mathbb{Q} = (Q, \lesssim_{\mathbb{Q}}, \mathrm{Con}_{\mathbb{Q}})$ be equivalent preorders with consistency, with equivalence $f : P \to Q$. Their downwards-closed consistent sets $(\mathcal{L}(\mathbb{P}), \subseteq)$ and $(\mathcal{L}(\mathbb{Q}), \subseteq)$ are isomorphic, under $x \mapsto \downarrow(fx)$. In particular, $(\mathcal{L}(\mathbb{P}), \subseteq)$ is isomorphic to $(\mathcal{L}(\mathbb{P}/\simeq), \subseteq)$, under $x \mapsto x/\simeq$.*

2.2 Characterisations

Our first characterisation of prime algebraic domains appears in [19] for the case of binary conflict/consistency. (The slightly more general case below was considered in [23].) A prime algebraic domain can always be represented as the downwards-closed consistent subsets of a partial order with consistency obtained from its complete primes.

Theorem 8 *(i) Let $\mathbb{P} = (P, \lesssim, \mathrm{Con})$ be a preorder with consistency. Its consistent downwards-closed subsets ordered by inclusion, $(\mathcal{L}(\mathbb{P}), \subseteq)$, form a prime algebraic domain; the complete primes of $(\mathcal{L}(\mathbb{P}), \subseteq)$ have the form $\downarrow p$ for $p \in P$. The preorder with consistency $\mathbb{P}$ is equivalent to $(\{\downarrow p \mid p \in P\}, \subseteq, \uparrow)$, where $\uparrow$ is the relation of compatibility in $\mathcal{L}(\mathbb{P})$, restricted to the complete primes; the equivalence given by the function $p \mapsto \downarrow p$, for $p \in P$.*

(ii) Let $\mathbb{D} = (D, \sqsubseteq)$ be a prime algebraic domain. Let $\mathbb{P} = (P, \leq, \mathrm{Con})$ comprise the partial order with consistency consisting of the complete primes of $\mathbb{D}$ ordered by the restriction $\leq$ of $\sqsubseteq$ to P and consistency Con given by compatibility in $\mathbb{D}$. Then $\theta : (\mathcal{L}(\mathbb{P}), \subseteq) \cong \mathbb{D}$ where $\theta(X) = \bigsqcup X$ for $X \in \mathcal{L}(\mathbb{P})$, with inverse ϕ given by $\phi(x) = \{p \in P \mid p \sqsubseteq x\}$ for $x \in D$.

Proof: The proof follows closely that of theorem 1 with the additional consideration of consistency. □

We can adapt theorems 4 and 6 for lattices to domains via the following proposition.

Proposition 9 *Let $\mathbb{D} = (D, \sqsubseteq)$ be a consistently complete partial order satisfying the distributive law $(2')$. For $d \in D$, define $\mathbb{L}_d$ to be the partial order of $\mathbb{D}$ restricted to $\{x \in D \mid x \sqsubseteq d\}$. Then,*
(i) $\mathbb{D}$ is algebraic iff $\mathbb{L}_d$ is an algebraic lattice for all $d \in D$.
(ii) $\mathbb{D}$ is prime algebraic iff $\mathbb{L}_d$ is a prime algebraic lattice for all $d \in D$.

Proof: (i) The key point is that the isolated elements of each $\mathbb{L}_d$ are precisely those isolated elements of $\mathbb{D}$ below or equal to d. Clearly any such element of $\mathbb{D}$ is automatically isolated in $\mathbb{L}_d$. Conversely, if e is isolated in $\mathbb{L}_d$ and $e \sqsubseteq \bigsqcup S$ for directed $S \subseteq D$, then by distributivity $(2')$ we obtain $e \sqsubseteq \bigsqcup\{s \sqcap d \mid s \in S\}$, a directed join in $\mathbb{L}_d$. Hence $e \sqsubseteq s \sqcap d$, so $e \sqsubseteq s$, for some $s \in S$. Now (i) follows.

(ii) By an argument echoing that we have just seen but this time using compatible subsets in place of directed sets, we can show that the complete primes of $\mathbb{L}_d$ are precisely those complete primes of $\mathbb{D}$ below or equal to d. Now (ii) follows. □

Theorem 10 *Let $\mathbb{D} = (D, \sqsubseteq)$ be a consistently complete partial order. Then, $\mathbb{D}$ is prime algebraic iff it is a Scott domain and satisfies the distributive laws* $(2')$ *and* $(3')$*.*

Proof: "*only if*": Straightforwardly from the representation given by theorem 8(ii). "*if*": Assume that $\mathbb{D}$ is algebraic and satisfies the distributive laws $(2')$ and $(3')$. For $d \in D$, define $\mathbb{L}_d$ to be the partial order of $\mathbb{D}$ restricted to $\{x \in D \mid x \sqsubseteq d\}$. By (i) of proposition 9, $\mathbb{L}_d$ is an algebraic lattice satisfying the distributive laws (2) and (3). By theorem 4, each $\mathbb{L}_d$ is a prime algebraic lattice. By (ii) of proposition 9, $\mathbb{D}$ is prime algebraic. □

Combining previous results:

Corollary 11 *Let $\mathbb{D}$ be a consistently complete partial order. The following are equivalent:*

(i) *$\mathbb{D}$ is a prime algebraic domain,*

(ii) *$\mathbb{D}$ is isomorphic to $(\mathcal{L}(\mathbb{P}), \subseteq)$ for some partial order with consistency $\mathbb{P}$,*

(iii) *$\mathbb{D}$ is isomorphic to $(\mathcal{L}(\mathbb{P}), \subseteq)$ for some preorder with consistency $\mathbb{P}$,*

(iv) *$\mathbb{D}$ is a completely distributive Scott domain,*

(v) *$\mathbb{D}$ is a Scott domain and satisfies the distributive laws (2) and (3).*

2.3 The finitary case

Definition: A Scott domain $\mathbb{D} = (D, \sqsubseteq)$ is said to satisfy *axiom F* when $\{y \in D \mid y \sqsubseteq x\}$ is finite for all isolated elements $x \in D$.

Theorem 12 *Let $\mathbb{D}$ be a Scott domain which satifies axiom F. Then $\mathbb{D}$ is prime algebraic iff $\mathbb{D}$ satisfies the finite distributive law* $(4')$*.*

Proof: The "only if" part follows from theorem 10. The converse, "if" part, follows by considering the lattices $\mathbb{L}_d$ got by restricting $\mathbb{D}$ to $\{x \in D \mid x \sqsubseteq d\}$, for $d \in D$. By (i) of proposition 9, each lattice $\mathbb{L}_d$ is algebraic. Each $\mathbb{L}_d$ satisfies axiom F and the distributive law (4). By theorem 6, each $\mathbb{L}_d$ is prime algebraic. By (ii) of proposition 9, the domain $\mathbb{D}$ is prime algebraic. □

In an prime algebraic domain, axiom F clearly implies that a complete prime only dominates finitely many complete primes; the converse also holds as any isolated element is easily shown to be a join of finitely many complete primes. Thus prime algebraic domains satisfying axiom F are represented by *finitary* partial orders (or indeed preorders) with consistency. Gathering facts together we obtain the following corollary.

Corollary 13 *Let $\mathbb{D}$ be a consistently complete partial order. The following are equivalent:*

(i) *$\mathbb{D}$ is a prime algebraic domain satisfying axiom F,*
(ii) *$\mathbb{D}$ is isomorphic to $(\mathcal{L}(\mathbb{P}), \subseteq)$ for some finitary partial order with consistency $\mathbb{P}$,*
(iii) *$\mathbb{D}$ is isomorphic to $(\mathcal{L}(\mathbb{P}), \subseteq)$ for some finitary preorder with consistency $\mathbb{P}$,*
(iv) *$\mathbb{D}$ is a distributive Scott domain satisfying axiom F.*

3 Some consequences

3.1 dI-domains and event structures

In the late 1970's Gérard Berry uncovered dI-domains as those domains which supported a rich *stable* domain theory, based on the stable ordering between stable continuous functions [2, 3]. dI-domains are Scott domains which are distributive and satisfy axiom I (*i.e.* axiom F). Berry's aim was to move to a domain theory that caught operational aspects such as sequentiality of functions more accurately. Recognising that capturing sequentiality at higher-order required radically new machinery, he settled initially on the more modest goal of a domain theory based on stable functions.[2] Berry proceeded axiomatically, narrowing down to the axioms on dI-domains from various mathematical requirements, chief being that he end up with a cartesian-closed category.

His work coincided with the earliest work on event structures [18], and it took a while to realise that his dI-domains were precisely those domains represented by simple event structures. Now we can see this directly from corollary 13, once we recognise that finitary partial orders with consistency are just (prime) event structures [27]. (More general 'stable' event structures provide a more workable representation of the constructions of stable domain

[2]A function $f : \mathbb{D} \to \mathbb{D}'$ between dI-domains is *stable* iff it is continuous and $f(x \sqcap y) = f(x) \sqcap f(y)$ for all compatible $x, y \in \mathbb{D}$.

theory, in particular stable function space [27].) According to the event-structure representation, the order on Berry's dI-domains is revealed as a temporal order, with more information corresponding to the occurrence of more events.

Stable domain theory was rediscovered in the work of Jean-Yves Girard [9, 10], though for more restricted *qualitative domains* (where in the representation by partial orders with consistency, the partial order is just the identity relation); the subcategory in which consistency is determined in a binary fashion (by a *coherence*) played a key role in suggesting the structure of *classical linear logic*. Girard's use of qualitative domains in the semantics of polymorphism extends to dI-domains [6].

3.2 Simple information systems

Preorders with consistency can play the role of simplified information systems [22, 26], that is, provided one is content to restrict to completely distributive Scott domains. The advantage comes from the simplified form of entailment as just given by a preorder between 'tokens'. Such representations have been shown helpful in solving recursive domain equations and in proofs of adequacy and full abstraction—see [28, 20] for example.

Given preorders with consistency $\mathbb{P} = (P, \lesssim_{\mathbb{P}}, \mathrm{C}on_{\mathbb{P}})$ and $\mathbb{Q} = (Q, \lesssim_{\mathbb{Q}}, \mathrm{C}on_{\mathbb{Q}})$, define

$$\begin{aligned} \mathbb{P} \trianglelefteq \mathbb{Q} \quad \text{iff} \quad & P \subseteq Q \quad \text{and} \\ & \forall p, p' \in P.\ p \lesssim_{\mathbb{P}} p' \iff p \lesssim_{\mathbb{Q}} p' \quad \text{and} \\ & \forall X \subseteq P.\ X \in \mathrm{C}on_{\mathbb{P}} \iff X \in \mathrm{C}on_{\mathbb{Q}}\,. \end{aligned}$$

The empty preorder with consistency is the $\trianglelefteq$-least element and ω-chains of preorders with consistency have joins given by unions. In this sense preorders with consistency form a 'large complete partial order' when ordered by the substructure relation $\trianglelefteq$. The $\trianglelefteq$ relation gives a very concrete representation of embedding-projection pairs between domains. With a little care type operations become $\trianglelefteq$-continuous so that solving a recursive domain equation becomes simply a matter of finding the least fixed point of a continuous operation. The relaxation to *preorders* with consistency can be necessary to ensure the monotonicity of some constructions w.r.t. $\trianglelefteq$.

As examples, we look at the function spaces of join-preserving and continuous functions afforded by the representation of prime algebraic domains. Let $\mathbb{P} = (P, \lesssim_{\mathbb{P}}, \mathrm{C}on_{\mathbb{P}})$ and $\mathbb{Q} = (Q, \lesssim_{\mathbb{Q}}, \mathrm{C}on_{\mathbb{Q}})$ be preorders with consistency.

Join-preserving functions from $\mathbb{P}$ to $\mathbb{Q}$ are in 1-1 correspondence with certain relations between $\mathbb{P}$ and $\mathbb{Q}$. Such relations, $R \subseteq P \times Q$, satisfy

$$p \lesssim_{\mathbb{P}} p' \ \& \ p \, R \, q \ \& \ q \lesssim_{\mathbb{Q}} q' \ \Rightarrow \ p' \, R \, q' \quad \text{and}$$
$$p_1 \, R \, q_1 \ \& \ \cdots \ \& \ p_k \, R \, q_k \ \& \ \{p_1, \cdots, p_k\} \in \mathrm{Con}_{\mathbb{P}} \ \Rightarrow \ \{q_1, \cdots, q_k\} \in \mathrm{Con}_{\mathbb{Q}} \,.$$

The relations can be presented as elements of $\mathcal{L}(\mathbb{P} \multimap \mathbb{Q})$, where $\mathbb{P} \multimap \mathbb{Q}$, a preorder with consistency, is the function space of join-preserving functions, given as follows. It comprises $\mathbb{P} \multimap \mathbb{Q} =_{def} (P \times Q, \lesssim, \mathrm{Con})$ where

$$(p', q') \lesssim (p, q) \ \text{ iff } \ p \lesssim_{\mathbb{P}} p' \ \& \ q' \lesssim_{\mathbb{Q}} q \,, \quad \text{and}$$
$$\{(p_1, q_1), \cdots, (p_n, q_n)\} \in \mathrm{Con} \ \text{ iff}$$
$$\forall I \subseteq \{1, \cdots, n\}. \ \{p_i \mid i \in I\} \in \mathrm{Con}_{\mathbb{P}} \Rightarrow \{q_i \mid i \in I\} \in \mathrm{Con}_{\mathbb{Q}} \,.$$

The operation $\multimap$ is $\trianglelefteq$-continuous in both arguments.

We can build the function space of all continuous functions from that for join-preserving functions using an operation ! which on $\mathbb{P}$ produces $!\mathbb{P} = (\mathrm{Con}_{\mathbb{P}}, \lesssim_{!\mathbb{P}}, \mathrm{Con}_{!\mathbb{P}})$, where

$$X \lesssim_{!\mathbb{P}} Y \ \text{ iff } \ \forall p \in X \exists p' \in Y. \ p \lesssim_{\mathbb{P}} p' \,, \quad \text{and}$$
$$X \in \mathrm{Con}_{!\mathbb{P}} \ \text{ iff } \ \bigcup X \in \mathrm{Con}_{\mathbb{P}} \,.$$

The preorder with consistency $!\mathbb{P}$ is equivalent to that which would be obtained by restricting the subset order and compatibility of $\mathcal{L}(\mathbb{P})$ to its isolated elements. (As it is defined the operation ! is $\trianglelefteq$-continuous, whereas the alternative definition based on isolated elements would not even be $\trianglelefteq$-monotonic.) Now define $(\mathbb{P} \to \mathbb{Q}) =_{def} (!\mathbb{P} \multimap \mathbb{Q})$. The elements of $\mathcal{L}(!\mathbb{P} \multimap \mathbb{Q})$ correspond to continuous functions from $\mathcal{L}(\mathbb{P})$ to $\mathcal{L}(\mathbb{Q})$.

The notation of linear logic is no accident. The category of preorders with consistency and join-preserving functions between their domains is part of a model of linear logic (in the sense of [1]), one where ! plays the role of a linear-logic exponential.

3.3 A domain theory for concurrency

Prime algebraic lattices have appeared in domain theories for concurrency. The potentially complicated structure of computation paths of processes has suggested building domain theories for concurrency directly on computation paths. This line has been followed in what seemed originally to be two different approaches. One is Matthew Hennessy's semantics for CCS with process passing [12], in which a process denotes the set of its computation paths. The other is that of categories of presheaf models [5] in which processes denote mappings from computation paths to sets of "realisers" saying how each computation path may be realised. This extra structure allows the incorporation of branching information, and the corresponding notion of process equivalence is a form of bisimulation. The two approaches are variations on a common idea: that a process denotes a form of characteristic function in which the truth values are sets of realisers. A path set may be viewed as a special presheaf that yields at most one realiser for each path. (Here we are seeing prime algebraic lattices as degenerate forms of presheaf category.)

In the path semantics, processes are intuitively represented as collections of their computation paths. Paths are elements of preorders $\mathbb{P}$, $\mathbb{Q}$, ... called path orders which function as process types, each describing the set of possible paths for processes of that type together with their sub-path ordering. A process of type $\mathbb{P}$ then denotes a downwards-closed subset of paths in $\mathbb{P}$, called a path set. Path sets ordered by inclusion form the elements of the prime algebraic domain $\mathcal{L}(\mathbb{P})$, the domain of processes of type $\mathbb{P}$.

Linear maps from path order $\mathbb{P}$ to path order $\mathbb{Q}$ are join-preserving functions from $\mathcal{L}(\mathbb{P})$ to $\mathcal{L}(\mathbb{Q})$. The ensuing category has enough structure to form a model of Girard's (classical) linear logic [1, 17, 29, 20]. As usual, one moves to more liberal maps through the use of a suitable comonad (an exponential of linear logic generally written !). Here, $!\mathbb{P}$ can be taken to be the finite subsets of path order $\mathbb{P} = (P, \lesssim_{\mathbb{P}})$ ordered by

$$X \lesssim_{!\mathbb{P}} Y \text{ iff } \forall p \in X \exists p' \in Y.\ p \lesssim_{\mathbb{P}} p' .$$

The path order $!\mathbb{P}$ should be thought of as consisting of compound paths, associated with several runs of a (replicated) process. As a preorder $!\mathbb{P}$ is equivalent to the partial order of isolated elements in $\mathcal{L}(\mathbb{P})$. Linear maps from $!\mathbb{P}$ to $\mathbb{Q}$ correspond to continuous functions from $\mathcal{L}(\mathbb{P})$ to $\mathcal{L}(\mathbb{Q})$.

This categorical situation yields a rich higher-order process languages, HOPLA and affine-HOPLA. The semantics of recursive types is given essentially as in the previous section. The representation of domains by preorders plays an important role in proofs of adequacy and full-abstraction [20].

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