Probabilistic and Quantum Event Structures

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Abstract. A mathematical theory of probabilistic and quantum event structures is developed. It has some claim to providing fundamental models of distributed probabilistic and quantum systems, and has formed the basis for distributed probabilistic and quantum games.

1 Introduction

Prakash Panangaden has been drawn to conceptual problems in computer science, logic and computation, how to structure and understand probabilistic computation, and the boundaries of computer science with physics. I hope here to be dealing with subjects close to Prakash's heart.

Event structures have emerged as a fundamental model of distributed computation, a model in which the traditional view of a history as a sequence of events is replaced by a view of a history as a partial order of events. This article studies the mathematics needed to take event structures into the realm of distributed probabilistic and distributed quantum computation. The lack of a sufficiently general definition of probabilistic event structure became apparent in work on concurrent games and strategies, in extending concurrent strategies to probabilistic strategies—see the companion work [1]. The description of a probabilistic event structure here meets that need and extends previous definitions, summarised in [2].

A probabilistic event structure essentially comprises an event structure together with a continuous valuation on the Scott open sets of its domain of configurations. The continuous valuation assigns a probability to each open set. However open sets are several levels removed from the events of an event structure, so a more workable definition is obtained by considering the probabilities of basic open sets, generated by single finite configurations; for each finite configuration this specifies the probability of a result which extends the finite configuration. Such valuations on configuration determine the continuous valuations from which they arise, and can be characterised through the device of "drop functions." The characterisation yields a workable definition of probabilistic event structure.

In a quantum event structure events are interpreted as unitary or projection operators in a Hilbert space. Unitary operators are associated with events of preparation, such as a change of coordinates with which to make a measurement or a time period over which the system is allowed to evolve undisturbed. Projection operators are associated with events of elementary tests. Causally independent (*i.e.* concurrent) events are interpreted by commuting operators. A configuration of the event structure is thought of as a distributed quantum experiment; it describes which events of preparation and tests to perform and their (partial) order of dependency. Once given an initial state as a density operator, a quantum event structure assigns an intrinsic weight to each finite configuration. This does not make the whole event structure into a probabilistic event structure, but it does do so locally: under each configuration there is a probabilistic event structure giving the probabilities over the outcomes of the experiment the configuration describes. Quantum theory is often described as a contextual theory, in that it is only sensible to consider outcomes w.r.t. a specified measurement context [3]. In a quantum event structure configurations assume the role of measurement contexts; w.r.t. a measurement context expressed as a configuration, the sub-configurations constitute the possible outcomes.

2 Event structures

An *event structure* comprises $(E, \leq, \mathrm{Con})$, consisting of a set E, of *events* which are partially ordered by $\leq$, the *causal dependency relation*, and a nonempty *consistency relation* Con consisting of finite subsets of E, which satisfy

$$\begin{aligned} &\{e' \mid e' \leq e\} \text{ is finite for all } e \in E, \\ &\{e\} \in \mathrm{Con} \text{ for all } e \in E, \\ &Y \subseteq X \in \mathrm{Con} \implies Y \in \mathrm{Con}, \quad \text{and} \\ &X \in \mathrm{Con} \ \&\ e \leq e' \in X \implies X \cup \{e\} \in \mathrm{Con}. \end{aligned}$$

The *configurations* $\mathcal{C}^\infty(E)$ of an event structure E consist of those subsets $x \subseteq E$ which are

(Consistent) $\forall X \subseteq x.\ X$ is finite $\Rightarrow X \in \mathrm{Con}\ x \in \mathrm{Con}$, and
(Down-closed) $\forall e, e'.\ e' \leq e \in x \implies e' \in x$.

Often we shall be concerned with just the finite configurations, $\mathcal{C}(E)$.

We say an event structure is *elementary* when the consistency relation consists of all finite subsets of events. Two events e, e' which are both consistent and incomparable w.r.t. causal dependency in an event structure are regarded as *concurrent*, written $e\, co\, e'$. We shall occasionally say events are in *conflict* when they are they are not consistent. For $X \subseteq E$ we write $[X]$ for $\{e \in E \mid \exists e' \in X.\ e \leq e'\}$, the down-closure of X; note if $X \in \mathrm{Con}$, then $[X] \in \mathrm{Con}$ so is a configuration.

Notation 1 Let E be an event structure. We use $x {-\!\!\subset} y$ to mean y covers x in $\mathcal{C}^\infty(E)$, *i.e.* $x \subsetneq y$ in $\mathcal{C}^\infty(E)$ with nothing in between, and $x \overset{e}{-\!\!\!-\!\!\subset} y$ to mean $x \cup \{e\} = y$ for $x, y \in \mathcal{C}^\infty(E)$ and event $e \notin x$. We use $x \overset{e}{-\!\!\!-\!\!\subset}$, expressing that event e is enabled at configuration x, when $x \overset{e}{-\!\!\!-\!\!\subset} y$ for some y. We write $\{x_i \mid i \in I\}\!\uparrow$ to indicate that a subset of configurations is compatible, *i.e.* bounded above by a configuration.

3 Probabilistic event structures

A probabilistic event structure comprises an event structure $(E, \leq, \mathrm{Con})$ with a continuous valuation on its Scott open sets of configurations. Recall a *continuous valuation* is a function w from the Scott-open subsets of $\mathcal{C}^\infty(E)$ to $[0,1]$ which is

(normalized) $w(\mathcal{C}^\infty(E)) = 1$; *(strict)* $w(\varnothing) = 0$;
(monotone) $U \subseteq V \implies w(U) \leq w(V)$;
(modular) $w(U \cup V) + w(U \cap V) = w(U) + w(V)$; and
(continuous) $w(\bigcup_{i\in I} U_i) = \sup_{i\in I} w(U_i)$ for *directed* unions $\bigcup_{i\in I} U_i$.

The value $w(U)$ of a continuous valuation w specifies the probability of a result in open set U. Continuous valuations traditionally play the role of elements in probabilistic powerdomains [4]. Continuous valuations are determined by their restrictions to basic open sets

$$\widehat{x} =_{\mathrm{def}} \{y \in \mathcal{C}^\infty(E) \mid x \subseteq y\}\,,$$

for x a finite configuration. A characterisation of such restrictions yields an equivalent, more workable definition of probabilistic event structure, that we present in Section 3.2. As preparation we first develop some machinery for assigning values to "general intervals."

3.1 General intervals and drop functions

Throughout this section assume E is an event structure and $v : \mathcal{C}(E) \to \mathbb{R}$. Extend $\mathcal{C}(E)$ to a lattice $\mathcal{C}(E)^\top$ by adjoining an extra top element $\top$. Write its order as $x \sqsubseteq y$ and its finite join operations as $x \vee y$ and $\bigvee_{i\in I} x_i$. Extend v to $v^\top : \mathcal{C}(E)^\top \to \mathbb{R}$ by taking $v^\top(\top) = 0$.

We are concerned with drops in value across general intervals $[y; x_1, \cdots, x_n]$, where $y, x_1, \cdots, x_n \in \mathcal{C}(E)^\top$ with $y \sqsubseteq x_1, \cdots, x_n$ in $\mathcal{C}(E)^\top$. The interval is thought of as specifying the set of configurations $\widehat{y} \smallsetminus (\widehat{x}_1 \cup \cdots \cup \widehat{x}_n)$, *viz.* those configurations above or equal to y and not above or equal to any $x_1, \cdots, x_n$. As such the intervals form a basis of the Lawson topology on $\mathcal{C}^\infty(E)^\top$.

Define the *drop functions* $d_v^{(n)}[y; x_1, \cdots, x_n] \in \mathbb{R}$ for $y, x_1, \cdots, x_n \in \mathcal{C}(E)^\top$ with $y \sqsubseteq x_1, \cdots, x_n$ in $\mathcal{C}(E)^\top$, by induction, taking

$$d_v^{(0)}[y;\,] =_{\mathrm{def}} v^\top(y) \text{ and}$$
$$d_v^{(n)}[y; x_1, \cdots, x_n] =_{\mathrm{def}} d_v^{(n-1)}[y; x_1, \cdots, x_{n-1}] - d_v^{(n-1)}[x_n; x_1 \vee x_n, \cdots, x_{n-1} \vee x_n]\,,$$

for $n > 0$.

The following proposition shows how drop functions assign to general intervals $[y; x_1, \cdots, x_n]$ the value of being in $\widehat{y}$ minus the value of being in $\widehat{x}_1 \cup \cdots \cup \widehat{x}_n$, and that the latter is calculated using the inclusion-exclusion principle for sets; notice that an overlap $\bigcap_{i\in I} \widehat{x}_i$ equals $\widehat{\bigvee_{i\in I} x_i}$, where $\varnothing \neq I \subseteq \{1, \cdots, n\}$.

Proposition 1. *Let $n \in \omega$. For $y, x_1, \cdots, x_n \in \mathcal{C}(E)^\top$ with $y \sqsubseteq x_1, \cdots, x_n$,*

$$d_v^{(n)}[y; x_1, \cdots, x_n] = v(y) - \sum_{\emptyset \neq I \subseteq \{1,\cdots,n\}} (-1)^{|I|+1} v(\bigvee_{i \in I} x_i)\,.$$

For $y, x_1, \cdots, x_n \in \mathcal{C}(E)$ with $y \subseteq x_1, \cdots, x_n$,

$$d_v^{(n)}[y; x_1, \cdots, x_n] = v(y) - \sum_I (-1)^{|I|+1} v(\bigcup_{i \in I} x_i)\,,$$

where the index I ranges over sets satisfying $\emptyset \neq I \subseteq \{1, \cdots, n\}$ s.t. $\{x_i \mid i \in I\}\uparrow$.

Proof. We prove the first statement by induction on n. For the basis, when $n = 0$, $d_v^{(n)}[y;] = v(y)$, as required. For the induction step, with $n > 0$, we reason

$$\begin{aligned} d_v^{(n)}[y; x_1, \cdots, x_n] &=_{\text{def}} d_v^{(n-1)}[y; x_1, \cdots, x_{n-1}] - d_v^{(n-1)}[x_n; x_1 \vee x_n, \cdots, x_{n-1} \vee x_n] \\ &= v(y) - \sum_{\emptyset \neq I \subseteq \{1,\cdots,n-1\}} (-1)^{|I|+1} v(\bigvee_{i \in I} x_i) \\ &\quad - v(x_n) + \sum_{\emptyset \neq J \subseteq \{1,\cdots,n-1\}} (-1)^{|I|+1} v(\bigvee_{j \in J} x_i \vee x_n)\,, \end{aligned}$$

making use of the induction hypothesis. Consider subsets K for which $\emptyset \neq K \subseteq \{1, \cdots, n\}$. Either $n \notin K$, in which case $\emptyset \neq K \subseteq \{1, \cdots, n-1\}$, or $n \in K$, in which case $K = \{n\}$ or $J =_{\text{def}} K \setminus \{n\}$ satisfies $\emptyset \neq J \subseteq \{1, \cdots, n-1\}$. From this observation, the sum above amounts to

$$v(y) - \sum_{\emptyset \neq K \subseteq \{1,\cdots,n\}} (-1)^{|K|+1} v(\bigvee_{k \in K} x_k)\,,$$

as required to maintain the induction hypothesis.

The second expression of the proposition is got by discarding all terms $v(\bigvee_{i \in I} x_i)$ for which $\bigvee_{i \in I} x_i = \top$ which leaves the sum unaffected as they contribute 0. □

Corollary 1. *Let $n \in \omega$ and $y, x_1, \cdots, x_n \in \mathcal{C}(E)^\top$ with $y \sqsubseteq x_1, \cdots, x_n$. For ρ an n-permutation,*

$$d_v^{(n)}[y; x_{\rho(1)}, \cdots, x_{\rho(n)}] = d_v^{(n)}[y; x_1, \cdots, x_n]\,.$$

Proof. As by Proposition 1, the value of $d_v^{(n)}[y; x_1, \cdots, x_n]$ is insensitive to permutations of its arguments. □

In the following results we lay out the fundamental properties of drop functions for later use.

Proposition 2. *Assume $n \geq 1$ and $y, x_1, \cdots, x_n \in \mathcal{C}(E)^\top$ with $y \sqsubseteq x_1, \cdots, x_n$. If $y = x_i$ for some i with $1 \leq i \leq n$ then $d_v^{(n)}[y; x_1, \cdots, x_n] = 0$.*

Proof. By Corollary 1, it suffices to show $d_v^{(n)}[y; x_1, \cdots, x_n] = 0$ when $y = x_n$. In this case,

$$\begin{aligned} d_v^{(n)}[y; x_1, \cdots, x_n] &= d_v^{(n-1)}[y; x_1, \cdots, x_{n-1}] - d_v^{(n-1)}[x_n; x_1 \vee x_n, \cdots, x_{n-1} \vee x_n] \\ &= d_v^{(n-1)}[y; x_1, \cdots, x_{n-1}] - d_v^{(n-1)}[y; x_1, \cdots, x_{n-1}] \\ &= 0 . \end{aligned}$$

□

Corollary 2. *Assume $n \geq 1$ and $y, x_1, \cdots, x_n \in \mathcal{C}(E)^\top$ with $y \sqsubseteq x_1, \cdots, x_n$. If $x_i \sqsubseteq x_j$ for distinct i, j with $1 \leq i, j \leq n$ then*

$$d_v^{(n)}[y; x_1, \cdots, x_n] = d_v^{(n-1)}[y; x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n] .$$

Proof. By Corollary 1, it suffices to show

$$d_v^{(n)}[y; x_1, \cdots, x_{n-1}, x_n] = d_v^{(n-1)}[y; x_1, \cdots, x_{n-1}]$$

when $x_{n-1} \sqsubseteq x_n$. Then,

$$\begin{aligned} d_v^{(n)}[y; x_1, \cdots, x_n] &= d_v^{(n-1)}[y; x_1, \cdots, x_{n-1}] - d_v^{(n-1)}[x_n; x_1 \vee x_n, \cdots, x_{n-1} \vee x_n] \\ &= d_v^{(n-1)}[y; x_1, \cdots, x_{n-1}] - d_v^{(n-1)}[x_n; x_1 \vee x_n, \cdots, x_{n-2}, x_n] \\ &= d_v^{(n-1)}[y; x_1, \cdots, x_{n-1}] - 0 , \end{aligned}$$

by Proposition 2. □

Proposition 3. *Assume $n \in \omega$ and $y, x_1, \cdots, x_n \in \mathcal{C}(E)^\top$ with $y \sqsubseteq x_1, \cdots, x_n$. Then, $d_v^{(n)}[y; x_1, \cdots, x_n] = 0$ if $y = \top$ and $d_v^{(n)}[y; x_1, \cdots, x_n] = d_v^{(n-1)}[y; x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n]$ if $x_i = \top$ with $1 \leq i \leq n$.*

Proof. When $n = 0$, $d_v^{(0)}[\top;] = v^\top(\top) = 0$. When $n \geq 1$, $d_v^{(n)}[\top; x_1, \cdots, x_n] = 0$ by Proposition 2 as *e.g.* $x_n = \top$. For the remaining statement, w.l.og. we may assume $i = n$ and that $x_n = \top$, yielding

$$\begin{aligned} & d_v^{(n)}[y; x_1, \cdots, \top] = \\ & d_v^{(n-1)}[y; x_1, \cdots, x_{n-1}] - d_v^{(n-1)}[\top; x_1 \vee \top, \cdots, x_{n-1} \vee \top] = d_v^{(n-1)}[y; x_1, \cdots, x_{n-1}] . \end{aligned}$$

□

It will be important that drops across general intervals can be reduced to sums of drops across intervals based on coverings, as explained in the next two results.

Lemma 1. *Let $n \geq 1$. Let $y, x_1, \cdots, x_n, x'_n \in \mathcal{C}(E)^\top$ with $y \sqsubseteq x_1, \cdots, x_n$. Assume $x_n \sqsubseteq x'_n$. Then,*

$$d_v^{(n)}[y; x_1, \cdots, x'_n] = d_v^{(n)}[y; x_1, \cdots, x_n] + d_v^{(n)}[x_n; x_1 \vee x_n, \cdots, x_{n-1} \vee x_n, x'_n] .$$

Proof. By definition,

$$
\begin{aligned}
\text{the r.h.s.} &= d_v^{(n-1)}[y; x_1, \cdots, x_{n-1}] - d_v^{(n-1)}[x_n; x_1 \vee x_n, \cdots, x_{n-1} \vee x_n] \\
&\quad + d_v^{(n-1)}[x_n; x_1 \vee x_n, \cdots, x_{n-1} \vee x_n] - d_v^{(n-1)}[x'_n; x_1 \vee x'_n, \cdots, x_{n-1} \vee x'_n] \\
&= d_v^{(n-1)}[y; x_1, \cdots, x_{n-1}] - d_v^{(n-1)}[x'_n; x_1 \vee x'_n, \cdots, x_{n-1} \vee x'_n] \\
&= d_v^{(n)}[y; x_1, \cdots, x_{n-1}, x'_n] \\
&= \text{the l.h.s.}.
\end{aligned}
$$

□

Lemma 2. *Let $y \subseteq x_1, \cdots, x_n$ in $\mathcal{C}(E)$. Then, $d_v^{(n)}[y; x_1, \cdots, x_n]$ is expressible as a sum of terms $d_v^{(k)}[u; w_1, \cdots, w_k]$ where $y \subseteq u \longrightarrow\!\!\!\!\subset w_i$ in $\mathcal{C}(E)$ and $w_i \subseteq x_1 \cup \cdots \cup x_n$, for all i with $1 \leq i \leq k$. (The set $x_1 \cup \cdots \cup x_n$ need not be in $\mathcal{C}(E)$.)*

Proof. Define the *weight* of a term $d_v^{(n)}[y; x_1, \cdots, x_n]$, where $y \subseteq x_1, \cdots, x_n$ in $\mathcal{C}(E)$, to be the product $|x_1 \smallsetminus y| \times \cdots \times |x_n \smallsetminus y|$.

Assume $y \subseteq x_1, \cdots, x'_n$ in $\mathcal{C}(E)$. By Proposition 2, if y equals x'_n or some x_i, then $d_v^{(n)}[y; x_1, \cdots, x'_n] = 0$, so may be deleted as a contribution to a sum. Otherwise, if $y \subsetneq x_n \subsetneq x'_n$, by Lemma 1 we can rewrite $d_v^{(n)}[y; x_1, \cdots, x'_n]$ to the sum

$$d_v^{(n)}[y; x_1, \cdots, x_n] + d_v^{(n)}[x_n; x_1 \vee x_n, \cdots, x_{n-1} \vee x_n, x'_n],$$

where we further observe

$$|x_n \smallsetminus y| < |x'_n \smallsetminus y|, \qquad |x'_n \smallsetminus x_n| < |x'_n \smallsetminus y|$$

and

$$|(x_i \cup x_n) \smallsetminus x_n| \leq |x_i \smallsetminus y|,$$

whenever $x_i \vee x_n \neq \top$. Using Proposition 3 we may tidy away any mentions of $\top$. This reduces $d_v^{(n)}[y; x_1, \cdots, x'_n]$ to the sum of at most two terms, each of lesser weight. For notational simplicity we have concentrated on the nth argument in $d_v^{(n)}[y; x_1, \cdots, x'_n]$, but by Corollary 1 an analogous reduction is possible w.r.t. any argument.

Repeated use of the reduction, rewrites $d_v^{(n)}[y; x_1, \cdots, x_n]$ to a sum of terms of the form

$$d_v^{(k)}[u; w_1, \cdots, w_k]$$

where $k \leq n$ and $u \longrightarrow\!\!\!\!\subset w_1, \cdots, w_k \subseteq x_1 \cup \cdots \cup x_n$. This justifies the claims of the lemma. □

3.2 Probabilistic event structures

A probabilistic event structure is an event structure associated with a $[0,1]$-valuation on configurations, normalised to 1 at the emptyset, such that no general interval has a negative drop.

Definition 1. Let E be an event structure. A *configuration-valuation* on E is function $v : \mathcal{C}(E) \to [0,1]$ such that $v(\emptyset) = 1$ and which satisfies the *drop condition*

$$d_v^{(n)}[y; x_1, \cdots, x_n] \geq 0$$

for all $n \geq 1$ and $y, x_1, \cdots, x_n \in \mathcal{C}(E)$ with $y \subseteq x_1, \cdots, x_n$. A *probabilistic event structure* comprises an event structure E together with a configuration-valuation $v : \mathcal{C}(E) \to [0,1]$.[1]

Proposition 4. *Let E be an event structure. Let $v : \mathcal{C}(E) \to [0,1]$. Then, v is a configuration-valuation iff $d_v^{(n)}[y; x_1, \cdots, x_n] \geq 0$ for all $n \in \omega$ and $y, x_1, \cdots, x_n \in \mathcal{C}(E)^\top$ with $y \sqsubseteq x_1, \cdots, x_n$. If v is a configuration-valuation, then*

$$y \sqsubseteq x \implies v^\top(y) \geq v^\top(x)\,,$$

for all $x, y \in \mathcal{C}(E)^\top$.

Proof. By Proposition 3 and as $d_v^{(1)}[y; x] = v^\top(y) - v^\top(x)$. □

By Lemma 2, in showing we have a probabilistic event structure it suffices to verify the "drop condition" only for special general intervals $[y; x_1, \cdots, x_n]$ in which the configurations $x_1, \cdots, x_n$ cover y.

Proposition 5. *Let E be an event structure. Let $v : \mathcal{C}(E) \to [0,1]$. v is a configuration-valuation iff $v(\emptyset) = 1$ and*

$$d_v^{(n)}[y; x_1, \cdots, x_n] \geq 0$$

for all $n \geq 1$ and $y {-\!\!\subset} x_1, \cdots, x_n$ in $\mathcal{C}(E)$.

4 The characterisation

Our goal is to prove that probabilistic event structures correspond to event structures with a continuous valuation. It is clear that a continuous valuation w on the Scott-open subsets of an event structure E gives rise to a configuration-valuation v on E: take $v(x) =_{\text{def}} w(\widehat{x})$, for $x \in \mathcal{C}(E)$. We will show that this construction has an inverse, that a configuration-valuation determines a continuous valuation.

For this we need a combinatorial lemma:[2]

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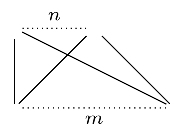
[1] Samy Abbes has pointed out that the "drop condition" appears in early work of the Russian mathematician V.A.Rohlin [5](as relation (6) of Section 3, p.7), and Klaus Keimel that functions satisfying the "drop condition" are called "totally convex" or "completely monotone" in the literature [6]. The rediscovery of the "drop condition" and its reuse in the context of event structures was motivated by Lemma 2, tying it to occurrences of events.

[2] The proof of the combinatorial lemma, due to the author, appears with acknowledgement as Lemma 6.App.1 in [7], the PhD thesis of my former student Daniele Varacca, whom I thank, both for the collaboration and the latex.

Lemma 3. *For all finite sets I, J,*

$$\sum_{\substack{\varnothing \neq K \subseteq I\times J \\ \pi_1(K)=I,\pi_2(K)=J}} (-1)^{|K|} = (-1)^{|I|+|J|-1} .$$

Proof. W.l.o.g. we can take $I = \{1,\dots,n\}$ and $J = \{1,\dots,m\}$. Also observe that a subset $K \subseteq I \times J$ such that $\pi_1(K) = I, \pi_2(K) = J$ is in fact a surjective and total relation between the two sets, pictured below.



Let

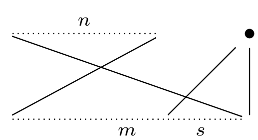
$$t_{n,m} =_{\text{def}} \sum_{\substack{\varnothing \neq K \subseteq I\times J \\ \pi_1(K)=I,\pi_2(K)=J}} (-1)^{|}K|;$$

$$t^o_{n,m} =_{\text{def}} |\{\varnothing \neq K \subseteq I\times J \mid |K| \text{ odd}, \pi_1(K) = I, \pi_2(K) = J\}|;$$

$$t^e_{n,m} := |\{\varnothing \neq K \subseteq I\times J \mid |K| \text{ even}, \pi_1(K) = I, \pi_2(K) = J\}|.$$

Clearly $t_{n,m} = t^e_{n,m} - t^o_{n,m}$. We want to prove that $t_{n,m} = (-1)^{n+m+1}$. We do this by induction on n. It is easy to check that this is true for $n = 1$. In this case, if m is even then $t^e_{1,m} = 1$ and $t^o_{1,m} = 0$, so that $t^e_{1,m} - t^o_{1,m} = (-1)^{1+m+1}$. Similarly if m is odd.

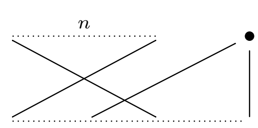
Now assume that $t_{n,p} = (-1)^{n+p+1}$, for every p, and compute $t_{n+1,m}$. To evaluate $t_{n+1,m}$ we count all surjective and total relations K between I and J together with their "sign." Consider the pairs in K of the form $(n+1,h)$ for $h \in J$. The result of removing them is a a total surjective relation between $\{1,\dots,n\}$ and a subset J_K of $\{1,\dots,m\}$.



Consider first the case where $J_K = \{1,\dots,m\}$. Consider the contribution of such K's to $t_{n+1,m}$. There are $\binom{m}{s}$ ways of choosing s pairs of the form $(n+1,h)$. For every such choice there are $t_{n,m}$ (signed) relations. Adding the pairs $(n+1,h)$ possibly modifies the sign of such relations. In all the contribution amounts to

$$\sum_{1\leq s\leq m} \binom{m}{s} (-1)^s t_{n,m} .$$

Suppose now that J_K is a proper subset of $\{1,\dots,m\}$ leaving out r elements.



Since K is surjective, all such elements h must be in a pair of the form $(n+1,h)$. Moreover there can be s pairs of the form $(n+1,h')$ with $h' \in J_K$. What is the contribution of such K's to $t_{n,m}$? There are $\binom{m}{r}$ ways of choosing the elements that are left out. For every such choice and for every s such that $0 \le s \le m-r$ there are $\binom{m-r}{s}$ ways of choosing the $h' \in J_K$. And for every such choice there are $t_{n,m-r}$ (signed) relations. Adding the pairs $(n+1,h)$ and $(n+1,h')$ possibly modifies the sign of such relations. In all, for every r such that $1 \le r \le m-1$, the contribution amounts to

$$\binom{m}{r} \sum_{1 \le s \le m-r} \binom{m}{s} (-1)^{s+r} t_{n,m-n} \,.$$

The (signed) sum of all these contribution will give us $t_{n+1,m}$. Now we use the induction hypothesis and we write $(-1)^{n+p+1}$ for $t_{n,p}$.

Thus,

$$\begin{aligned}
t_{n+1,m} &= \sum_{1 \le s \le m} \binom{m}{s} (-1)^s t_{n,m} \\
&\quad + \sum_{1 \le r \le m-1} \binom{m}{r} \sum_{0 \le s \le m-r} \binom{m-r}{s} (-1)^{s+r} t_{n,m-r} \\
&= \sum_{1 \le s \le m} \binom{m}{s} (-1)^{s+n+m+1} \\
&\quad + \sum_{1 \le r \le m-1} \binom{m}{r} \sum_{0 \le s \le m-r} \binom{m-r}{s} (-1)^{s+n+m+1} \\
&= (-1)^{n+m+1} \left(\sum_{1 \le s \le m} \binom{m}{s} (-1)^s \right. \\
&\quad \left. + \sum_{1 \le r \le m-1} \binom{m}{r} \sum_{0 \le s \le m-r} \binom{m-r}{s} (-1)^s \right) .
\end{aligned}$$

By the binomial formula, for $1 \le r \le m-1$ we have

$$0 = (1-1)^{m-r} = \sum_{0 \le s \le m-r} \binom{m-r}{s} (-1)^s \,.$$

So we are left with

$$\begin{aligned}
t_{n+1,m} &= (-1)^{n+m+1} \left(\sum_{1 \le s \le m} \binom{m}{s} (-1)^s \right) \\
&= (-1)^{n+m+1} \left(\sum_{0 \le s \le m} \binom{m}{s} (-1)^s - \binom{m}{0} (-1)^0 \right) \\
&= (-1)^{n+m+1} (0-1) \\
&= (-1)^{n+1+m+1} ,
\end{aligned}$$

as required. □

Theorem 1. *A configuration-valuation v on an event structure E extends to a unique continuous valuation w_v on the open sets of $\mathcal{C}^\infty(E)$, so that $w_v(\widehat{x}) = v(x)$, for all $x \in \mathcal{C}(E)$.*

Conversely, a continuous valuation w on the open sets of $\mathcal{C}^\infty(E)$ restricts to a configuration-valuation v_w on E, assigning $v_w(x) = w(\widehat{x})$, for all $x \in \mathcal{C}(E)$.

Proof. The proof is inspired by the proofs in the appendix of [2] and the thesis [7].

First, a continuous valuation w on the open sets of $\mathcal{C}^\infty(E)$ restricts to a configuration-valuation v defined as $v(x) =_{\text{def}} w(\widehat{x})$ for $x \in \mathcal{C}(E)$. Note that any extension of a configuration-valuation to a continuous valuation is bound to be unique by continuity.

To show the converse we first define a function w from the basic open sets $Bs =_{\text{def}} \{\widehat{x_1} \cup \cdots \cup \widehat{x_n} \mid x_1, \cdots, x_n \in \mathcal{C}(E)\}$ to $[0,1]$ and show that it is normalised, strict, monotone and modular. Define

$$\begin{aligned} w(\widehat{x_1} \cup \cdots \cup \widehat{x_n}) =_{\text{def}} &\ 1 - d_v^{(n)}[\varnothing; x_1, \cdots, x_n] \\ = &\sum_{\varnothing \neq I \subseteq \{1,\cdots,n\}} (-1)^{|I|+1} v(\bigvee_{i \in I} x_i) \end{aligned}$$

—this can be shown to be well-defined using Corollaries 1 and 2.

Clearly, w is normalised in the sense that $w(\mathcal{C}^\infty(E)) = w(\widehat{\varnothing}) = 1$ and strict in that $w(\varnothing) = 1 - v(\varnothing) = 0$.

To see that it is monotone, first observe that

$$w(\widehat{x_1} \cup \cdots \cup \widehat{x_n}) \le w(\widehat{x_1} \cup \cdots \cup \widehat{x_{n+1}})$$

as

$$\begin{aligned} w(\widehat{x_1} \cup \cdots \cup \widehat{x_{n+1}}) - w(\widehat{x_1} \cup \cdots \cup \widehat{x_n}) &= d_v^{(n)}[\varnothing; x_1, \cdots, x_n] - d_v^{(n+1)}[\varnothing; x_1, \cdots, x_{n+1}] \\ &= d_v^{(n)}[x_{n+1}; x_1 \vee x_{n+1}, \cdots, x_n \vee x_{n+1}] \ge 0\,. \end{aligned}$$

By a simple induction (on m),

$$w(\widehat{x_1} \cup \cdots \cup \widehat{x_n}) \le w(\widehat{x_1} \cup \cdots \cup \widehat{x_n} \cup \widehat{y_1} \cup \cdots \cup \widehat{y_m})\,.$$

Suppose that $\widehat{x_1} \cup \cdots \cup \widehat{x_n} \subseteq \widehat{y_1} \cup \cdots \cup \widehat{y_m}$. Then $\widehat{y_1} \cup \cdots \cup \widehat{y_m} = \widehat{x_1} \cup \cdots \cup \widehat{x_n} \cup \widehat{y_1} \cup \cdots \cup \widehat{y_m}$. By the above,

$$\begin{aligned} w(\widehat{x_1} \cup \cdots \cup \widehat{x_n}) &\le w(\widehat{x_1} \cup \cdots \cup \widehat{x_n} \cup \widehat{y_1} \cup \cdots \cup \widehat{y_m}) \\ &= w(\widehat{y_1} \cup \cdots \cup \widehat{y_m})\,, \end{aligned}$$

as required to show w is monotone.

To show modularity we require

$$\begin{aligned} &w(\widehat{x_1} \cup \cdots \cup \widehat{x_n}) + w(\widehat{y_1} \cup \cdots \cup \widehat{y_m}) \\ =&w(\widehat{x_1} \cup \cdots \cup \widehat{x_n} \cup \widehat{y_1} \cup \cdots \cup \widehat{y_m}) + w((\widehat{x_1} \cup \cdots \cup \widehat{x_n}) \cap (\widehat{y_1} \cup \cdots \cup \widehat{y_m}))\,. \end{aligned}$$

Note

$$(\widehat{x_1} \cup \cdots \cup \widehat{x_n}) \cap (\widehat{y_1} \cup \cdots \cup \widehat{y_m}) = (\widehat{x_1} \cap \widehat{y_1}) \cup \cdots \cup (\widehat{x_i} \cap \widehat{y_j}) \cdots \cup (\widehat{x_n} \cap \widehat{y_m})$$
$$= \widehat{x_1 \vee y_1} \cup \cdots \cup \widehat{x_i \vee y_j} \cdots \cup \widehat{x_n \vee y_m}\,.$$

From the definition of w we require

$$\begin{aligned} & w(\widehat{x_1} \cup \cdots \cup \widehat{x_n} \cup \widehat{y_1} \cup \cdots \cup \widehat{y_m}) \\ = & \sum_{\emptyset \neq I \subseteq \{1,\cdots,n\}} (-1)^{|I|+1} v(\bigvee_{i \in I} x_i) + \sum_{\emptyset \neq J \subseteq \{1,\cdots,m\}} (-1)^{|J|+1} v(\bigvee_{j \in J} y_j) \\ & - \sum_{\emptyset \neq R \subseteq \{1,\cdots,n\} \times \{1,\cdots,m\}} (-1)^{|R|+1} v(\bigvee_{(i,j) \in R} x_i \vee y_j)\,. \end{aligned} \tag{1}$$

Consider the definition of $w(\widehat{x_1} \cup \cdots \cup \widehat{x_n} \cup \widehat{y_1} \cup \cdots \cup \widehat{y_m})$ as a sum. Its components are associated with indices which either lie entirely within $\{1, \cdots, n\}$, entirely within $\{1, \cdots, m\}$, or overlap both. Hence

$$\begin{aligned} & w(\widehat{x_1} \cup \cdots \cup \widehat{x_n} \cup \widehat{y_1} \cup \cdots \cup \widehat{y_m}) \\ = & \sum_{\emptyset \neq I \subseteq \{1,\cdots,n\}} (-1)^{|I|+1} v(\bigvee_{i \in I} x_i) + \sum_{\emptyset \neq J \subseteq \{1,\cdots,m\}} (-1)^{|J|+1} v(\bigvee_{j \in J} y_j) \\ & + \sum_{\emptyset \neq I \subseteq \{1,\cdots,n\}, \emptyset \neq J \subseteq \{1,\cdots,m\}} (-1)^{|I|+|J|+1} v(\bigvee_{i \in I} x_i \vee \bigvee_{j \in J} y_j)\,. \end{aligned} \tag{2}$$

Comparing (1) and (2), we require

$$\begin{aligned} & - \sum_{\emptyset \neq R \subseteq \{1,\cdots,n\} \times \{1,\cdots,m\}} (-1)^{|R|+1} v(\bigvee_{(i,j) \in R} x_i \vee y_j) \\ = & \sum_{\emptyset \neq I \subseteq \{1,\cdots,n\}, \emptyset \neq J \subseteq \{1,\cdots,m\}} (-1)^{|I|+|J|+1} v(\bigvee_{i \in I} x_i \vee \bigvee_{j \in J} y_j)\,. \end{aligned} \tag{3}$$

Observe that

$$\bigvee_{(i,j) \in R} x_i \vee y_j = \bigvee_{i \in I} x_i \vee \bigvee_{j \in J} y_j$$

when $I = R_1 =_{\text{def}} \{i \in I \mid \exists j \in J.\ (i,j) \in R\}$ and $J = R_2 =_{\text{def}} \{j \in J \mid \exists i \in I.\ (i,j) \in R\}$ for a relation $R \subseteq \{1, \cdots, n\} \times \{1, \cdots, m\}$. With this observation we see that equality (3) follows from the combinatorial lemma, Lemma 3 above. This shows modularity.

Finally, we can extend w to all open sets by taking an open set U to $\sup_{b \in Bs \,\&\, b \subseteq U} w(b)$. The verification that w is indeed a continuous valuation extending v is now straightforward. □

The above theorem also holds (with the same proof) for Scott domains. Now, by [8], Corollary 4.3:

Theorem 2. *For a configuration-valuation v on E there is a unique probability measure μ_v on the Borel subsets of $\mathcal{C}^\infty(E)$ extending w_v.*

When x a finite configuration has $v(x) > 0$ and $\mu_v(\{x\}) = 0$ we can understand x as being a transient configuration on the way to a final with probability $v(x)$. In general, there is a simple expression for the probability of terminating at a finite configuration, helpful in the examples that follow.

Proposition 6. *Let E, v be a probabilistic event structure. For any finite configuration $y \in \mathcal{C}(E)$, the singleton set $\{y\}$ is a Borel subset with probability measure*

$$\mu_v(\{y\}) = \inf\{d_v^{(n)}[y; x_1, \cdots, x_n] \mid n \in \omega \ \& \ y \subsetneq x_1, \cdots, x_n \in \mathcal{C}(E)\}\,.$$

Proof. Let $y \in \mathcal{C}(E)$. Then $\{y\} = \widehat{y} \setminus U_y$ is clearly Borel as $U_y =_{\text{def}} \{x \in \mathcal{C}^\infty(E) \mid y \subsetneq x\}$ is open. Let w be the continuous valuation extending v. Then

$$w(U_y) = \sup\{w(\widehat{x}_1 \cup \cdots \cup \widehat{x}_n) \mid y \subsetneq x_1, \cdots, x_n \in \mathcal{C}(E)\}$$

as U_y is the directed union $\bigcup\{\widehat{x}_1 \cup \cdots \cup \widehat{x}_n \mid y \subsetneq x_1, \cdots, x_n \in \mathcal{C}(E)\}$. Hence

$$\begin{aligned}\mu_v(\{y\}) = v(y) - w(U_y) &= v(y) - \sup\{w(\widehat{x}_1 \cup \cdots \cup \widehat{x}_n) \mid y \subsetneq x_1, \cdots, x_n \in \mathcal{C}(E)\}\\ &= \inf\{v(y) - \sum_{\emptyset \neq I \subseteq \{1,\cdots,n\}} (-1)^{|I|+1} v(\bigvee_{i \in I} x_i) \mid y \subsetneq x_1, \cdots, x_n \in \mathcal{C}(E)\}\\ &= \inf\{d_v^{(n)}[y; x_1, \cdots, x_n] \mid n \in \omega \ \& \ y \subsetneq x_1, \cdots, x_n \in \mathcal{C}(E)\}\,.\end{aligned}$$

□

Example 1. Consider the event structure comprising two concurrent events e_1, e_2 with configuration-valuation v for which $v(\emptyset) = 1, v(\{e_1\}) = 1/3, v(\{e_2\}) = 1/2$ and $v(\{e_1, e_2\}) = 1/12$. This means in particular that there is a probability of 1/3 of a result within the Scott open set consisting of both the configuration $\{e_1\}$ and the configuration $\{e_1, e_2\}$. In other words, there is a probability of 1/3 of observing e_1 (possibly with or possibly without e_2). The induced probability measure p assigns a probability to any Borel set, in this simple case any subset of configurations, and is determined by its value on single configurations: $p(\emptyset) = 1 - 4/12 - 6/12 + 1/12 = 3/12$, $p(\{e_1\}) = 4/12 - 1/12 = 3/12$, $p(\{e_2\}) = 6/12 - 1/12 = 5/12$ and $p(\{e_1, e_2\}) = 1/12$. Thus there is a probability of 3/12 of observing neither e_1 nor e_2, and a probability of 5/12 of observing just the event e_2 (and not e_1). There is a drop $d_v^{(0)}[\emptyset; \{e_1\}, \{e_2\}] = 1 - 4/12 - 6/12 + 1/12 = 3/12$ corresponding to the probability of remaining at the empty configuration and not observing any event. Sometimes it's said that probability "leaks" at the empty configuration, but it's more accurate to think of this leak in probability as associated with a non-zero chance that the initial observation of no events will not improve. □

Example 2. Consider the event structure with events $\mathbb{N}^+$ with causal dependency $n \leq n+1$, with all finite subsets consistent. It is not hard to check that all subsets of $\mathcal{C}^\infty(\mathbb{N}^+)$ are Borel sets. Consider the ensuing probability distributions w.r.t. the following configuration-valuations:
(i) $v_0(x) = 1$ for all $x \in \mathcal{C}(\mathbb{N}^+)$. The resulting probability distribution assigns probability 1 to the singleton set $\{\mathbb{N}^+\}$, comprising the single infinite configuration $\mathbb{N}^+$, and 0 to $\emptyset$ and all other singleton sets of configurations.

(ii) $v_1(\varnothing) = v_1(\{1\}) = 1$ and $v_1(x) = 0$ for all other $x \in \mathcal{C}(\mathbb{N}^+)$. The resulting probability distribution assigns probability 0 to $\varnothing$ and probability 1 to the singleton set $\{1\}$, and 0 to all other singleton sets of configurations.
(iii) $v_2(\varnothing) = 1$ and $v_2(\{1, \cdots, n\}) = (1/2)^n$ for all $n \in \mathbb{N}^+$. The resulting probability distribution assigns probability $1/2$ to $\varnothing$ and $(1/2)^{n+1}$ to each singleton $\{\{1, \cdots, n\}\}$ and 0 to the singleton set $\{\mathbb{N}^+\}$, comprising the single infinite configuration $\mathbb{N}^+$. □

Remark. There is a seeming redundancy in the definition of purely probabilistic event structures, in that there are two different ways to say, for example, that events e_1 and e_2 do not occur together at a finite configuration y where $y \xrightarrow{e_1}\!\subset x_1$ and $y \xrightarrow{e_2}\!\subset x_2$: either through $y \cup \{e_1, e_2\} \notin \text{Con}$; or via the configuration-valuation v through $v(x_1 \cup x_2) = 0$. However, when we mix probability with nondeterminism [1], we make use of both consistency and the valuation. In the next section, for a quantum event structure, consistency will be important in determining when there is a sensible intrinsic probability distribution on a family of configurations, even though the probability of the union of the configurations ends up being zero.

5 Quantum event structures

Event structures are a model of distributed computation in which the causal dependence and independence of events is made explicit. By associating events with the most basic operators on a Hilbert space, *viz.* projection and unitary operators, so that independent (*i.e.* concurrent) events are associated with independent (*i.e.* commuting) operators, we obtain quantum event structures.

An event associated with a projection is thought of as an elementary positive test; its occurrence leaves the system in the eigenspace associated with eigenvalue 1 (rather than 0) of the projection. An event associated with a unitary operator is an event of preparation; the preparation might be a change of the direction in which to make a measurement, or the undisturbed evolution of the system over a time interval. A configuration is thought of as specifying a distributed quantum experiment. As we shall see, w.r.t. an initial state given as a density operator, each configuration w of a quantum event structure determines a probabilistic event structure, giving a probability distribution on its sub-configurations—the possible results of the experiment w.

Throughout let $\mathcal{H}$ be a separable Hilbert space over the complex numbers. For operators A, B on $\mathcal{H}$ we write $[A, B] =_{\text{def}} AB - BA$.

5.1 Events as operators

Formally, we obtain a quantum event structure from an event structure by interpreting its events as unitary or projection operators which must commute when events are concurrent.

Definition 2. A *quantum event structure* (over $\mathcal{H}$) comprises an event structure $(E, \leq, \mathrm{Con})$ together with an assignment Q_e of projection or unitary operators on $\mathcal{H}$ to events $e \in E$ such that for all $e_1, e_2 \in E$,

$$e_1 \; co \; e_2 \implies [Q_{e_1}, Q_{e_2}] = 0 \, .$$

Given a finite configuration, $x \in \mathcal{C}(E)$, define the operator A_x to be the composition $Q_{e_n} Q_{e_{n-1}} \cdots Q_{e_2} Q_{e_1}$ for some covering chain

$$\varnothing \xrightarrow{e_1}\!\!\subset x_1 \xrightarrow{e_2}\!\!\subset x_2 \cdots \xrightarrow{e_n}\!\!\subset x_n = x$$

in $\mathcal{C}(E)$. This is well-defined as for any two covering chains up to x the sequences of events are Mazurkiewicz trace equivalent, *i.e.* obtainable, one from the other, by successively interchanging concurrent events. In particular $A_\varnothing$ is the identity operator on $\mathcal{H}$. An *initial state* is given by a density operator ρ on $\mathcal{H}$.

Interpretation Consider first the simpler situation where in a quantum event structure E, Q the event structure E is elementary (*i.e.* all finite subsets are consistent). We regard E, Q as specifying a, possibly distributed, quantum experiment. The experiment says which unitary operators (events of preparation) and projection operators (elementary positive tests) to apply and in which order. The order being partial permits commuting operators to be applied concurrently, independently of each other, perhaps in a distributed fashion.

For a quantum event structure, E, Q, in general, an individual configuration $w \in \mathcal{C}^\infty(E)$ inherits the order of the ambient event structure E to become an elementary event structure, and can itself be regarded as a quantum experiment. The quantum event structure E, Q represents a collection of quantum experiments which may extend or overlap each other: when $w \subseteq w'$ in $\mathcal{C}^\infty(E)$ the experiment w' extends the experiment w, or equivalently w is a restriction of the experiment w'. In this sense a quantum event structure in general represents a nondeterministic quantum experiment. The extra generality will be crucial later in interpreting probabilistic quantum experiments.

5.2 From quantum to probabilistic

Consider a quantum event structure with initial state. A configuration w stands for an experiment and specifies which tests and preparations to try and in which order. In general, not all the tests in w need succeed, yielding as final result a possibly proper sub-configuration x of w. Theorem 3 below explains how there is an inherent probability distribution q_w over such final results. So an experiment provides a context for measurement w.r.t. which there is an intrinsic probability distribution over the possible outcomes. In particular, when the event structure is elementary it itself becomes a probabilistic event structure. (Below, by an unnormalised density operator we mean a positive, self-adjoint operator with trace less than or equal to one.)

Theorem 3. *Let E, Q be a quantum event structure with initial state ρ. Each configuration $x \in \mathcal{C}(E)$ is associated with an unnormalised density operator $\rho_x =_{\text{def}} A_x \rho A_x^\dagger$ and a value in $[0,1]$ given by $v(x) =_{\text{def}} \text{Tr}(\rho_x) = \text{Tr}(A_x^\dagger A_x \rho)$. For any $w \in \mathcal{C}^\infty(E)$, the function v restricts to a configuration-valuation v_w on the elementary event structure w (viz. the event structure with events w, and causal dependency and (trivial) consistency inherited from E); hence v_w extends to a probability measure q_w on $\mathcal{F}_w =_{\text{def}} \{x \in \mathcal{C}^\infty(E) \mid x \subseteq w\}$.*

Proof. We show v restricts to a configuration-valuation on $\mathcal{F}_w$. As $A_\varnothing = \text{id}_\mathcal{H}$, $v(\varnothing) = \text{Tr}(\rho) = 1$. By Lemma 2, we need only to show $d_v^{(n)}[y; x_1, \cdots, x_n] \geq 0$ when $y \overset{e_1}{\longrightarrow\!\!\subset} x_1, \cdots, y \overset{e_n}{\longrightarrow\!\!\subset} x_n$ in $\mathcal{F}_w$.

First, observe that if for some event e_i the operator Q_{e_i} is unitary, then $d_v^{(n)}[y; x_1, \cdots, x_n] = 0$. W.l.o.g. suppose e_n is assigned the unitary operator U. Then, $A_{x_n} = U A_y$ so

$$v(x_n) = \text{Tr}(A_{x_n}^\dagger A_{x_n} \rho) = \text{Tr}(A_y^\dagger U^\dagger U A_y \rho) = \text{Tr}(A_y^\dagger A_y \rho) = v(y)\,.$$

Let $\varnothing \neq I \subseteq \{1, \cdots, n\}$. Then, either $\bigcup_{i \in I} x_i = \bigcup_{i \in I} x_i \cup x_n$ or $\bigcup_{i \in I} x_i \overset{e_n}{\longrightarrow\!\!\subset} \bigcup_{i \in I} x_i \cup x_n$. In the either case—in the latter case by an argument similar to that above,

$$v(\bigcup_{i \in I} x_i) = v(\bigcup_{i \in I} x_i \cup x_n)\,.$$

Consequently,

$$\begin{aligned} d_v^{(n)}[y; x_1, \cdots, x_n] =& d_v^{(n-1)}[y; x_1, \cdots, x_{n-1}] - d_v^{(n-1)}[x_n; x_1 \cup x_n, \cdots, x_{n-1} \cup x_n] \\ =& v(y) - \sum_I (-1)^{|I|+1} v(\bigcup_{i \in I} x_i) - v(x_n) + \sum_I (-1)^{|I|+1} v(\bigcup_{i \in I} x_i \cup x_n) \\ =& \ 0 \end{aligned}$$

—above index I is understood to range over sets for which $\varnothing \neq I \subseteq \{1, \cdots, n\}$.

It remains to consider the case where all events e_i are assigned projection operators P_{e_i}. As $x_1, \cdots, x_n \subseteq w$ we must have that all the projection operators $P_{e_1}, \cdots, P_{e_n}$ commute.

As $[P_{e_i}, P_{e_j}] = 0$, for $1 \leq i, j \leq n$, we can assume an orthonormal basis which extends the sub-basis of eigenvectors of all the projection operators P_{e_i}, for $1 \leq i \leq n$. Let $y \subseteq x \subseteq \bigcup_{1 \leq i \leq n} x_i$. Define P_x to be the projection operator got as the composition of all the projection operators P_e for $e \in x \setminus y$—this is a projection operator, well-defined irrespective of the order of composition as the relevant projection operators commute. Define B_x to be the set of those basis vectors fixed by the projection operator P_x. In particular, P_y is the identity operator and B_y the set of all basis vectors. When $x, x' \in \mathcal{C}(E)$ with $y \subseteq x \subseteq \bigcup_{1 \leq i \leq n} x_i$ and $y \subseteq x' \subseteq \bigcup_{1 \leq i \leq n} x_i$,

$$B_{x \cup x'} = B_x \cap B_{x'}\,.$$

Also,

$$P_x |\psi\rangle = \sum_{i \in B_x} \langle i|\psi\rangle\, |i\rangle\,,$$

so

$$\langle\psi|P_x|\psi\rangle = \sum_{i\in B_x} \langle i|\psi\rangle\langle\psi|i\rangle = \sum_{i\in B_x} |\langle i|\psi\rangle|^2 ,$$

for all $|\psi\rangle \in \mathcal{H}$.

Assume $\rho = \sum_k p_k|\psi_k\rangle\langle\psi_k|$, where the ψ_k are normalised and all the p_k are positive with sum $\sum_k p_k = 1$. For x with $y \subseteq x \subseteq \bigcup_{1\leq i\leq n} x_i$,

$$\begin{aligned} v(x) &= \mathrm{Tr}(A_x^\dagger A_x \rho) \\ &= \mathrm{Tr}(A_y^\dagger P_x^\dagger P_x A_y \rho) \\ &= \mathrm{Tr}(A_y^\dagger P_x A_y \sum_k p_k |\psi_k\rangle\langle\psi_k|) \\ &= \sum_k p_k \,\mathrm{Tr}(A_y^\dagger P_x A_y |\psi_k\rangle\langle\psi_k|) \\ &= \sum_k p_k \langle A_y\psi_k|P_x|A_y\psi_k\rangle \\ &= \sum_{i\in B_x}\sum_k p_k |\langle i|A_y\psi_k\rangle|^2 = \sum_{i\in B_x} r_i , \end{aligned}$$

where we define $r_i =_{\text{def}} \sum_k p_k|\langle i|A_y\psi_k\rangle|^2$, necessarily a non-negative real for $i \in B_x$.

We now establish that

$$d_v^{(n)}[y; x_1, \cdots, x_n] = \sum_{i\in B_y \smallsetminus B_{x_1}\cup\cdots\cup B_{x_n}} r_i ,$$

for all $n \in \omega$, by mathematical induction—it then follows directly that its value is non-negative.

The base case of the induction, when $n = 0$, follows as

$$d_v^{(0)}[y;] = v(y) = \sum_{i\in B_y} r_i ,$$

a special case of the result we have just established.

For the induction step, assume $n > 0$. Observe that

$$B_y \smallsetminus B_{x_1} \cup\cdots\cup B_{x_{n-1}} = (B_y \smallsetminus B_{x_1} \cup\cdots\cup B_{x_n}) \,\dot\cup\, (B_{x_n} \smallsetminus B_{x_1\cup x_n} \cup\cdots\cup B_{x_{n-1}\cup x_n}) ,$$

where as signified the outer union is disjoint. Hence,

$$\sum_{i\in B_y\smallsetminus B_{x_1}\cup\cdots\cup B_{x_{n-1}}} r_i = \sum_{i\in B_y\smallsetminus B_{x_1}\cup\cdots\cup B_{x_n}} r_i + \sum_{i\in B_{x_n}\smallsetminus B_{x_1\cup x_n}\cup\cdots\cup B_{x_{n-1}\cup x_n}} r_i ,$$

By definition,

$$d_v^{(n)}[y; x_1, \cdots, x_n] =_{\text{def}} d_v^{(n-1)}[y; x_1, \cdots, x_{n-1}] - d_v^{(n-1)}[x_n; x_1 \cup x_n, \cdots, x_{n-1} \cup x_n]$$

—making use of the fact that we are only forming unions of compatible configurations. From the induction hypothesis,

$$d_v^{(n-1)}[y; x_1, \cdots, x_{n-1}] = \sum_{i\in B_y\smallsetminus B_{x_1}\cup\cdots\cup B_{x_{n-1}}} r_i$$

$$\text{and } d_v^{(n-1)}[x_n; x_1 \cup x_n, \cdots, x_{n-1} \cup x_n] = \sum_{i\in B_{x_n}\smallsetminus B_{x_1\cup x_n}\cup\cdots\cup B_{x_{n-1}\cup x_n}} r_i .$$

Hence

$$d_v^{(n)}[y; x_1, \cdots, x_n] = \sum_{i \in B_y \smallsetminus B_{x_1} \cup \cdots \cup B_{x_n}} r_i \,,$$

ensuring $d_v^{(n)}[y; x_1, \cdots, x_n] \geq 0$, as required.

By Theorem 2, the configuration-valuation v_w extends to a unique probability measure on $\mathcal{F}_w$. □

Corollary 3. *Let E, Q be a quantum event structure in which E is elementary. Assume an initial state ρ. Then, $x \mapsto \mathrm{Tr}(A_x^\dagger A_x \rho)$, for $x \in \mathcal{C}(E)$, is a configuration-valuation on E. It extends to a probability measure on the Borel sets of $\mathcal{C}^\infty(E)$.*

Theorem 3 is reminiscent of the consistent-histories approach to quantum theory [9] once we understand configurations as partial-order histories. The traditional decoherence/consistency conditions on histories, saying when a family of histories supports a probability distribution, have been replaced by $\subseteq$-compatibility.

Example 3. Let E comprise the quantum event structure with two concurrent events e_0 and e_1 associated with projectors P_0 and P_1, where necessarily $[P_0, P_1] = 0$. Assume an initial state $|\psi\rangle\langle\psi|$, corresponding to the pure state $|\psi\rangle$. The configuration $\{e_0, e_1\}$ is associated with the following probability distribution. The probability that e_0 succeeds is $||P_0|\psi\rangle||^2$, that e_1 succeeds $||P_1|\psi\rangle||^2$, and that both succeed is $||P_1 P_0|\psi\rangle||^2$.

In the case where P_0 and P_1 commute because $P_0 P_1 = P_1 P_0 = 0$, the events e_0 and e_1 are mutually exclusive in the sense that there is probability zero of both events e_0 and e_1 succeeding, probability $||P_0|\psi\rangle||^2$ of e_0 succeeding, $||P_1|\psi\rangle||^2$ of e_1 succeeding, and probability $1 - ||P_0|\psi\rangle||^2 - ||P_1|\psi\rangle||^2$ of getting stuck at the empty configuration where neither event succeeds.

A special case of this is the measurement of a qubit in state ψ, the measurement of 0 where $P_0 = |0\rangle\langle 0|$, and the measurement of 1 where $P_1 = |1\rangle\langle 1|$, though here $||P_0|\psi\rangle||^2 + ||P_1|\psi\rangle||^2 = 1$, as a measurement of the qubit will determine a result of either 0 or 1. □

Example 4. Let E comprise the event structure with three events e_1, e_2, e_3 with trivial causal dependency and consistency relation generated by taking $\{e_1, e_2\} \in \mathrm{Con}$ and $\{e_2, e_3\} \in \mathrm{Con}$—so $\{e_1, e_3\} \notin \mathrm{Con}$. To be a quantum event structure we must have $[Q_{e_1}, Q_{e_2}] = 0$, $[Q_{e_2}, Q_{e_3}] = 0$. The maximal configurations are $\{e_1, e_2\}$ and $\{e_2, e_3\}$. Assume an initial state $|\psi\rangle\langle\psi|$. The first maximal configuration is associated with a probability distribution where e_1 occurs with probability $||Q_{e_1}|\psi\rangle||^2$ and e_2 occurs with probability $||Q_{e_2}|\psi\rangle||^2$. The second maximal configuration is associated with a probability distribution where e_2 occurs with probability $||Q_{e_2}|\psi\rangle||^2$ and e_3 occurs with probability $||Q_{e_3}|\psi\rangle||^2$. □

5.3 Measurement

To support measurements yielding values we associate values with configurations of a quantum event structure E, Q, in the form of a measurable function, $V : \mathcal{C}^\infty(E) \to \mathbb{R}$. If the experiment results in $x \in \mathcal{C}^\infty(E)$ we obtain $V(x)$ as the measurement value resulting from the experiment. By Theorem 3, assuming an initial state given by a density operator ρ, we obtain a probability measure q_w on the sub-configurations of $w \in \mathcal{C}^\infty(E)$. This is interpreted as giving a probability distribution on the final results of an experiment w. Accordingly, w.r.t. an experiment $w \in \mathcal{C}^\infty(E)$, the expected value is

$$\mathbf{E}_w(V) =_{\text{def}} \int_{x \in \mathcal{F}_w} V(x)\ dq_w(x)\,.$$

Traditionally quantum measurement is associated with an Hermitian operator A on $\mathcal{H}$ where the possible values of a measurement are eigenvalues of A. How is this realized by a quantum event structure? Suppose the Hermitian operator has spectral decomposition

$$A = \sum_{i \in I} \lambda_i P_i$$

where orthogonal projection operators P_i are associated with eigenvalue λ_i. The projection operators satisfy $\sum_{i \in I} P_i = \text{id}_{\mathcal{H}}$ and $P_i P_j = 0$ if $i \neq j$.

Form the quantum event structure with concurrent events e_i, for $i \in I$, and $Q(e_i) = P_i$. Because the projection operators are orthogonal, $[P_i, P_j] = 0$ when $i \neq j$, so we do indeed obtain a quantum event structure. Let $V(\{e_i\}) = \lambda_i$, and take arbitrary values on all other configurations. The event structure has a single, maximum configuration $w =_{\text{def}} \{e_i \mid i \in I\}$. It is the experiment w which will correspond to traditional measurement via A. Assume an initial state $|\psi\rangle\langle\psi|$. It can be checked that the probability ascribed to each of the singleton configurations $\{e_i\}$ is $\langle\psi|P_i|\psi\rangle$, and is zero elsewhere. Consequently,

$$\mathbf{E}_w(V) = \sum_{i \in I} \lambda_i \langle\psi|P_i|\psi\rangle = \langle\psi|A|\psi\rangle$$

—the well-known expression for the expected value of the measurement A on pure state $|\psi\rangle$.

Example 5. The spin state of a spin-1/2 particle is an element of two-dimensional Hilbert space, $\mathcal{H}_2$. Traditionally the Hermitian operator for measuring spin in a particular fixed direction is

$$|{\uparrow}\rangle\langle{\uparrow}| \ - \ |{\downarrow}\rangle\langle{\downarrow}|\,.$$

It has eigenvectors $|{\uparrow}\rangle$ (spin up) with eigenvalue $+1$ and $|{\downarrow}\rangle$ (spin down) with eigenvalue -1. Accordingly, its quantum event structure comprises the two concurrent events u associated with projector $|{\uparrow}\rangle\langle{\uparrow}|$ and d with projector $|{\downarrow}\rangle\langle{\downarrow}|$. Its configurations are: $\varnothing$, $\{u\}$, $\{d\}$ and $\{u, d\}$. The value associated with the configuration $\{u\}$ is $+1$, and that with $\{d\}$ is -1. Given an initial pure state $a|{\uparrow}\rangle + b|{\downarrow}\rangle$, the probability of the experiment $\{u, d\}$ yielding value $+1$ is $|a|^2$ and that of yielding -1 is $|b|^2$. The probability that the experiment ends in configurations $\varnothing$ or $\{u, d\}$ is zero. Its expected value is $|a|^2 - |b|^2$. This would be the average value resulting from measuring the spin of a large number of particles initially in the pure state. □

An event logic One way to assign values to configurations is via logic of which the assertions will be true (taken as 1) or false (0) at a configuration. Given a countable event structure E, we can build terms for events and assertions in a straightforward way. Event terms are given by $\epsilon ::= e \in E \mid v \in \mathrm{Var}$, where Var is a set of variables over events, and assertions by

$$L ::= \epsilon \mid \mathrm{T} \mid \mathrm{F} \mid L_1 \wedge L_2 \mid L_1 \vee L_2 \mid \neg L \mid \forall v.L \mid \exists v.L\,.$$

W.r.t. an environment $\zeta : \mathrm{Var} \to E$, an assertion L denotes $[\![L]\!]\zeta$, a Borel subset of $\mathcal{C}^\infty(E)$, for example:

$$[\![e]\!]\zeta = \{x \in \mathcal{C}^\infty(E) \mid e \in x\} \qquad [\![v]\!]\zeta = \{x \in \mathcal{C}^\infty(E) \mid \zeta(v) \in x\}$$
$$[\![\forall v.L]\!]\zeta = \{x \in \mathcal{C}^\infty(E) \mid \forall e \in x.\ x \in [\![L]\!]\zeta[e/v]\}$$
$$[\![\exists v.L]\!]\zeta = \{x \in \mathcal{C}^\infty(E) \mid \exists e \in x.\ x \in [\![L]\!]\zeta[e/v]\}$$

with T, F, $\wedge$, $\vee$ and $\neg$ interpreted standardly by the set of all configurations, the emptyset, intersection, union and complement. In this logic, for example, $\neg(a{\downarrow} \wedge b{\downarrow}) \wedge \neg(a{\uparrow} \wedge b{\uparrow})$ could express the anti-correlation of the spin of particles a and b.

W.r.t. a quantum event structure with initial state, for an experiment the configuration w, the probability of the result of the quantum experiment satisfying L, a closed assertion of the logic, is

$$q_w(L \cap \mathcal{F}_w)\,,$$

which coincides with the expected value of the characteristic function for L.

5.4 Probabilistic quantum experiments

It can be useful, or even necessary, to allow the choice of which quantum measurements to perform to be made probabilistically. For example, experiments to invalidate the Bell inequalities, to demonstrate the non-locality of quantum physics, may make use of probabilistic quantum experiments.

Formally, a probability distribution over quantum experiments can be realized by a total map of event structures $f : P \to E$ where P, v is a probabilistic event structure and E, Q is a quantum event structure; the configurations of E correspond to quantum experiments assigned probabilities through P. Through the map f we can integrate the probabilistic and quantum features. Via the map f, the event structure E inherits a configuration valuation, making it itself a probabilistic event structure; we can see this indirectly by noting that if v_o is a continuous valuation on the open sets of P then $v_o f^{-1}$ is a continuous valuation on the open sets of E. On the other hand, via f the event structure P becomes a quantum event structure; an event $p \in P$ is interpreted as operation $Q(f(p))$. Of course, f can be the identity map, as is so in Example 6 below.

Suppose E, Q is a quantum event structure with initial state ρ and a measurable value function $V : \mathcal{C}^\infty(E) \to \mathbb{R}$. Recall, from Section 5.3, that the expected value of a quantum experiment $w \in \mathcal{C}^\infty(E)$ is

$$\mathbf{E}_w(V) =_{\mathrm{def}} \int_{x \in \mathcal{F}_w} V(x)\ dq_w(x)\,,$$

where q_w is the probability measure induced on $\mathcal{F}_w$ by Q and ρ. The expected value of a probabilistic quantum experiment $f : P \to E$, where P, v is a probabilistic event structure is

$$\int_{w \in \mathcal{C}^\infty(E)} \mathbf{E}_w(V)\ d\mu f^{-1}(w)\,,$$

where μ is the probability measure induced on $\mathcal{C}^\infty(P)$ by the configuration-valuation v. Specialising the value function to the characteristic function of a Borel subset $L \subseteq \mathcal{C}^\infty(E)$, perhaps given by an assertion of the event logic of Section 5.3, the probability of the result of the probabilistic experiment satisfying L is

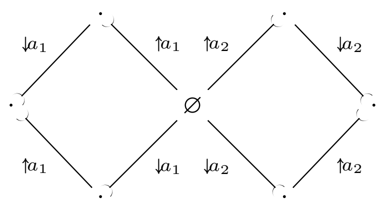
$$\int_{w \in \mathcal{C}^\infty(E)} q_w(L \cap \mathcal{F}_w)\ d\mu f^{-1}(w)\,.$$

The following example illustrates how a very simple form of probabilistic quantum experiment (in which the event structure has a discrete partial order of causal dependency) provides a basis for the analysis of Bell and EPR experiments.

Example 6. Imagine an observer who randomly chooses between measuring spin in a first fixed direction $\mathbf{a_1}$ or in a second fixed direction $\mathbf{a_2}$. Assume that the probability of measuring in the $\mathbf{a_1}$-direction is p_1 and in the $\mathbf{a_2}$-direction is p_2, where $p_1 + p_2 = 1$. The two directions $\mathbf{a_1}$ and $\mathbf{a_2}$ correspond to choices of bases $|{\uparrow} a_1\rangle$, $|{\downarrow} a_1\rangle$ and $|{\uparrow} a_2\rangle$, $|{\downarrow} a_2\rangle$ in $\mathcal{H}_2$. We describe this scenario as a probabilistic quantum experiment. The quantum event structure has four events, ${\uparrow} a_1, {\downarrow} a_1, {\uparrow} a_2, {\downarrow} a_2$, in which ${\uparrow} a_1, {\downarrow} a_1$ are concurrent, as are ${\uparrow} a_2, {\downarrow} a_2$; all other pairs of events are in conflict. The event ${\uparrow} a_1$ is associated with measuring spin up in direction $\mathbf{a_1}$ and the event ${\downarrow} a_1$ with measuring spin down in direction $\mathbf{a_1}$. Similarly, events ${\uparrow} a_2$ and ${\downarrow} a_2$ correspond to measuring spin up and down, respectively, in direction $\mathbf{a_2}$. Correspondingly, we associate events with the following projection operators:

$$Q({\uparrow} a_1) = |{\uparrow} a_1\rangle\langle{\uparrow} a_1|\,, \qquad Q({\downarrow} a_1) = |{\downarrow} a_1\rangle\langle{\downarrow} a_1|\,,$$
$$Q(u_2) = |{\uparrow} a_2\rangle\langle{\uparrow} a_2|\,, \qquad Q(d_2) = |{\downarrow} a_2\rangle\langle{\downarrow} a_2|\,.$$

The configurations of the event structure take the form



where we have taken the liberty of inscribing the events just on the covering intervals. Measurement in the $\mathbf{a_1}$-direction corresponds to the configuration $\{\uparrow a_1, \downarrow a_1\}$—the configuration to the far left in the diagram—and in the $\mathbf{a_2}$-direction to the configuration $\{\uparrow a_2, \downarrow a_2\}$—that to the far right. To describe that the probability of the measurement in the $\mathbf{a_1}$-direction is p_1 and that in the $\mathbf{a_2}$-direction is p_2, we assign a configuration valuation v for which

$$v(\{\uparrow a_1, \downarrow a_1\}) = v(\{\uparrow a_1\}) = v(\{\downarrow a_1\}) = p_1\,,$$
$$v(\{\uparrow a_2, \downarrow a_2\}) = v(\{\uparrow a_2\}) = v(\{\downarrow a_2\}) = p_2 \quad \text{and} \quad v(\varnothing) = 1\,.$$

Such a probabilistic quantum experiment is not very interesting on its own. But imagine that there are two similar observers A and B measuring the spins in directions $\mathbf{a_1}$, $\mathbf{a_2}$ and $\mathbf{b_1}$, $\mathbf{b_2}$, respectively, of two particles created so that together they have zero angular momentum, ensuring they have a total spin of zero in any direction. Then quantum mechanics predicts some remarkable correlations between the observations of A and B, even at distances where their individual choices of what directions to perform their measurements could not possibly be communicated from one observer to another. For example, were both observers to choose the same direction to measure spin, then if one measured spin up then other would have to measure spin down even though the observers were light years apart.

We can describe such scenarios by a probabilistic quantum experiment which is essentially a simple parallel composition of two versions of the (single-observer) experiment above. In more detail, make two copies of the single-observer event structure: that for A, the event structure E_A, has events $\uparrow a_1, \downarrow a_1, \uparrow a_2, \downarrow a_2$, while that for B, the event structure E_B, has events $\uparrow b_1, \downarrow b_1, \uparrow b_2, \downarrow b_2$. Assume they possess configuration valuations v_A and v_B, respectively, determined by the probabilistic choices of directions made by A and B. Write Q_A and Q_B for the respective assignments of projection operators to events of E_A and E_B. The probabilistic event structure for the two observers together is got as $E_A \| E_B$, their simple parallel composition got by juxtaposition, with configuration valuation $v(x) = v_A(x_A) \times v_B(x_B)$, for $x \in \mathcal{C}(E_A \| E_B)$, where x_A and x_B are projections of x to configurations of A and B. In this compound system an event such as *e.g.* $\uparrow a_1$ is interpreted as the projection operator $Q_A(\uparrow a_1) \otimes \mathrm{id}_{\mathcal{H}_2}$ on the Hilbert space $\mathcal{H}_2 \otimes \mathcal{H}_2$, where the combined state of the two particles belongs. We can capture the correlation or anti-correlation of the observers' measurements of spin through a value function on configurations, given by

$$V(\{\uparrow a_i, \uparrow b_j\}) = V(\{\downarrow a_i, \downarrow b_j\}) = 1\,, \quad V(\{\uparrow a_i, \downarrow b_j\}) = V(\{\downarrow a_i, \uparrow b_j\}) = -1\,, \text{ and } V(x) = 0 \text{ otherwise,}$$

and study their expectations under various initial states and choices of measurement. In this way probabilistic quantum experiments, as formalised through probabilistic and quantum event structures, provide a basis for the analysis of Bell or EPR experiments. □

The ideas of probabilistic and quantum event structures carry over to probabilistic and quantum games and their strategies [1]; the result of the play of quantum strategy against a counterstrategy is a probabilistic event structure. This is yielding operations and languages which should be helpful in a structured development and analysis of experiments on quantum systems.

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