Mini-course on proof theory

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Cambridge, June 1, 3, and 4, 2010

Supported by a Leverhulme grant

What the course is about

Term languages for proofs

Main proof system styles : Hilbert, natural deduction, **sequent calculus**

Main logics : **classical**, intuitionistic, **linear**

Semantics : operational (cut-elimination), denotational (categories, realisability/ludics)

We concentrate our attention on *propositional* logic

Structure of the course

First part (today) :
- Styles of sequent calculus rules (reversible/irreversible, multiplicative/additive)
- Completeness proof of classical logic (for provability) based on a reversible presentation.
- A syntax for sequent caculus proofs (cf. Urban's thesis)
- Non confluence (Lafont's critical pair) $\rightarrow$ **focalised system** L
- Completeness of focalised proofs

Second part (Thursday)
- Linear logic, polarised linear logic
- Translations
- Relation with Levi's CBPV
- Categorical semantics for linear, intuitionistic, and (focalised) classical logic

Third part (Friday)
- Synthetic connectives $\rightarrow$ **synthetic system** L
- Ludics as a realisability semantics
- Full completeness (via non-deterministic observers) (**Terui**)

Part I

Systems à la Hilbert

$$\frac{A \Rightarrow B \qquad A}{B}$$

plus axioms. For implication :

$$\frac{}{A \Rightarrow (B \Rightarrow A)} \qquad \frac{}{(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))}$$

$A \Rightarrow A$ is a consequence :

$$\frac{\dfrac{\dfrac{}{(A \Rightarrow ((B \Rightarrow A) \Rightarrow A)) \Rightarrow ((A \Rightarrow (B \Rightarrow A)) \Rightarrow (A \Rightarrow A))} \qquad \dfrac{}{A \Rightarrow ((B \Rightarrow A) \Rightarrow A)}}{(A \Rightarrow (B \Rightarrow A)) \Rightarrow (A \Rightarrow A)} \qquad \dfrac{}{A \Rightarrow (B \Rightarrow A)}}{A \Rightarrow A}$$

Combinatory logic

$$t ::= K \parallel S \parallel t\; t$$

$$\dfrac{\dfrac{\overline{S : (A \Rightarrow ((B \Rightarrow A) \Rightarrow A)) \Rightarrow ((A \Rightarrow (B \Rightarrow A)) \Rightarrow (A \Rightarrow A))} \qquad \overline{K : A \Rightarrow ((B \Rightarrow A) \Rightarrow A)}}{SK : (A \Rightarrow (B \Rightarrow A)) \Rightarrow (A \Rightarrow A)} \qquad \overline{K : A \Rightarrow (B \Rightarrow A)}}{SKK : A \Rightarrow A}$$

One-to-one correspondence between proofs and typing proofs

It is the first step of the Curry-Howard isomorphism

The second is to read these proof terms as programs (not a focus of this course)

Classical sequents

$$A ::= X \parallel A \wedge A \parallel A \vee A \parallel \neg A$$

A (bilateral) sequent is a pair of two (finite) multi-sets of formulas, written

$$\Gamma \vdash \Delta$$

A presentation of classical sequent calculus LK

Axiom and cut :

$$\frac{}{\Gamma, A \vdash A, \Delta} \qquad \frac{\Gamma \vdash A, \Delta \qquad \Gamma, A \vdash \Delta}{\Gamma \vdash \Delta}$$

Right introduction rules :

$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta} \qquad \frac{\Gamma \vdash A_1, \Delta \qquad \Gamma \vdash A_2, \Delta}{\Gamma \vdash A_1 \wedge A_2, \Delta}$$

$$\frac{\Gamma \vdash A_1, \Delta}{\Gamma \vdash A_1 \vee A_2, \Delta} \qquad \frac{\Gamma \vdash A_2, \Delta}{\Gamma \vdash A_1 \vee A_2, \Delta}$$

Left introduction rules

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta} \qquad \frac{\Gamma, A_1, A_2 \vdash \Delta}{\Gamma, A_1 \wedge A_2 \vdash \Delta} \qquad \frac{\Gamma, A_1 \vdash \Delta \qquad \Gamma, A_2 \vdash \Delta}{\Gamma, A_1 \vee A_2 \vdash \Delta}$$

We say that A, A_1, A_2, $\neg A$, $A_1 \wedge A_2$, $A_1 \vee A_2$ are the *active* formulas of the rules.

Implication as a derived connective

Set $A \Rightarrow B = \neg(A \wedge \neg B)$

$$\frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \Rightarrow B} \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \Rightarrow B \vdash \Delta}$$

$$\dfrac{\dfrac{\dfrac{\Gamma, A \vdash B, \Delta}{\Gamma, A, \neg B \vdash \Delta}}{\Gamma, A \wedge \neg B \vdash \Delta}}{\Gamma \vdash \neg(A \wedge \neg B), \Delta} \qquad \dfrac{\dfrac{\dfrac{}{\Gamma \vdash A, \Delta} \quad \dfrac{\Gamma, B \vdash \Delta}{\Gamma \vdash \neg B, \Delta}}{\Gamma \vdash A \wedge \neg B, \Delta}}{\Gamma, \neg(A \wedge \neg B) \vdash \Delta}$$

Why sequents ?

$A \vdash B$ as $\vdash A \Rightarrow B$ both read as "A implies B", which does not help...

Proof search : formula decomposition

Other motivation : back to secondary school, think of a polynom, say

$$p(x) = x^2 + 3mx + (1 - m)$$

that depends on *variable* x and *parameter* m, and whose roots are expressed as formal expressions depending on m.

$$m : \mathbb{R} \vdash (x \mapsto p(x)) : \mathbb{R} \Rightarrow \mathbb{R}$$

We also have :

$\vdash (m \mapsto (x \mapsto x^2 + 3mx + (1 - m)) : \mathbb{R} \to (\mathbb{R} \to \mathbb{R})$ and
$m : \mathbb{R}, x : \mathbb{R} \vdash x^2 + 3mx + (1 - m)) : \mathbb{R}$

but only the first typing reflects the different roles played by m and x

Weakening and Contraction

Weakening   Contraction

$$\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta} \qquad \frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta} \qquad \frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}$$

In our presentation of LK :

– Weakening is admissible : add the weakened formulas everywhere in the sequents of the proof. In fact, our terms do not distinguish a proof of $\Gamma, A \vdash \Delta$ where A is never active from the proof of $\Gamma \vdash \Delta$ of which the former is a weakening. We say that weakening is transparent.

– Contraction is derivable :

$$\frac{\Gamma \vdash A, A, \Delta \qquad \overline{\Gamma, A \vdash A, \Delta}}{\Gamma \vdash A, \Delta}$$

Hence we call our rule the cut/contraction

Additive versus multiplicative

$$\frac{\Gamma_1 \vdash A, \Delta_1 \quad \Gamma_2 \vdash B, \Delta_2}{\Gamma_1, \Gamma_2 \vdash A \wedge B, \Delta_1, \Delta_2}$$

is “derivable" (if weakening is viewed as transparent)

Note that multiplicative cut is just cut (and interestingly, only the cut is multiplicative in Gentzen’s original presentation)

Reversible versus irreversible

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash A_1, A_2, \Delta}{\Gamma \vdash A_1 \vee A_2, A_2, \Delta}}{\Gamma \vdash A_1 \vee A_2, A_1 \vee A_2, \Delta}}{\Gamma \vdash A_1 \vee A_2, \Delta} \qquad \dfrac{\dfrac{\Gamma, A_1 \vdash \Delta}{\Gamma, A_1, A_2 \vdash \Delta}}{\Gamma, A_1 \wedge A_2 \vdash \Delta} \qquad \dfrac{\dfrac{\Gamma, A_2 \vdash \Delta}{\Gamma, A_1, A_2 \vdash \Delta}}{\Gamma, A_1 \wedge A_2 \vdash \Delta}$$

We have chosen an irreversible disjunction on the right and a reversible conjunction on the left, as an anticipation of focalisation

Elimination vs left introduction : natural deduction

$$\frac{\Gamma \vdash A \Rightarrow B, \Delta \qquad \Gamma \vdash A, \Delta}{\Gamma \vdash B, \Delta}$$

$$\frac{\Gamma \vdash A \Rightarrow B, \Delta \qquad \dfrac{\Gamma \vdash A, B, \Delta \quad \overline{\Gamma, B \vdash B, \Delta}}{\Gamma, A \Rightarrow B \vdash B, \Delta}}{\Gamma \vdash B, \Delta}$$

For conjunction :

$$\frac{\Gamma \vdash A \wedge B, \Delta}{\Gamma \vdash A, \Delta} \qquad \frac{\Gamma \vdash A \wedge B, \Delta}{\Gamma \vdash B, \Delta}$$

Atomic axioms

$$\frac{}{\Gamma, X \vdash X, \Delta}$$

Indeed

$$\dfrac{\dfrac{\dfrac{}{\Gamma, A \vdash A, \Delta}}{\Gamma, \neg A, A \vdash \Delta}}{\Gamma, \neg A \vdash \neg A, \Delta} \qquad \dfrac{\dfrac{\dfrac{}{\Gamma, A_1, A_2 \vdash A_1, \Delta} \quad \dfrac{}{\Gamma, A_1, A_2 \vdash A_2, \Delta}}{\Gamma, A_1, A_2 \vdash A_1 \wedge A_2, \Delta}}{\Gamma, A_1 \wedge A_2 \vdash A_1 \wedge A_2, \Delta} \qquad \dfrac{\dfrac{\dfrac{}{\Gamma, A_1 \vdash A_1, \Delta}}{\Gamma, A_1 \vdash A_1 \vee A_2, \Delta} \quad \dfrac{\dfrac{}{\Gamma, A_2 \vdash A_2, \Delta}}{\Gamma, A_2 \vdash A_1 \vee A_2, \Delta}}{\Gamma, A_1 \vee A_2 \vdash A_1 \vee A_2, \Delta}$$

Completeness of LK for provability

Lemma : A sequent $A_1, \ldots, A_m \vdash B_1, \ldots, B_n$ is satisfied iff one of the B_j's is satisfied or one of the A_i's is satisfied.

Corollary : An atomic sequent $X_1, \ldots, X_m \vdash Y_1, \ldots, Y_n$ is valid iff there exists i, j s.t. $X_i = Y_j$

Theorem : Every valid sequent admits a (cut-free) proof in the following presentation of LK :

$$\frac{}{\Gamma, X \vdash X, \Delta}$$

$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta} \qquad \frac{\Gamma \vdash A_1, \Delta \qquad \Gamma \vdash A_2, \Delta}{\Gamma \vdash A_1 \wedge A_2, \Delta} \qquad \frac{\Gamma \vdash A_1, A_2, \Delta}{\Gamma \vdash A_1 \vee A_2, \Delta}$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta} \qquad \frac{\Gamma, A_1, A_2 \vdash \Delta}{\Gamma, A_1 \wedge A_2 \vdash \Delta} \qquad \frac{\Gamma, A_1 \vdash \Delta \qquad \Gamma, A_2 \vdash \Delta}{\Gamma, A_1 \vee A_2 \vdash \Delta}$$

“Cut elimination” via completeness.

Various presentations of LK

1) Pushing weakening in the axiom makes weakening transparent, whatever style is used for all other rules. Assuming such transparent weakening, we have :

2) The cut/contraction rule is equivalent to the multiplicative cut rule + the contraction rules

3) then we have choices as to the reversibility or irreversibility for $\vee$ on the right and for $\wedge$ on the left :

1. Symmetric, both reversible : friendly for completeness
2. Symmetric, both irreversible : Gentzen's original choice
3. Dissymmetric. There are dual choices. The one presented here ($\vee$ irreversible on the right and $\wedge$ reversible on the left) is friendly to the *call-by-value* encoding of implication $A \Rightarrow B = \neg(A \wedge \neg B)$. It is our guide all along the course
4. Dissymmetric. The dual choice is friendly to the call-by-name encoding of implication $A \Rightarrow B = (\neg A) \vee B$

Cut elimination : logical cuts

$$\dfrac{\dfrac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta} \qquad \dfrac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}}{\Gamma \vdash \Delta} \quad \longrightarrow \quad \dfrac{\Gamma, A \vdash \Delta \qquad \Gamma \vdash A, \Delta}{\Gamma \vdash \Delta}$$

$$\dfrac{\dfrac{\Gamma, A_1 \vdash \Delta \quad \Gamma, A_1 \vdash \Delta}{\Gamma, A_1 \vee A_2 \vdash \Delta} \qquad \dfrac{\Gamma \vdash A_1, \Delta}{\Gamma \vdash A_1 \vee A_2, \Delta}}{\Gamma \vdash \Delta} \quad \longrightarrow \quad \dfrac{\Gamma, A_1 \vdash \Delta \qquad \Gamma \vdash A_1, \Delta}{\Gamma \vdash \Delta}$$

$$\dfrac{\dfrac{\Gamma, A_1, A_2 \vdash \Delta}{\Gamma, A_1 \wedge A_2 \vdash \Delta} \qquad \dfrac{\Gamma \vdash A_1, \Delta \quad \Gamma \vdash A_2, \Delta}{\Gamma \vdash A_1 \wedge A_2, \Delta}}{\Gamma \vdash \Delta} \quad \longrightarrow \quad \dfrac{\dfrac{\Gamma, A_1, A_2 \vdash \Delta \quad \Gamma, A_2 \vdash A_1, \Delta}{\Gamma, A_2 \vdash \Delta} \qquad \Gamma \vdash A_2, \Delta}{\Gamma \vdash \Delta}$$

Cut elimination : commutative cuts

$$\dfrac{\dfrac{\Gamma, A, B \vdash \Delta}{\Gamma, A \vdash \neg B, \Delta} \qquad \Gamma \vdash A, \neg B, \Delta}{\Gamma \vdash \neg B, \Delta} \quad \longrightarrow \quad \dfrac{\dfrac{\dfrac{\Gamma, A, B \vdash \neg B, \Delta \qquad \Gamma, B \vdash A, \neg B, \Delta}{\Gamma, B \vdash \neg B, \Delta}}{\Gamma \vdash \neg B, \neg B, \Delta}}{\Gamma \vdash \neg B, \Delta}$$

Erasing :

$$\dfrac{\dfrac{}{\Gamma, A, B \vdash B, \Delta} \qquad \Gamma, B \vdash A, B, \Delta}{\Gamma, B \vdash B, \Delta} \quad \longrightarrow \quad \dfrac{}{\Gamma, B \vdash B, \Delta}$$

Duplication :

$$\dfrac{\dfrac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta} \qquad \Gamma \vdash A, \Delta}{\Gamma \vdash \Delta} \quad \longrightarrow \quad \dfrac{\dfrac{\Gamma, A, A \vdash \Delta \quad \Gamma, A \vdash A, \Delta}{\Gamma, A \vdash, \Delta} \qquad \Gamma \vdash A, \Delta}{\Gamma \vdash \Delta}$$

Curien-Herbelin's syntactic kit

Expressions	Contexts	Commands
$\Gamma \vdash v : A \,\vert\, \Delta$	$\Gamma \,\vert\, e : A \vdash \Delta$	$c : (\Gamma \vdash \Delta)$

where Γ is a set of pairs $x : N$ and Δ is a set of pairs $\alpha : P$ (ordinary variables, continuation variables)

$$\frac{\dfrac{c_2 : (\Gamma, x : A \vdash \Delta)}{\Gamma \,|\, \tilde{\mu}x.c_2 : A \vdash \Delta} \qquad \dfrac{c_1 : (\Gamma \vdash \alpha : A, \Delta)}{\Gamma \vdash \mu\alpha.c_1 : A \,|\, \Delta}}{\langle \mu\alpha.c_1 \mid \tilde{\mu}x.c_2 : A \rangle : (\Gamma \vdash \Delta)}$$

$$\begin{aligned} v &::= x \parallel \mu\alpha.c \parallel \ldots \\ e &::= \alpha \parallel \tilde{\mu}x.c \parallel \ldots \\ c &::= \langle v \mid e \rangle \end{aligned}$$

The variable x is bound in $\tilde{\mu}x.c$ (likewise for $\mu\alpha.c$)

We give the collective name of “system L for syntaxes based on this kit

All proofs are equal...

Operational semantics (first try) :

$$\langle \mu\alpha.c \mid e \rangle \longrightarrow c[e/\alpha] \qquad\qquad \langle v \mid \tilde{\mu}x.c \rangle \longrightarrow c[v/x]$$

Lafont's critical pair (if α is not free in c_1 and x is not free in c_2) :

$$c_1 = c_1[\tilde{\mu}x.c_2 : A/\alpha] \longleftarrow \langle \mu\alpha.c_1 \mid \tilde{\mu}x.c_2 : A \rangle \longrightarrow c_2[\mu\alpha.c_1/x] = c_2$$

A faithful (uninspiring) proof language for LK 1/2

Commands	$c ::= \langle x \mid \alpha \rangle \parallel \langle v \mid \alpha \rangle \parallel \langle x \mid e \rangle \parallel \langle \mu\alpha.c \mid \tilde{\mu}x.c \rangle$
Expressions	$v ::= (\tilde{\mu}x.c)^\bullet \parallel (\mu\alpha.c, \mu\alpha.c) \parallel inl(\mu\alpha.c) \parallel inr(\mu\alpha.c)$
Contexts	$e ::= \tilde{\mu}\alpha^\bullet.c \parallel \tilde{\mu}(x_1, x_2).c \parallel \tilde{\mu}[inl(x_1).c_1 \vert inr(x_2).c_2]$

(In $\langle v \mid \alpha \rangle$ (resp. $\langle x \mid e \rangle$), we suppose α (resp. x) fresh for v (resp. e).)

$$\frac{}{\langle x \mid \alpha \rangle : (\Gamma, x : A \vdash \alpha : A, \Delta)} \qquad \frac{c : (\Gamma \vdash \alpha : A, \Delta) \qquad d : (\Gamma, x : A \vdash \Delta)}{\langle \mu\alpha.c \mid \tilde{\mu}x.d \rangle : (\Gamma \vdash \Delta)}$$

$$\frac{c : (\Gamma, x : A \vdash \Delta)}{\Gamma \vdash (\tilde{\mu}x.c)^\bullet : \neg A \mid \Delta} \qquad \frac{c_1 : (\Gamma \vdash \alpha_1 : A_1, \Delta) \qquad c_2 : (\Gamma \vdash \alpha_2 : A_2, \Delta)}{\Gamma \vdash (\mu\alpha_1.c_1, \mu\alpha_2.c_2) : A_1 \wedge A_2 \mid \Delta} \qquad \frac{c_1 : (\Gamma \vdash \alpha_1 : A_1, \Delta)}{\Gamma \vdash inl(\mu\alpha_1.c_1) : A_1 \vee A_2 \mid \Delta}$$

$$\frac{c : (\Gamma \vdash \alpha : A, \Delta)}{\Gamma \mid \tilde{\mu}\alpha^\bullet.c : \neg A \vdash \Delta} \qquad \frac{c : (\Gamma, x_1 : A_1, x_2 : A_2 \vdash \Delta)}{\Gamma \mid \tilde{\mu}(x_1, x_2).c : A_1 \wedge A_2 \vdash \Delta} \qquad \frac{c_1 : (\Gamma, x_1 : A_1 \vdash \Delta) \qquad c_2 : (\Gamma, x_2 : A_2 \vdash \Delta)}{\Gamma \mid \tilde{\mu}[inl(x_1).c_1 \vert inr(x_2).c_2] : A_1 \vee A_2 \vdash \Delta}$$

$$\frac{\Gamma \vdash v : A \mid \Delta}{\langle v \mid \alpha \rangle : (\Gamma \vdash \alpha : A, \Delta)} \qquad \frac{\Gamma \mid e : A \vdash \Delta}{\langle x \mid e \rangle : (\Gamma, x : A \vdash \Delta)}$$

A faithful (uninspiring) proof language for LK 2/2

Logical rules (redexes of the form $\langle \mu\alpha.\langle v \mid \alpha\rangle \mid \tilde{\mu}x.\langle x \mid e\rangle\rangle$) :

$\langle \mu\alpha.\langle (\tilde{\mu}x.c)^\bullet \mid \alpha\rangle \mid \tilde{\mu}y.\langle y \mid \tilde{\mu}\alpha^\bullet.d\rangle\rangle \longrightarrow \langle \mu\alpha.d \mid \tilde{\mu}x.c\rangle$ (similar rules for conjunction and disjunction)

Commutative rules (going "up left", redexes of the form $\langle \mu\alpha.\langle v \mid \beta\rangle \mid \tilde{\mu}x.c\rangle$) :

$\langle \mu\alpha.\langle (\tilde{\mu}y.c)^\bullet \mid \beta\rangle \mid \tilde{\mu}x.d\rangle \longrightarrow \langle \mu\beta'.\langle (\tilde{\mu}y.\langle \mu\alpha.c \mid \tilde{\mu}x.d\rangle)^\bullet \mid \beta'\rangle \mid \tilde{\mu}y.\langle y \mid \beta\rangle\rangle$ ($\neg$ right)
(similar rules of commutation with the other right introduction rules and with the left introduction rules)
$\langle \mu\alpha.\langle \mu\beta.\langle y \mid \beta\rangle \mid \tilde{\mu}y'.c\rangle \mid \tilde{\mu}x.d\rangle \longrightarrow \langle \mu\beta.\langle y \mid \beta\rangle \mid \tilde{\mu}y'.\langle \mu\alpha.c \mid \tilde{\mu}x.d\rangle\rangle$ (contraction right)
$\langle \mu\alpha.\langle \mu\beta'.c \mid \tilde{\mu}y.\langle y \mid \beta\rangle\rangle \mid \tilde{\mu}x.d\rangle \longrightarrow \langle \mu\beta'.\langle \mu\alpha.c \mid \tilde{\mu}x.d\rangle \mid \tilde{\mu}y.\langle y \mid \beta\rangle\rangle$ (contraction left)
$\langle \mu\alpha.\langle \mu\alpha'.c \mid \tilde{\mu}x'.\langle x' \mid \alpha\rangle\rangle \mid \tilde{\mu}x.d\rangle \longrightarrow \langle \mu\alpha.\langle \mu\alpha'.c \mid \tilde{\mu}x.d\rangle \mid \tilde{\mu}x.d\rangle$ (duplication)
$\langle \mu\alpha.\langle y \mid \beta\rangle \mid \tilde{\mu}x.d\rangle \longrightarrow \langle y \mid \beta\rangle$ (erasing)

Commutative rules (going "up right", redexes of the form $\langle \mu\alpha.c \mid \tilde{\mu}x.\langle y \mid e\rangle\rangle$) : similar rules.

A simple twist makes it more inspiring !

Making activation "first class"

Commands	$c ::= \langle v \mid e \rangle \parallel c[\sigma]$
Expressions	$v ::= x \parallel \mu\alpha.c \parallel e^{\bullet} \parallel (v,v) \parallel inl(v) \parallel inr(v) \parallel v[\sigma]$
Contexts	$e ::= \alpha \parallel \tilde{\mu}x.c \parallel \tilde{\mu}\alpha^{\bullet}.c \parallel \tilde{\mu}(x_1,x_2).c \parallel \tilde{\mu}[inl(x_1).c_1 \vert inr(x_2).c_2] \parallel e[\sigma]$

where σ is a list $v_1/x_1, \ldots, v_m/x_m, e_1/\alpha_1, \ldots, e_n/\alpha_n$

$$\frac{}{\Gamma\, ,\, x : A \vdash x : A \,|\, \Delta} \qquad \frac{}{\Gamma \,|\, \alpha : A \vdash \alpha : A\, ,\, \Delta} \qquad \frac{\Gamma \vdash v : A \,|\, \Delta \qquad \Gamma \,|\, e : A \vdash \Delta}{\langle v \mid e \rangle : (\Gamma \vdash \Delta)}$$

$$\frac{c : (\Gamma\, ,\, x : A \vdash \Delta)}{\Gamma \,|\, \tilde{\mu}x.c : A \vdash \Delta} \qquad \frac{c : (\Gamma \vdash \alpha : A\, ,\, \Delta)}{\Gamma \vdash \mu\alpha.c : A \,|\, \Delta}$$

$$\frac{\Gamma \,|\, e : A \vdash \Delta}{\Gamma \vdash e^{\bullet} : \neg A \,|\, \Delta} \qquad \frac{\Gamma \vdash v_1 : A_1 \,|\, \Delta \qquad \Gamma \vdash v_2 : A_2 \,|\, \Delta}{\Gamma \vdash (v_1, v_2) : A_1 \wedge A_2 \,|\, \Delta} \qquad \frac{\Gamma \vdash v_1 : A_1 \,|\, \Delta}{\Gamma \vdash inl(v_1) : A_1 \vee A_2 \,|\, \Delta}$$

$$\frac{c : (\Gamma, x_1 : A_1, \ldots, x_m : A_m \vdash \alpha_1 : B_1, \ldots, \alpha_n : B_n) \ \ldots\ \Gamma \vdash v_i : A_i \,|\, \Delta \ \ldots\ \Gamma \,|\, e_j : B_j \vdash \Delta \ \ldots}{c[v_1/x_1, \ldots, v_m/x_m, e_1/\alpha_1, \ldots, e_n/\alpha_n] : (\Gamma \vdash \Delta)} \quad \text{(idem } v[\sigma], e[\sigma])$$

(rules unchanged for the $\tilde{\mu}$'s)

Commutative cuts as explicit substitutions !

(control)	$\langle \mu\alpha.c \mid e\rangle \longrightarrow c[e/\alpha]$
	$\langle v \mid \tilde{\mu}x.c\rangle \longrightarrow c[v/x]$
(logical)	$\langle e^\bullet \mid \tilde{\mu}\alpha^\bullet.c\rangle \longrightarrow c[e/\alpha]$
	$\langle (v_1, v_2) \mid \tilde{\mu}(x_1, x_2).c\rangle \longrightarrow c[v_1/x_1, v_2/x_2]$
	$\langle inl(v_1) \mid \tilde{\mu}[inl(x_1).c_1 \mid inr(x_2).c_2]\rangle \longrightarrow c_1[v_1/x_1]$
(commutation)	$\langle v \mid e\rangle[\sigma] \longrightarrow \langle v[\sigma] \mid e[\sigma]\rangle$
	$x[\sigma] \longrightarrow x$ (x not declared in σ)
	$x[v/x, \sigma] \longrightarrow v$ (idem $\alpha[\sigma]$)
	$(\mu\alpha.c)[\sigma] \longrightarrow \mu\alpha.(c[\sigma])$ (capture avoiding)
	$\vdots$

Relation with the previous rules : for all s_1, s_2 such that $s_1 \longrightarrow s_2$ in the first system, there exists s such that

$s_1 \longrightarrow^* s \ {}^*\!\!\longleftarrow s_2$ in the new system

Focalisation

A focalised proof search alternates between right and left phases, as follows :

- *Left phase* : Decompose (copies of) formulas on the left, in any order. Every decomposition of a negation on the left feeds the right part of the sequent. At any moment, one can change the phase from left to right.

- *Right phase* : Choose a formula A on the right, and *hereditarily* decompose a copy of it in all branches of the proof search. This *focusing* in any branch can only end with an axiom (which ends the proof search in that branch), or with a decomposition of a negation, which prompts a phase change back to the left. Etc. . .

Polarisation

To account for right focalisation, we introduce a fourth kind of judgement : the *values*, typed as $(\Gamma \vdash V : A \,;\, \Delta)$

We also make official the existence of two disjunctions (since the behaviours of the conjunction on the left and of the disjunction on the right are different) and two conjunctions, by renaming $\wedge, \vee, \neg$ as $\otimes, \oplus, \neg^+$, respectively (**positive** formulas) :

$$P ::= X \parallel P \otimes P \parallel P \oplus P \parallel \neg^+ P$$

We can define their De Morgan duals (**negative** formulas) :

$$N ::= \overline{X} \parallel N \invamp N \parallel N \& N \parallel \neg^- N$$

They restore the duality of connectives (think of P on the left as being a $\overline{P}$ in a unilateral sequent $\vdash \overline{\Gamma}, \Delta$).

Syntax of focalising system L

Commands	$c ::= \langle v \mid e \rangle \parallel c[\sigma]$
Expressions	$v ::= V^{\Diamond} \parallel \mu\alpha.c \parallel v[\sigma]$
Values	$V ::= x \parallel (V, V) \parallel inl(V) \parallel inr(V) \parallel e^{\bullet} \parallel V[\sigma]$
Contexts	$e ::= \alpha \parallel \tilde{\mu}x.c \parallel e[\sigma] \parallel$
	$\tilde{\mu}\alpha^{\bullet}.c \parallel \tilde{\mu}(x_1, x_2).c \parallel \tilde{\mu}[inl(x_1).c_1 \mid inr(x_2).c_2]$

(control)	$\langle \mu\alpha.c \mid e \rangle \longrightarrow c[e/\alpha]$
	$\langle V^{\Diamond} \mid \tilde{\mu}x.c \rangle \longrightarrow c[V/x]$
(logical)	$\langle (e^{\bullet})^{\Diamond} \mid \tilde{\mu}\alpha^{\bullet}.c \rangle \longrightarrow c[e/\alpha]$
	$\langle (V_1, V_2)^{\Diamond} \mid \tilde{\mu}(x_1, x_2).c \rangle \longrightarrow c[V_1/x_1, V_2/x_2]$
	$\langle inl(V_1)^{\Diamond} \mid \tilde{\mu}[inl(x_1).c_1 \mid inr(x_2).c_2] \rangle \longrightarrow c_1[V_1/x_1]$
(commutation)	$\langle v \mid e \rangle[\sigma] \longrightarrow \langle v[\sigma] \mid e[\sigma] \rangle$ etc ...

System LKQ

$$\frac{}{\Gamma\,,\,x:P\vdash x:P\,;\,\Delta}\qquad\frac{}{\Gamma\,|\,\alpha:P\vdash\alpha:P\,,\,\Delta}\qquad\frac{\Gamma\vdash v:P\,|\,\Delta\qquad\Gamma\,|\,e:P\vdash\Delta}{\langle v\,|\,e\rangle:(\Gamma\vdash\Delta)}$$

$$\frac{c:(\Gamma\,,\,x:P\vdash\Delta)}{\Gamma\,|\,\tilde{\mu}x.c:P\vdash\Delta}\qquad\frac{c:(\Gamma\vdash\alpha:P\,,\,\Delta)}{\Gamma\vdash\mu\alpha.c:P\,|\,\Delta}\qquad\frac{\Gamma\vdash V:P\,;\,\Delta}{\Gamma\vdash V^{\diamond}:P\,|\,\Delta}$$

$$\frac{\Gamma\,|\,e:P\vdash\Delta}{\Gamma\vdash e^{\bullet}:\neg^{+}P\,;\,\Delta}\qquad\frac{\Gamma\vdash V_1:P_1\,;\,\Delta\qquad\Gamma\vdash V_2:P_2\,;\,\Delta}{\Gamma\vdash(V_1,V_2):P_1\otimes P_2\,;\,\Delta}\qquad\frac{\Gamma\vdash V_1:P_1\,;\,\Delta}{\Gamma\vdash inl(V_1):P_1\oplus P_2\,;\,\Delta}$$

$$\frac{c:(\Gamma\vdash\alpha:P,\Delta)}{\Gamma\,|\,\tilde{\mu}\alpha^{\bullet}.c:\neg^{+}P\vdash\Delta}\qquad\frac{c:(\Gamma,x_1:P_1,x_2:P_2\vdash\Delta)}{\Gamma\,|\,\tilde{\mu}(x_1,x_2).c:P_1\otimes P_2\vdash\Delta}\qquad\frac{c_1:(\Gamma,x_1:P_1\vdash\Delta)\qquad c_2:(\Gamma,x_2:P_2\vdash\Delta)}{\Gamma\,|\,\tilde{\mu}[inl(x_1).c_1|inr(x_2).c_2]:P_1\oplus P_2\vdash\Delta}$$

$$\frac{\ldots\quad\Gamma\vdash V:P\,;\,\Delta\ \ldots\ \Gamma\,|\,e:Q\vdash\Delta\ \ldots\ c:(\Gamma\ \ldots,q:P,\ldots\vdash\Delta,\ldots,\alpha:Q,\ldots)}{c[\ldots,V/q,\ldots,e/\alpha]:(\Gamma\vdash\Delta)}$$

(idem $v[\sigma], V[\sigma], e[\sigma]$)

Completeness of LKQ

If $\Gamma \vdash \Delta$ is provable in LK, then it is provable in LKQ.

We can define $inl(\mu\alpha_1.c_1)$ as

$$\Gamma \vdash \mu\alpha.\langle \mu\alpha_1.c_1 \mid \tilde{\mu}x_1.\langle (inl(x_1))^{\lozenge} \mid \alpha\rangle\rangle : P_1 \oplus P_2 \mid \Delta \qquad \text{(idem } \textit{inr}\text{)}$$

and $(\mu\alpha_1.c_1, \mu\alpha_2.c_2)$ as

$$(\Gamma \vdash \mu\alpha.\langle \mu\alpha_2.c_2 \mid \tilde{\mu}x_2.\langle \mu\alpha_1.c_1 \mid \tilde{\mu}x_1.\langle (x_1, x_2)^{\lozenge} \mid \alpha\rangle\rangle\rangle : P_1 \otimes P_2 \mid \Delta)$$

Note that the translation introduces cuts (that are then eliminated, yielding a *cut-free* focalised proof)

Part II

Linear logic 1/2

$$A ::= X \parallel X^{\perp} \parallel A \otimes A \parallel 1 \parallel A \parr A \parallel \perp \parallel A \oplus A \parallel 0 \parallel A \& A \parallel \top \parallel !A \parallel ?A$$

Negation implicit except on atoms

AXIOM

$$\frac{}{\vdash A, A^{\perp}}$$

CUT

$$\frac{\vdash A, \Gamma_1 \quad \vdash A^{\perp}, \Gamma_2}{\vdash \Gamma_1, \Gamma_2}$$

Linear logic 2/2

MULTIPLICATIVES

$$\frac{\vdash A, B, \Gamma}{\vdash A \parr B, \Gamma} \qquad \frac{\vdash A, \Gamma_1 \quad \vdash B, \Gamma_2}{\vdash A \otimes B, \Gamma_1, \Gamma_2}$$

ADDITIVES

$$\frac{\vdash A, \Gamma}{\vdash A \oplus B, \Gamma} \quad \frac{\vdash B, \Gamma}{\vdash A \oplus B, \Gamma} \qquad \frac{\vdash A, \Gamma \quad \vdash B, \Gamma}{\vdash A \& B, \Gamma}$$

UNITS

$$\frac{\vdash \Gamma}{\vdash \bot, \Gamma} \qquad \frac{}{\vdash 1} \qquad \text{no rule for } 0 \qquad \frac{}{\vdash \top, \Gamma}$$

EXPONENTIALS

Contraction	**Weakening**	**Dereliction**	**Promotion**
$\dfrac{\vdash ?A, ?A, \Gamma}{\vdash ?A, \Gamma}$	$\dfrac{\vdash \Gamma}{\vdash ?A, \Gamma}$	$\dfrac{\vdash \Gamma, A}{\vdash \Gamma, ?A}$	$\dfrac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}$

Girard's (call-by-name) translation 1/2

This translation takes (a proof of) a judgement $\Gamma \vdash M : A$ and turns it into

a proof $[\![\Gamma \vdash M : A]\!]$ of $\vdash ?(\Gamma^*)^\perp, A^*$,

where $A^* = A$ (A atomic), $(B \to C)^* = ?(B^*)^\perp \parr C^*$,
and $?(\Gamma^*)^\perp = \{?(A^*)^\perp \mid A \in \Gamma\}$

Variable

$$[\![\Gamma, x : A \vdash x : A]\!] = \dfrac{\dfrac{\dfrac{}{\vdash A^\perp, A}}{\vdash ?\Gamma^\perp, A^\perp, A}}{\vdash ?\Gamma^\perp, ?A^\perp, A}$$

Abstraction

$$[\![\Gamma \vdash \lambda x.M : A \to B]\!] = \dfrac{\begin{array}{c}[\![\Gamma, x : A \vdash M : B]\!] \\ \vdots \\ \vdash ?\Gamma^\perp, ?A^\perp, B\end{array}}{\vdash ?\Gamma^\perp, (?A^\perp \parr B)}$$

Girard's (call-by-name) translation 2/2

Application

$$[\![\Gamma \vdash MN : B]\!] \;=\; \dfrac{\dfrac{\begin{array}{c}[\![\Gamma \vdash M : A \to B]\!]\\ \vdots \\ \vdash ?\Gamma^{\perp}, ?A^{\perp} \parr B\end{array} \qquad \dfrac{\dfrac{\begin{array}{c}[\![\Gamma \vdash N : A]\!]\\ \vdots \\ \vdash ?\Gamma^{\perp}, A\end{array}}{\vdash ?\Gamma^{\perp}, !A} \qquad \dfrac{}{\vdash B^{\perp}, B}}{\vdash ?\Gamma^{\perp}, !A \otimes B^{\perp}, B}}{\dfrac{\vdash ?\Gamma^{\perp}, ?\Gamma^{\perp}, B}{\vdash ?\Gamma^{\perp}, B}}$$

Encoding CBV $\lambda(\mu)$-calculus into LKQ

We define the following derived CBV implication and terms :

$$P \to^v Q = \neg^{+}(P \otimes \neg^{+}Q)$$

$$\lambda x.v = ((\tilde{\mu}(x, \alpha^{\bullet}).\langle v \mid \alpha \rangle)^{\bullet})^{\lozenge} \qquad v_1 v_2 = \mu\alpha.\langle v_2 \mid \tilde{\mu}x.\langle v_1 \mid ((x, \alpha^{\bullet})^{\lozenge})^{\blacklozenge} \rangle\rangle$$

where $\tilde{\mu}(x, \alpha^{\bullet}).c$ is an abbreviation for $\tilde{\mu}(x, y).\langle y^{\lozenge} \mid \tilde{\mu}\alpha^{\bullet}.c\rangle$ and where $V^{\blacklozenge}$ stands for $\tilde{\mu}\alpha^{\bullet}.\langle V^{\lozenge} \mid \alpha \rangle$

The translation extends to (call-by-value) $\lambda\mu$-calculus

The translation makes also sense in the untyped setting

Encoding CBN $\lambda(\mu)$-calculus 1/2

What about CBN ? We can translate it to LKQ, but at the price of translating terms to contexts, which is kind of a violence...

But keeping the *same* term language, we can type sequents of negative formulas, giving rise to a dual logic LKT :

$$N := \overline{X} \parallel N \parr N \parallel N \& N \parallel \neg^{-} N$$

Four kinds of judgements :

$$c : (\Gamma \vdash \Delta) \quad \Gamma\,;\, E : N \vdash \Delta \quad \Gamma \,|\, e : N \vdash \Delta \quad \Gamma \vdash v : N \,|\, \Delta$$

We would have arrived to this logic naturally if we had chosen to present LK with a reversible disjunction on the right and an irreversible conjunction on the left (cf. above)

Focalising system L (negatively-minded repainting)

Commands	$c ::= \langle v \mid e \rangle$
Covalues	$E ::= \alpha \parallel [E, E] \parallel fst(E) \parallel snd(E) \parallel v^\bullet$
Contexts	$e ::= E^\Diamond \parallel \tilde{\mu}x.c$
Expressions	$v ::= x \parallel \mu\alpha.c \parallel \mu x^\bullet.c \parallel \ldots$

$$
\begin{array}{l}
\langle v \mid \tilde{\mu}x.c \rangle \longrightarrow c[v/x] \\
\langle \mu\alpha.c \mid E^\Diamond \rangle \longrightarrow c[E/\alpha] \\
\langle \mu x^\bullet.c \mid (v^\bullet)^\Diamond \rangle \longrightarrow c[v/x] \\
\vdots
\end{array}
$$

The system LKT

$$\frac{}{\Gamma\,;\,\alpha : N \vdash \Delta\,,\,\alpha : N} \qquad \frac{\Gamma \vdash v : N \mid \Delta}{\Gamma\,;\,v^{\bullet} : \neg^{-} N \vdash \Delta}$$

$$\frac{\Gamma\,;\,E_1 : N_1 \vdash \Delta \qquad \Gamma\,;\,E_2 : N_2 \vdash \Delta}{\Gamma\,;\,[E_1, E_2] : N_1 \invamp N_2 \vdash \Delta} \qquad \frac{\Gamma\,;\,E_1 : N_1 \vdash \Delta}{\Gamma\,;\,fst(E_1) : N_1 \& N_2 \vdash \Delta}$$

$$\frac{\Gamma\,;\,E : N \vdash \Delta}{\Gamma \mid E^{\Diamond} : N \vdash \Delta} \qquad \frac{c : (\Gamma\,,\,x : N \vdash \Delta)}{\Gamma \mid \tilde{\mu}x.c : N \vdash \Delta}$$

$$\frac{}{\Gamma\,,\,x : N \vdash x : N \mid \Delta} \qquad \frac{c : (\Gamma \vdash \alpha : N\,,\,\Delta)}{\Gamma \vdash \mu\alpha.c : N \mid \Delta} \qquad \frac{c : (\Gamma\,,\,x : N \vdash \Delta)}{\Gamma \vdash \mu x^{\bullet}.c : \neg^{-} N \mid \Delta} \qquad \ldots$$

$$\frac{\Gamma \vdash v : N \mid \Delta \qquad \Gamma \mid e : N \vdash \Delta}{\langle v \mid e \rangle : (\Gamma \vdash \Delta)}$$

Encoding CBN $\lambda(\mu)$-calculus 2/2

In LKT we can define the following derived CBN implication and terms :

$$M \rightarrow^n N = (\neg^- M) \parr N$$

$$\lambda x.v = \mu(x^\bullet, \alpha).\langle v \mid \alpha^\Diamond\rangle \qquad v_1 v_2 = \mu\alpha.\langle v_1 \mid (v_2^\bullet, \alpha)^\Diamond\rangle$$

The translation extends to $\lambda\mu$-calculus, and also to left introduction of implication :

$$\frac{\Gamma \vdash v : N_1 \mid \Delta \quad \Gamma\,;\, E : N_2 \vdash \Delta}{\Gamma\,;\, v \cdot E : N_1 \Rightarrow N_2 \vdash \Delta}$$

with $v \cdot E = (v^\bullet, E)$ (read covalues as stacks, and this one as obtained by pushing v on top of E)

With these definitions, we have :

$$\langle \lambda x.v_1 \mid (v_2 \cdot E)^\Diamond\rangle = \langle \mu(x^\bullet, \alpha).\langle v_1 \mid \alpha^\Diamond\rangle \mid (v_2^\bullet, E)^\Diamond\rangle \longrightarrow \langle v_1[v_2/x] \mid E^\Diamond\rangle$$
$$\langle v_1 v_2 \mid E^\Diamond\rangle = \langle \mu\alpha.\langle v_1 \mid (v_2^\bullet, \alpha)^\Diamond\rangle \mid E^\Diamond\rangle \longrightarrow \langle v_1 \mid (v_2^\bullet, E)^\Diamond\rangle = \langle v_1 \mid (v_2 \cdot E)^\Diamond\rangle$$

(Krivine CBN abstract machine)

Translating LKQ to intuitionistic logic 1/3

Our target language will be intuitionistic logic with the following connectives :

$$\neg^i \quad \text{(negation)} \qquad \times \quad \text{(conjunction)} \qquad + \quad \text{(disjunction)}$$

$$\begin{array}{lcl} c & ::= & t\,t \\ t & ::= & x \parallel (t,t) \parallel inl(t) \parallel inr(t) \\ & & \lambda x.c \parallel \lambda(x_1,x_2).c \parallel \lambda z.\textit{case}\ z\ [inl(x_1)\cdot c_1, inr(x_2)\cdot c_2] \end{array}$$

Two typing judgements :

$$c : (\Gamma \vdash) \qquad\qquad \Gamma \vdash t : A$$

System NJ_0

N for Natural, J for Intuitionistic, $_0$ for not having full implication : think of $\neg^i A$ as $A \Rightarrow R$ for some fixed R, considered as "false", or as "the type of final results"

$$\frac{}{\Gamma, x : A \vdash x : A} \qquad \frac{\Gamma \vdash t_1 : \neg^i A \quad \Gamma \vdash t_2 : A}{t_1 t_2 : (\Gamma \vdash)} \qquad \frac{c : (\Gamma, x : A \vdash)}{\Gamma \vdash \lambda x.c : \neg^i A}$$

$$\frac{\Gamma \vdash t_1 : A_1 \quad \Gamma \vdash t_2 : A_2}{\Gamma \vdash (t_1, t_2) : A_1 \times A_2} \qquad \frac{\Gamma \vdash t_1 : A_1}{\Gamma \vdash inl(t_1) : A_1 + A_2}$$

$$\frac{c : (\Gamma, x_1 : A_1, x_2 : A_2 \vdash)}{\Gamma \vdash \lambda(x_1, x_2).c : \neg^i (A_1 \times A_2)}$$

$$\frac{c_1 : (\Gamma, x_1 : A_1 \vdash) \quad c_2 : (\Gamma, x_2 : A_2 \vdash)}{\Gamma \vdash \lambda z.\textit{case}\ z\ [inl(x_1) \cdot c_1, inr(x_2) \cdot c_2] : \neg^i (A_1 + A_2)}$$

Translating LKQ to intuitionistic logic 2/3

Translation of formulas :

$$X_{cps} = X \qquad (\neg^{+} P)_{cps} = \neg^{i}(P_{cps})$$
$$(P \otimes Q)_{cps} = (P_{cps}) \times (Q_{cps}) \quad (P \oplus Q)_{cps} = (P_{cps}) + (Q_{cps})$$

Translation of terms :

$$\langle v \mid e \rangle_{cps} = (v_{cps})(e_{cps})$$
$$(V^{\Diamond})_{cps} = \lambda k.k(V_{cps}) \qquad (\mu\alpha.c)_{cps} = \lambda k_{\alpha}.(c_{cps}) = (\tilde{\mu}\alpha^{\bullet}.c)_{cps}$$
$$x_{cps} = x \qquad (V_1, V_2)_{cps} = ((V_1)_{cps}, (V_2)_{cps})$$
$$inl(V_1)_{cps} = inl((V_1)_{cps}) \qquad (e^{\bullet})_{cps} = e_{cps}$$
$$\alpha_{cps} = k_{\alpha} \qquad (\tilde{\mu}x.c)_{cps} = \lambda x.(c_{cps}) \qquad (\tilde{\mu}(x_1, x_2).c)_{cps} = \lambda(x_1, x_2).(c_{cps})$$
$$(\tilde{\mu}[inl(x_1).c_1 | inr(x_2).c_2])_{cps} = \lambda z.\textit{case}\ z\ [inl(x_1) \cdot (c_1)_{cps}, inr(x_2) \cdot (c_2)_{cps}]$$

Translating LKQ to intuitionistic logic 3/3

We set

$$\begin{array}{l} \Gamma_{cps} = \{x : P_{cps} \mid x : P \in \Gamma\} \\ \neg^i(\Delta_{cps}) = \{k_\alpha : \neg^i(P_{cps}) \mid \alpha : P \in \Delta\} \end{array}$$

We have :

$$\begin{array}{rcl} c : (\Gamma \vdash \Delta) & \Rightarrow & c_{cps} : (\Gamma_{cps} \, , \, \neg^i(\Delta_{cps}) \vdash) \\ \Gamma \vdash V : P \, ; \, \Delta & \Rightarrow & \Gamma_{cps} \, , \, \neg^i(\Delta_{cps}) \vdash V_{cps} : P_{cps} \\ \Gamma \vdash v : P \mid \Delta & \Rightarrow & \Gamma_{cps} \, , \, \neg^i(\Delta_{cps}) \vdash v_{cps} : \neg^i(\neg^i(P_{cps})) \\ \Gamma \mid e : P \vdash \Delta & \Rightarrow & \Gamma_{cps} \, , \, \neg^i(\Delta_{cps}) \vdash e_{cps} : \neg^i(P_{cps}) \end{array}$$

Moreover, the translation preserves reduction

CPS translation

By composition, we get a translation from $\lambda\mu$-calculus (CBN or CBV) into intuitionistic logic. Specifically, for the CBN case,

starting from the simply-typed λ-term $(\Gamma \vdash M : A)$,

– we view M as an expression $(\Gamma \vdash M : A \,|\,)$ of LKT (using the CBN encoding of implication)

– and then as a context $(\,|\, M : \overline{A} \vdash \overline{\Gamma})$ of LKQ,

– and we arrive to the Hofmann-Streicher CPS-transform of M :

$$\neg^{+}(\overline{\Gamma}) \vdash M_{cps} : \neg^{+}(\overline{A})$$

Hofmann-Streicher translation on types goes as follows :

$$(A \to B)_{\mathrm{HS}} = \neg^{i}(A_{\mathrm{HS}}) \times B_{\mathrm{HS}}$$

and we have indeed $(\overline{A})_{cps} = A_{\mathrm{HS}}$

Polarised linear logic $\mathsf{LL}_{\mathbf{pol}}$

$$P ::= X \parallel P \otimes P \parallel P \oplus P \parallel !N$$
$$N ::= X^{\perp} \parallel N \parr N \parallel N \& N \parallel ?P$$

Key observations :

– Defining $\neg^{+}P$ as $!(P^{\perp})$, the formulas of $\mathsf{LL}_{\mathrm{pol}}$ are exactly the formulas of LKQ, but in fact of (the positive reading of) J_0 (without N because we do not care whether the style is natural deduction or sequent calculus)

– Moreover, the sequents consisting of $\mathsf{LL}_{\mathrm{pol}}$ formulas that are provable in LL are in fact intuitionistically provable in, say LJ_0 (read positively), which is exactly Laurent's Polarised Linear Logic LLP

In other words :

$$\mathsf{LL}_{\mathrm{pol}} \subseteq \mathsf{J}_0$$

And as a matter of fact, Girard's translation of the (CBN) λ-calculus, which is polarised, coincides with Hofmann-Streicher's one – an observation that may have been obvious for only a happy few !

Positive translation of J_0 to $\mathsf{LL}_{\mathbf{pol}}$ (reversing)

Keeping the same rules (in N style as above, or in L style as in a later slide), we read $\neg^i, \times, +$ as $\neg^+, \otimes, \oplus$ and we call J_0^+ the result of this repainting

$$
\begin{aligned}
X^+ &= \neg^+ X \\
(P \otimes Q)^+ &= (P^+) \otimes (Q^+) \\
(P \oplus Q)^+ &= (P^+) \oplus (Q^+) \\
(\neg^+ P)^+ &= \neg^+ (P^+)
\end{aligned}
$$

If $\Gamma \vdash$ (resp. $\Gamma \vdash P$) is provable in J_0^+, then $\Gamma^+ \vdash$ (resp. $\Gamma^+ \vdash P^+$) is provable in $\mathsf{LL}_{\mathrm{pol}}$

Negative translation of J_0 to $\mathsf{LL}_{\mathbf{pol}}$ ("Girard")

Still keeping the same rules, we read $\neg^i, \times, +$ as $\neg^-, \&, \parr$ and we call J_0^- the result of this repainting

$$
\begin{aligned}
(\overline{X})^- &= \overline{X} \\
(M \parr N)^- &= (?!(M^-)) \parr (?!(N^-)) \\
(M \& N)^- &= (M^-) \& (N^-) \\
(\neg^{-} N)^- &= \neg^{-} (N^-)
\end{aligned}
$$

If $\Gamma \vdash$ (resp. $\Gamma \vdash N$) is provable in J_0^-, then $!\Gamma^- \vdash$ (resp. $!\Gamma^- \vdash N^-$) is provable in $\mathsf{LL}_{\mathrm{pol}}$

A lozenge of translations

LKT, CBN $\lambda\mu$

J_0^+ $\qquad\qquad$ J_0^-

LL_{pol}

$\nearrow$ translations = “Girard-Hofmann-Streicher”
Lower $\searrow$ translation = reversing

$\searrow$ (resp. $\nearrow$) allows to recover contraction on negative (resp. positive) formulas

Categorical models

(for LKT, CBN $\lambda\mu$)
control categories
(Selinger)

(for J_0 read positively, LLP)
response categories
(Lafont, Reus, Streicher)

(for J_0 read negatively)
cartesian closed categories

(for linear logic)
$\star$**-autonomous categories**
+ comonad
(Seely, Biermann, Benton, Lafont)

Call-By-Push-Value (P. B. Levy) 1/3

Different perspective (Moggi's monadic approach to the semantics of programming languages), leading to similar ideas.

We show how to define *textually* Levy's framework in the polarised language.

CBPV "lives" (but see note two slides below !) in LLP (= LJ_0).

Also, Levy proposes a quite interesting formulation of categorical models based on indexing (or presheaf enrichment) which allows to "see" at the semantic level the differences and coercions relating command, context and expression judgements (and should also allow to distinguish a context from an expression of the dual type). I wish I can say more on this later !

LLP **(O. Laurent)**

We give a system L syntax for Laurent's polarised linear logic (which as we have seen is LJ_0 read positively).

$$
\begin{array}{l}
c ::= \langle V \mid e \rangle \quad V ::= x \parallel e^{\bullet} \parallel (V, V) \parallel \mathit{inl}(V) \parallel \mathit{inr}(V) \\
e ::= V^{\diamond} \parallel \tilde{\mu}x.c \parallel \tilde{\mu}(x_1, x_2).c \parallel \tilde{\mu}[\mathit{inl}(x_1).c_1 | \mathit{inr}(c_2).c_2]
\end{array}
$$

$$
\frac{}{\Gamma\,,\ x : P \vdash x : P\,;} \qquad \frac{\Gamma \vdash V : P\,; \qquad \Gamma \mid e : P \vdash}{\langle V \mid e \rangle : (\Gamma \vdash)} \qquad \frac{c : (\Gamma\,,\ x : P \vdash)}{\Gamma \mid \tilde{\mu}x.c : P \vdash}
$$

$$
\frac{\Gamma \mid e : P \vdash}{\Gamma \vdash e^{\bullet} : \neg^{+} P\,;} \qquad \frac{\Gamma \vdash V_1 : P_1\,; \qquad \Gamma \vdash V_2 : P_2\,;}{\Gamma \vdash (V_1, V_2) : P_1 \otimes P_2\,;} \qquad \frac{\Gamma \vdash V_1 : P_1\,;}{\Gamma \vdash \mathit{inl}(V_1) : P_1 \oplus P_2\,;}
$$

$$
\frac{\Gamma \vdash V : P\,;}{\Gamma \mid V^{\diamond} : \neg^{+} P \vdash} \qquad \frac{c : (\Gamma, x_1 : P_1, x_2 : P_2 \vdash)}{\Gamma \mid \tilde{\mu}(x_1, x_2).c : P_1 \otimes P_2 \vdash} \qquad \frac{c_1 : (\Gamma, x_1 : P_1 \vdash) \qquad c_2 : (\Gamma, x_2 : P_2 \vdash)}{\Gamma \mid \tilde{\mu}[\mathit{inl}(x_1).c_1 | \mathit{inr}(x_2).c_2] : P_1 \oplus P_2 \vdash}
$$

$$
\begin{array}{l}
\langle V \mid \tilde{\mu}x.c \rangle \longrightarrow c[V/x] \\
\langle e^{\bullet} \mid V^{\diamond} \rangle \longrightarrow \langle V \mid e \rangle \\
\langle (V_1, V_2) \mid \tilde{\mu}(x_1, x_2).c \rangle \longrightarrow c[V_1/x_1, V_2/x_2] \\
\langle \mathit{inl}(V_1) \mid \tilde{\mu}[\mathit{inl}(x_1).c_1 | \mathit{inr}(x_2).c_2] \rangle \longrightarrow c_1[V_1/x_1]
\end{array}
$$

Call-By-Push-Value (CBPV) 2/3

value types	$A ::= U\underline{B} \parallel \Sigma_i A_i \parallel A \parallel A \times A$
computation types	$\underline{B} ::= FA \parallel \Pi_i \underline{B}_i \parallel A \to \underline{B}$

Dictionary :

value	computation	Σ	$\times$	UN	FP	Π	$P \to N$
positive	negative	$\oplus$	$\otimes$	$\neg^{+}(\overline{(N)})$	$\neg^{-}(\overline{P})$	$\&$	$\overline{P} ⅋ N$

Judgements (and dictionary)

values	computations	stacks
$\Gamma \vdash^{\mathrm{v}} V : A$	$\Gamma \vdash^{\mathrm{c}} M : \underline{B}$	$\Gamma \vert \underline{B} \vdash^{\mathrm{k}} K : \underline{C}$
values	contexts	values
$\Gamma \vdash V : A\,;$	$\Gamma \mid M : \overline{(\underline{B})} \vdash$	$\Gamma, [\cdot] : \overline{(\underline{C})} \vdash K : \overline{(\underline{B})}\,;$

Note that stacks are values depending on a special variable $[\cdot]$ (This view seems well-prepared to account for composable continuations / delimited control, a hot topic !)

Note. It would be more appropriate to see computations as expressions of negative type rather than as contexts of positive type, and likewise for stacks (cf. the discussion on the encoding of CBN in LKQ). So it is more appropriate to say that CBPV lives in a version of LLP where the distinctions between, say $\Gamma \mid P \vdash$ and $\Gamma \vdash \overline{P} \mid$ would not be blurred.

Call-By-Push-Value 3/3

x	$\rightsquigarrow$	x
let V be $x.M$	$\rightsquigarrow$	$\tilde{\mu}y.\langle V \mid \tilde{\mu}x.\langle y \mid M\rangle\rangle$
return V	$\rightsquigarrow$	$V^{\diamond}$
M to $x.N$	$\rightsquigarrow$	$\tilde{\mu}y.\langle(\tilde{\mu}x.\langle y \mid N\rangle)^{\bullet} \mid M\rangle$
thunk M	$\rightsquigarrow$	$M^{\bullet}$
force V	$\rightsquigarrow$	$\tilde{\mu}x.\langle V \mid x^{\blacklozenge}\rangle$ (where $V^{\blacklozenge} = \tilde{\mu}\alpha^{\bullet}.\langle V \mid \alpha\rangle$)
Σ introduction	$\rightsquigarrow$	inl, inr
pm V as $\{(1, x_1).M_1, (2, x_2).M_2\}$	$\rightsquigarrow$	$\tilde{\mu}y.\langle V \mid \tilde{\mu}[inl(x_1).\langle y \mid M_1\rangle \mid inr(x_2).\langle y \mid M_2\rangle]\rangle$
(V, V')	$\rightsquigarrow$	(V, V')
pm V as $(x, y).M$	$\rightsquigarrow$	$\tilde{\mu}y.\langle V \mid \tilde{\mu}(x, y).\langle y \mid M\rangle\rangle$
$\lambda\{1.M_1, 2.M_2\}$	$\rightsquigarrow$	$\tilde{\mu}[inl(x_1).\langle x_1 \mid M_1\rangle \mid inr(x_2).\langle x_2 \mid M_2\rangle]$
$\hat{ß}$‘M	$\rightsquigarrow$	$\tilde{\mu}x.\langle inl(x) \mid M\rangle$
$\lambda x.M$	$\rightsquigarrow$	$\tilde{\mu}(x, y).\langle y \mid M\rangle$
V‘M	$\rightsquigarrow$	$\tilde{\mu}x.\langle(V, x) \mid M\rangle$
nil	$\rightsquigarrow$	$[\cdot]$
$[\cdot]$ to $x.M :: K$	$\rightsquigarrow$	$(\tilde{\mu}x.\langle K \mid M\rangle)^{\bullet}$
$\hat{1} :: K$	$\rightsquigarrow$	$inl(K)$ (idem $\hat{2}, inr$)
$V :: K$	$\rightsquigarrow$	(V, K)

Part III

Motivations : two related goals 1/2

First, we want to account for the full (or strong) focalisation : carrying the phases maximally, all the way up to the atoms on the left, up to atomic axioms on the right. This is of interest in a proof search perspective, since the stronger discipline further reduces the search space

Motivations : two related goals 1/2

Second, we would like our syntax to quotient proofs over the order of decomposition of negative formulas. The use of a structured pattern-matching is relevant, as we can describe the construction of a proof of

$$(\Gamma, x : (P_1 \otimes P_2) \otimes (P_3 \otimes P_4) \vdash \Delta)$$

out of a proof of

$$c : (\Gamma, x_1 : P_1, x_2 : P_2, x_3 : P_3, x_4 : P_4 \vdash \Delta)$$

"synthetically", by writing

$$\langle x^\diamond \mid \tilde{\mu}((x_1, x_2), (x_3, x_4)).c\rangle$$

standing for an abbreviation of either of the following two commands :

$$\langle x^\diamond \mid \tilde{\mu}(y, z).\langle y^\diamond \mid \tilde{\mu}(x_1, x_2).\langle z^\diamond \mid \tilde{\mu}(x_3, x_4).c\rangle\rangle\rangle$$
$$\langle x^\diamond \mid \tilde{\mu}(y, z).\langle z^\diamond \mid \tilde{\mu}(x_3, x_4).\langle y^\diamond \mid \tilde{\mu}(x_1, x_2).c\rangle\rangle\rangle$$

The two goals are connected, since applying strong focalisation will forbid the formation of these two terms (because y, z are values appearing with non atomic types), keeping the synthetic form only... provided we make it first class.

First step : introducing first class counterpatterns

Simple commands	$c ::= \langle v \mid e \rangle$	Commands	$C ::= c \parallel [C \; {}^{q,q} \; C]$
Expressions	$v ::= V^{\diamond} \parallel \mu\alpha.C$	Values	$V ::= x \parallel (V, V) \parallel inl(V) \parallel inr(V) \parallel e^{\bullet}$
Contexts	$\boxed{e ::= \alpha \parallel \tilde{\mu}q.C}$	Counterpatterns	$\boxed{q ::= x \parallel \alpha^{\bullet} \parallel (q, q) \parallel [q, q]}$

Let $\Xi = x_1 : X_1, \ldots, x_n : X_n$ denote a left context consisting of *atomic formulas only*.
The rules are as follows :

$$\frac{}{\Xi \, , \, x : X \vdash x : X \, ; \, \Delta} \qquad \frac{C : (\Xi \, , \, q : P \vdash \Delta)}{\Xi \mid \tilde{\mu}q.C : P \vdash \Delta} \qquad \frac{C : (\Xi \vdash \alpha : P \, , \, \Delta)}{\Xi \vdash \mu\alpha.C : P \mid \Delta}$$

$$\frac{C : (\Gamma \vdash \alpha : P \, , \, \Delta)}{C : (\Gamma \, , \, \alpha^{\bullet} : \neg^{+}P \vdash \Delta)} \qquad \frac{C : (\Gamma \, , \, q_1 : P_1 \, , \, q_2 : P_2 \vdash \Delta)}{C : (\Gamma \, , \, (q_1, q_2) : P_1 \otimes P_2 \vdash \Delta)}$$

$$\frac{C_1 : (\Gamma \, , \, q_1 : P_1 \vdash \Delta) \quad C_2 : (\Gamma \, , \, q_2 : P_2 \vdash \Delta)}{[C_1 \; {}^{q_1,q_2} \; C_2] : (\Gamma \, , \, [q_1, q_2] : P_1 \oplus P_2 \vdash \Delta)}$$

(all the other rules as before, with Ξ in place of Γ)

But wait a minut...

We introduced a new mess, in the form of these ugly new (compound) commands. We did a good job for tensors on the left, but not for plus' on the left.

If $c_{ij} : (\Gamma, x_i : P_i, x_j : P_j \vdash_S \Delta)$ $(i = 1, 2, j = 3, 4)$, we want to identify

$$[[c_{13} \; {}^{x_3,x_4} \; c_{14}] \; {}^{x_1,x_2} \; [c_{23} \; {}^{x_3,x_4} \; c_{24}]]$$
$$[[c_{13} \; {}^{x_1,x_2} \; c_{23}] \; {}^{x_3,x_4} \; [c_{14} \; {}^{x_1,x_2} \; c_{24}]]$$

For this, we need a last ingredient : patterns.

Towards the second step : introducing first class patterns

we redefine the syntax of values, as follows :

$$\mathcal{V} ::= x \parallel e^{\bullet} \quad V ::= p\langle \mathcal{V}_i / i \mid i \in p\rangle \quad p ::= x \parallel \alpha^{\bullet} \parallel (p,p) \parallel inl(p) \parallel inr(p)$$

where $i \in p$ is defined by :

$$\frac{}{x \in x} \quad \frac{}{\alpha^{\bullet} \in \alpha^{\bullet}} \quad \frac{i \in p_1}{i \in (p_1, p_2)} \quad \frac{i \in p_2}{i \in (p_1, p_2)} \quad \frac{i \in p_1}{i \in inl(p_1)} \quad \frac{i \in p_2}{i \in inr(p_2)}$$

Moreover, $\mathcal{V}_i$ must be of the form y (resp. $e^{\bullet}$) if $i = x$ (resp. $i = \alpha^{\bullet}$).

Patterns are required to be linear, as well as the counterpatterns, for which the definition of “linear” is adjusted in the case $[q_1, q_2]$, in which a variable can occur (but recursively linearly so) in both q_1 and q_2

Values are defined up to α-conversion, e.g. $\alpha^{\bullet}\langle e^{\bullet}/\alpha^{\bullet}\rangle = \beta^{\bullet}\langle e^{\bullet}/\beta^{\bullet}\rangle$

Pattern-counterpattern interaction

We rephrase the logical reduction rules in terms of pattern/counterpattern interaction :

$$\frac{V = p\,\langle \ldots y/x, \ldots, e^\bullet/\alpha^\bullet, \ldots \rangle \qquad C[p/q] \longrightarrow^* c}{\langle V^\Diamond \mid \tilde{\mu} q.C \rangle \longrightarrow c\{\ldots, y/x, \ldots, e/\alpha, \ldots\}}$$

where $c\{\sigma\}$ is the usual, implicit substitution, and where c (see the next proposition) is the normal form of $C[p/q]$ with respect to the following set of rules :

$$\begin{array}{l}
C[(p_1, p_2)/(q_1, q_2), \sigma] \longrightarrow C[p_1/q_1, p_2/q_2, \sigma] \\
C[\alpha^\bullet/\alpha^\bullet, \sigma] \longrightarrow C[\sigma] \\
C[x/x, \sigma] \longrightarrow C[\sigma] \\
[C_1 \;{}^{q_1,q_2}\; C_2][inl(p_1)/[q_1, q_2], \sigma] \longrightarrow C_1[p_1/q_1, \sigma] \\
[C_1 \;{}^{q_1,q_2}\; C_2][inr(p_2)/[q_1, q_2], \sigma] \longrightarrow C_2[p_2/q_2, \sigma]
\end{array}$$

Logically, this means that we now consider each formula as made of blocks of *synthetic* connectives.

An example

Patterns for $P = X \otimes (Y \oplus \neg^+ Q)$. Focusing on the right yields two possible proof searches :

$$\frac{\Gamma \vdash x'\{\mathcal{V}_{x'}\} : X \,;\, \Delta \quad \Gamma \vdash y'\{\mathcal{V}_{y'}\} : Y \,;\, \Delta}{\Gamma \vdash (x', inl(y'))\{\mathcal{V}_{x'}, \mathcal{V}_{y'}\} : X \otimes (Y \oplus \neg^+ Q) \,;\, \Delta}$$

$$\frac{\Gamma \vdash x'\{\mathcal{V}_{x'}\} : X \,;\, \Delta \quad \Gamma \vdash \alpha'^{\bullet}\{\mathcal{V}_{\alpha'^{\bullet}}\} : \neg^+ Q \,;\, \Delta}{\Gamma \vdash (x', inr(\alpha'^{\bullet}))\{\mathcal{V}_{x'}, \mathcal{V}_{\alpha'^{\bullet}}\} : X \otimes (Y \oplus \neg^+ Q) \,;\, \Delta}$$

Counterpattern for $P = X \otimes (Y \oplus \neg^+ Q)$. The counterpattern describes the tree structure of P :

$$\frac{c_1 : (\Gamma \,,\, x : X \,,\, y : Y \vdash \Delta) \qquad c_2 : (\Gamma \,,\, x : X \,,\, \alpha^{\bullet} : \neg^+ Q \vdash \Delta)}{[c_1 \;{}^{y,\alpha^{\bullet}}\; c_2] : (\Gamma \,,\, (x, [y, \alpha^{\bullet}]) : X \otimes (Y \oplus \neg^+ Q) \vdash \Delta)}$$

We observe that the leaves of the decomposition of P pon the left are in one-to-one correspondence with the patterns p for the (irreversible) decomposition of P on the right :

$$[c_1 \;{}^{y,\alpha^{\bullet}}\; c_2][p_1/q] \longrightarrow^* c_1 \qquad [c_1 \;{}^{y,\alpha^{\bullet}}\; c_2][p_2/q] \longrightarrow^* c_2$$

where $q = (x, [y, \alpha^{\bullet}])$, $p_1 = (x, inl(y))$, $p_2 = (x, inr(\alpha^{\bullet}))$.

A key one-to-one correspondence

This correspondence is general. We define two predicates $c \in C$ and $q \perp p$ ("q is orthogonal to p") as follows :

$$\frac{}{c \in c} \qquad \frac{c \in C_1}{c \in [C_1 \; {}^{q_1,q_2} \; C_2]} \qquad \frac{c \in C_2}{c \in [C_1 \; {}^{q_1,q_2} \; C_2]}$$

$$\boxed{\begin{array}{ccc} \dfrac{}{x \perp x} & \dfrac{}{\alpha^\bullet \perp \alpha^\bullet} & \dfrac{q_1 \perp p_1 \quad q_2 \perp p_2}{(q_1, q_2) \perp (p_1, p_2)} \\ \multicolumn{2}{c}{\dfrac{q_1 \perp p_1}{[q_1, q_2] \perp inl(p_1)}} & \dfrac{q_2 \perp p_2}{[q_1, q_2] \perp inr(p_2)} \end{array}}$$

Proposition Let $C : (\Xi \,, \, q : P \vdash \Delta)$ and let p be such that q is orthogonal to p. Then the normal form c of $C[p/q]$ is a simple command, and the mapping $p \mapsto c$ $(q, C$ fixed) from $\{p \mid q \perp p\}$ to $\{c \mid c \in C\}$ is one-to-one and onto.

Synthetic system L 1/2

$$c ::= \langle v \mid e \rangle \qquad v ::= V^{\Diamond} \parallel \mu\alpha.c$$
$$V ::= p\,\langle \mathcal{V}_i/i \mid i \in p \rangle \qquad \mathcal{V} ::= x \parallel e^{\bullet} \qquad p ::= x \parallel \alpha^{\bullet} \parallel (p,p) \parallel inl(p) \parallel inr(p)$$
$$e ::= \alpha \parallel \tilde{\mu} q.\{p \mapsto c_p \mid q \perp p\} \qquad q ::= x \parallel \alpha^{\bullet} \parallel (q,q) \parallel [q,q]$$

$$\boxed{\begin{array}{c} \langle (p\,\langle \ldots, y/x, \ldots, e^{\bullet}/\alpha^{\bullet} \ldots \rangle)^{\Diamond} \mid \tilde{\mu} q.\{p \mapsto c_p \mid q \perp p\} \rangle \\ \downarrow \\ c_p\,\{\ldots, y/x, \ldots, e/\alpha, \ldots \rangle\} \end{array}}$$

and the μ rule, unchanged

Cf. N. Zeilberger's *unity of duality*

Synthetic system L 2/2

Typing rules : the old ones for $\alpha, x, e^\bullet, c$, plus the following ones :

$$\frac{\ldots \qquad \Xi \vdash \mathcal{V}_i : P_i \,;\, \Delta \quad ((i : P_i) \in \Gamma(p, P)) \qquad \ldots}{\Xi \vdash p \,\langle \mathcal{V}_i / i \mid i \in p \rangle : P \,;\, \Delta}$$

$$\frac{\ldots \qquad c_p : (\Xi \,,\, \Xi(p, P) \vdash \Delta(p, P), \,,\Delta) \quad (q \perp p) \qquad \ldots}{\Gamma \mid \tilde{\mu} q.\{p \mapsto c_p \mid q \perp p\} : P \vdash \Delta}$$

where $\Gamma(p, P)$ must be successfully defined as follows :

$$\begin{array}{l}\Gamma(x, X) = (x : X) \qquad \Gamma(\alpha^\bullet, \neg^+ P) = (\alpha^\bullet : \neg^+ P) \\ \Gamma((p_1, p_2), P_1 \otimes P_2) = \Gamma(p_1, P_1) \,,\, \Gamma(p_2, P_2) \\ \Gamma(inl(p_1), P_1 \oplus P_2) = \Gamma(p_1, P_1) \qquad \Gamma(inr(p_2), P_1 \oplus P_2) = \Gamma(p_2, P_2)\end{array}$$

and where

$$\Xi(p, P) = \{x : X \mid x : X \in \Gamma(p, P)\} \qquad \Delta(p, P) = \{a : P \mid \alpha^\bullet : \neg^+ P \in \Gamma(p, P)\}$$

Towards ludics (à la Terui)

Applying Occam's razor, we arrive at Terui's syntax for a (non locative version) of ludics :

$$\begin{array}{l} P ::= \Omega \parallel \maltese \parallel (N_0|\overline{a}\langle N_1, \ldots, N_n\rangle \\ N ::= x \parallel \Sigma a(\vec{x}).P \end{array}$$

where a ranges over an alphabet of symbols, each given an arity (the length of $\vec{x}$)

Dictionary :

$$\begin{array}{ccccc} N & P & x & \Sigma a(\vec{x}).P & (N_0|\overline{a}\langle N_1, \ldots, N_n\rangle \\ e & c & \alpha & \tilde{\mu}q..\{p \mapsto c_p \mid q \perp p\} & \langle (p\,\langle \ldots, x/x, \ldots, e_1^\bullet/\alpha_1^\bullet, \ldots, e_n^\bullet/\alpha_n^\bullet\rangle)^\Diamond \mid e_0\rangle \end{array}$$

What has disappeared : the structure of patterns (no big loss, can be encoded)

What has appeared : divergence (Ω) and convergence ($\maltese$), which play a key role for an observation / realisability semantics

But what is ludics about (for our concerns) ? 1/2

1. Start with a raw syntax of "would-be proofs" (if the syntax is distilled from a typed one, chances are higher to make something sensible !). It is also helpful that the raw syntax is divided in positive and negative terms (P, N)
2. Define reduction rules, and say that P (with only one free variable x_0) is orthogonal to N, or *passes the test* N when $P[N/x_0] \longrightarrow^* \maltese$.
3. Define a semantic type, or *behaviour* (in Girard's terminology) as a set **P** or **N** of raw terms of the same polarity which is closed under bi-orthogonal, i.e., that behave the same wrt a fixed set of observers. Say that, say P *realises* (in the terminology of Krivine) **P** if $P \in \mathbf{P}$

But what is ludics about ? 2/2

4. Interpret your favourite (preferably polarised) connectives as constructions on behaviours. The idea is that these constructions define the meaning of connectives internally, interactively. They are forced upon us just as, say continuity / computability arises for free in the effective topos.
5. Given a sensible typing system on your raw terms, it is going to be sound (fundamental lemma of logical relations !), i.e. if $\vdash P : A$, then $P \Vdash \mathbf{P}$ (where $\mathbf{P}$ is the behaviour interpreting A).
6. "The cherry on the cake" (nicer than icing...) : If the converse holds, we have **full completeness** : our realisability model (which in fact is built over the very syntax we started with) has a tight fit with the syntax, that is, our language has no junk nor redundancy, everything fits, plays a distinctive rule. Reaching that "eden" has been a popular goal in the 90's (game semantics).

The price of full completeness for ludics

There are two full completeness results for ludics :

1. Girard : no exponentials, i.e. only linear terms.
2. Basaldella-Terui : no axiom (constant-only logic)

There is no reason in principle why one could not have both, it is just that the difficulties are of different order and benefit from being treated separately :

1. Axioms : one needs the behaviours to incorporate notions of uniformity (infinite, uniform η-expansions of untyped variables)
2. Exponentials : one needs to give extra power to the observers : non-determinism (like in differential linear logic). The fact that Böhm's theorem (tighlty related to the completeness issue) holds for the λ-calculus is a kind of little miracle which does not extend to the syntax of ludics (named arguments versus sequence of arguments).

Basaldella-Terui's proof of full completeness

Remember the proof of “ordinary" completeness (for provability) : Take a non provable formula A, and build a (maximal) cut-free proof attempt P for it. Then there is one branch of P that ends with a "non-axiom", from which a counter-model is built.

One notes here that the quality of counter-model is relative to A, not to P. Full completeness looks for a term N that would be directly a “counter-model” for P. Basaldella and Terui prolong the completeness proof as follows :

1. (upwards) Find a faulty branch (like above).
2. (downwards) Starting from the leaf (or reasoning coinductively if the branch is infinite), synthesize a counter-proof N (all the way down to the root). It is here that non determinism is needed if the same head variable appears twice and the branch chooses different sons at these different occurrences.
3. (upwards) Run cut-elimination between P and N : this normalisation does not end up with ✠ but either diverges or ends up with Ω.

Basaldella-Terui's generalised connectives

Let $\mathbf{N}_1, \ldots, \mathbf{N}_m$ be negative behaviours. One sets (a of arity m) :

$$\overline{a}\langle \mathbf{N}_1, \ldots, \mathbf{N}_m \rangle = \{x_0 | \overline{a}\langle N_1, \ldots, N_m \rangle \mid N_1 \in \mathbf{N}_1, \ldots, N_m \in \mathbf{N}_m\}$$

The following data $\alpha = (\vec{z}, \{\ldots, a(z_{i_1}, \ldots, z_{i_m}), \ldots\})$ define dual n-ary constructions of types / behaviours :

- a sequence of n distinct variable names $z_1, \ldots, z_n$,
- alphabet symbols $a_1, \ldots, a_m$, each of arity $\leq n$, for each of which a subsequence $i_1, \ldots, i_m$ of $1, \ldots, n$ is associated

Given α and negative behaviours $\mathbf{N}_1, \ldots, \mathbf{N}_n$, one defines a positive behaviour as follows :

$$\overline{\alpha}\langle \mathbf{N}_1, \ldots, \mathbf{N}_n \rangle = \Big(\bigcup_{a(z_{i_1}, \ldots, z_{i_m}) \in \alpha} \overline{a}\langle \mathbf{N}_{i_1}, \ldots \mathbf{N}_{i_m} \rangle \Big)^{\perp\perp}$$

and by duality we have a constructor over positive behaviours :

$$\alpha(\mathbf{P}_1, \ldots, \mathbf{P}_n) = (\overline{\alpha}\langle (\mathbf{P}_1)^{\perp}, \ldots, \mathbf{P}_n)^{\perp} \rangle)^{\perp}$$

Examples :

$$\parr = ((x_1, x_2), \{\mathcal{P}(x_1, x_2)\}) \ , \ \& = ((x_1, x_2), \{\pi_1(x_1), \pi_2(x_2)\}) \ , \ \otimes = \overline{\parr} \ , \ \oplus = \overline{\&}$$

Some readings 1/2

The seminal papers on constructive (or Curry-Howard for) classical logic :

– T. Griffin, A formulae-as-types notion of control, Proc. ACM Principles of Prog. Lang. (1990)
– J.-Y. Girard, A new constructive logic : classical logic, Math. Struct. in Computer Science 1, 255-296 (1991)
– M. Parigot, $\lambda\mu$-calculus : An algorithmic interpretation of classical natural deduction, in Proc. of the Int. Conf. on Logic Programming and Automated Reasoning, St. Petersburg, LNCS 624 (1992)

Some readings 2/2

– "Proofs and types", J.-Y. Girard, Y. Lafont, P. Taylor, Cambridge University Press, available from http://www.paultaylor.eu/stable/Proofs+Types.html
– P.-L. Curien, Introduction to linear logic and ludics, part I and part II http://www.pps.jussieu.fr/~curien
– P.-L. Curien and G. Munch-Maccagnoni, The duality of computation under focus (same url)
– G. Munch, Focalisation and classical realisability, http://perso.ens-lyon.fr/guillaume.munch
– P. Selinger, Control categories and duality . . ., http://www.mscs.dal.ca/~selinger
– O. Laurent and Laurent Regnier, About translations of classical logic into polarized linear logic, http://perso.ens-lyon.fr/olivier.laurent
– O. Laurent, Intuitionistic dual-intuitionistic nets (same url)
– P. B. Levy, Adjunction models for call-by-push-value with stacks, Theory and Applications of Categories (2005)
– N. Zeilberger, On the unity of duality, http://www.cs.cmu.edu/~noam
– M. Basaldella and K. Terui, On the meaning of logical completeness, http://www.kurims.kyoto-u.ac.jp/~mbasalde

and also

– Olivier Laurent : Théorie de la démonstration, url as above.
– J.-Y. Girard, Locus solum : from the rules of logic to the logic of rules, MSCS (2001)
– "Le point aveugle I et II", J.-Y. Girard, Editions Hermann (English version available from http://iml.univ-mrs.fr/~girard)
– "Proof theory and automated deduction", J. Goubault-Larrecq, I. Mackie, Kluwer
– G. Dowek, Introduction to proof theory, http://www.lix.polytechnique.fr/~dowek/cours.html
– D. Miller, Proof search and computation, http://www.lix.polytechnique.fr/Labo/Dale.Miller/mpri/mono-jan-2009.pdf

Gratefully acknowledging the support of the Leverhulme trust

which enabled two fruitful three-month visits
to the Computer Laboratory at Cambridge
in 2009 and 2010