Causal Unfoldings

Marc de Visme[1] and Glynn Winskel[2]

[1] ENS Lyon, France
[2] Computer Laboratory, University of Cambridge, UK

Abstract. In the simplest form of event structure, a *prime* event structure, an event is associated with a unique causal history, its prime cause. However, it is quite common for an event to have disjunctive causes in that it can be enabled by any one of multiple sets of causes. Sometimes the sets of causes may be mutually exclusive, inconsistent one with another, and sometimes not, in which case they coexist consistently and constitute *parallel* causes of the event. The established model of *general* event structures can model parallel causes. On occasion however such a model abstracts too far away from the precise causal histories of events to be directly useful. For example, sometimes one needs to associate probabilities with different, possibly coexisting, causal histories of a common event. Ideally, the causal histories of a general event structure would correspond to the configurations of its *causal unfolding* to a prime event structure; and the causal unfolding would arise as a right adjoint to the embedding of prime in general event structures. But there is no such adjunction. However, a slight extension of prime event structures remedies this defect and provides a causal unfolding as a universal construction. Prime event structures are extended with an equivalence relation in order to dissociate the two roles, that of an event and its enabling; in effect, prime causes are labelled by a disjunctive event, the equivalence class of its prime causes. With this enrichment a suitable causal unfolding appears as a pseudo right adjoint. The adjunction relies critically on the central and subtle notion of *extremal causal realisation* as an embodiment of causal history.

1 Introduction

Work on probabilistic distributed strategies based on event structures brought us face to face with a limitation in existing models of concurrent computation, and in particular with the theory of event structures as it had been developed. In order to adequately express certain intuitively natural optimal probabilistic strategies, it was necessary to simultaneously support: probability on event structures with opponent moves, itself rather subtle; parallel causes, in which an event may be enabled in several distinct but compatible ways; and a hiding operation crucial in the composition of strategies. The difficulties did not show up in the less refined development of nondeterministic strategies; there the simplest form of event structure, prime event structures, sufficed. The “obvious” remedy, to base strategies on more general event structures, which do support parallel causes, failed to support probability and hiding adequately. The problems and a solution are documented in a recent article [3].

That work uncovered a central construction, what we here call the *causal unfolding* of a model with parallel causes. It is based on the notion of *extremal causal realisation* and attendant *extremal prime realisation* which plays a role analogous to that of complete prime in distributive orders. Both concepts deserve to be better known and are expanded on comprehensively with full proofs here. Intuitively, an extremal prime realisation is a finite partial order expressing a minimal causal history for an event to occur, even in the presence of several parallel causes for the event. Extremal realisations provide us with a way to unfold a model supporting parallel causes (general event structures—Section 2.2, or equivalence families—Section 3) into a structure describing all its causal histories—its causal unfolding. As is to be hoped, the unfolding will be a form of right adjoint giving the causal unfolding and extremal realisations a categorical significance.[3]

The new adjunction, with its right adjoint the causal unfolding, supplies a missing link in the landscape of models for concurrency [12], connecting models with parallel causes to those based on partial orders of events. In systems with parallel causes it is often necessary to associate probabilities with causal histories, and the causal unfolding provides a suitable structure on which to do this systematically [3]. Outside probability, there is a similar need for causal unfoldings, for example, when reversible computing encounters parallel causes [1, 4], and in extracting biochemical pathways, forms of causal history in biochemical systems where parallel causes are rife [2].

2 Event structures and their maps

We briefly review two well-established forms of event structure and explain the absence of an adjunction associated with the embedding of prime into general event structures. It is through such an adjunction one might otherwise have thought to find a causal unfolding of general event structures to prime event structures. The absence motivates a new model.

2.1 Prime event structures

The causal unfolding essentially produces a prime event structure as the unfolding. A *prime event structure* comprises $(E, \leq, \mathrm{Con})$, consisting of a set E of *events* which are partially ordered by $\leq$, the *causal dependency relation*, and a nonempty *consistency relation* Con consisting of finite subsets of E. The relation $e' \leq e$ expresses that event e causally depends on the previous occurrence of event e'. Write $[X]$ for the $\leq$-down-closure of a subset of events X. That a

[3] A forewarning: only in very special circumstances do extremal prime realisations coincide with complete irreducibles, a customary generalisation of complete primes to the nondistributive orders such as those of configurations of general event structures—see Example 2.

finite subset of events is consistent conveys that its events can occur together by some stage in the evolution of the process. Together the relations satisfy several axioms:

$$[e] = \{e' \mid e' \leq e\} \text{ is finite, for all } e \in E,$$
$$\{e\} \in \mathrm{Con}, \text{ for all } e \in E,$$
$$X \subseteq Y \in \mathrm{Con} \implies X \in \mathrm{Con}, \text{ and}$$
$$X \in \mathrm{Con} \ \& \ e \leq e' \in X \implies X \cup \{e\} \in \mathrm{Con}.$$

A *configuration* is a, possibly infinite, set of events $x \subseteq E$ which is: *consistent,* $X \subseteq x$ and X is finite implies $X \in \mathrm{Con}$; and *down-closed,* $[x] = x$. It is part and parcel of prime event structures that an event e is associated with a unique causal history $[e]$.

2.2 General event structures

A *general event structure* [10, 11] permits an event to be caused disjunctively in several ways, possibly coexisting in parallel, as parallel causes. A general event structure comprises $(E, \mathrm{Con}, \vdash)$ where E is a set of event occurrences, the consistency relation Con is a non-empty collection of finite subsets of E, and the *enabling relation* $\vdash$ is a relation in $\mathrm{Con} \times E$ such that

$$X \subseteq Y \in \mathrm{Con} \implies X \in \mathrm{Con}\,, \text{ and}$$
$$Y \in \mathrm{Con} \ \& \ Y \supseteq X \ \& \ X \vdash e \implies Y \vdash e\,.$$

A *configuration* is a subset x of E which is: *consistent,* $X \subseteq_{\mathrm{fin}} x \implies X \in \mathrm{Con}$; and *secured,* $\forall e \in x \exists e_1, \cdots, e_n \in x.\ \ e_n = e \ \ \& \ \forall i \leq n. \{e_1, \cdots, e_{i-1}\} \vdash \ e_i$. We write $\mathcal{C}^\infty(E)$ for the configurations of E and $\mathcal{C}(E)$ for its finite configurations.

An event e being enabled in a configuration has been expressed through the existence of a securing chain $e_1, \cdots, e_n$, with $e_n = e$, within the configuration. The chain represents a *complete enabling* of e in the sense that every event in the chain is itself enabled by earlier members of the chain. Just as mathematical proofs need not be sequences, so later complete enablings expressed more generally as partial orders—“causal realisations”—will play a central role.

A *map* $f : (E, \mathrm{Con}, \vdash) \to (E', \mathrm{Con}', \vdash')$ of general event structures is a partial function $f : E \rightharpoonup E'$ such that

$$\forall X \in \mathrm{Con}\,.\ fX \in \mathrm{Con}'\,,$$
$$\forall e_1, e_2 \in X \in \mathrm{Con}.\ f(e_1) = f(e_2) \implies e_1 = e_2\,, \text{ and}$$
$$X \vdash e \ \& \ f(e) \text{ is defined } \implies fX \vdash' f(e)\,.$$

Maps compose as partial functions. Write $\mathcal{G}$ for the category of general event structures.

We can characterise those families of configurations arising from a general event structure [11]. W.r.t. a family of subsets $\mathcal{F}$, a subset X of $\mathcal{F}$ is *compatible* (in $\mathcal{F}$), written $X\uparrow$, if there is $y \in \mathcal{F}$ such that $x \subseteq y$ for all $x \in X$; we write $x \uparrow y$ for $\{x, y\}\uparrow$. Say a subset is *finitely compatible* iff every finite subset is compatible.

A *family of configurations* comprises a family $\mathcal{F}$ of sets such that if $X \subseteq \mathcal{F}$ is finitely compatible in $\mathcal{F}$ then $\bigcup X \in F$; and if $e \in x \in \mathcal{F}$ there is a securing chain $e_1, \cdots, e_n = e$ in x such that $\{e_1, \cdots, e_i\} \in \mathcal{F}$ for all $i \leq n$.[4] Its *events* are elements of the underlying set $\bigcup \mathcal{F}$. A *map* between families of configurations from $\mathcal{A}$ to $\mathcal{B}$ is a partial function $f : \bigcup \mathcal{A} \rightharpoonup \bigcup \mathcal{B}$ between their events such that $fx \in \mathcal{B}$ if $x \in \mathcal{A}$ and any event of fx arises as the image of a unique event of x. Maps compose as partial functions.

2.3 A coreflection and non-coreflection

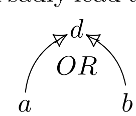
There is a forgetful functor taking a general event structure to its family of configurations. It has a left adjoint, which constructs a canonical general event structure from a family: given $\mathcal{A}$, a family of configurations with underlying events A, construct a general event structure $(A, \mathrm{Con}, \vdash)$ with $X \in \mathrm{Con}$ iff $X \subseteq_{\mathrm{fin}} y$, for some $y \in \mathcal{A}$; and with $X \vdash a$ iff $a \in A$, $X \in \mathrm{Con}$ and $a \in y \subseteq X \cup \{a\}$, for some $y \in \mathcal{A}$.

The above yields a coreflection[5] of families of configurations in general event structures. It cuts down to an equivalence between families of configurations and *replete* general event structures. A general event structure $(E, \mathrm{Con}, \vdash)$ is *replete* iff

$$\forall e \in E\, \exists X \in \mathrm{Con}.\ X \vdash e\ , \quad \forall X \in \mathrm{Con} \exists x \in \mathcal{C}(E).\ X \subseteq x \ \text{ and}$$
$$X \vdash e \implies \exists x \in \mathcal{C}(E).\ e \in x \ \&\ x \subseteq X \cup \{e\}\,.$$

A *map* of prime event structures is a map of their families of configurations. (A map need not preserve causal dependency; when it does and is total it is called *rigid*.) There is an obvious "inclusion" functor from the category of prime event structures into the category of families of configurations. We might expect this to form a coreflection, with right adjoint unfolding a (replete) general event structure to a prime event structure [11, 12]. However under reasonable assumptions this cannot exist, as the following example indicates.

Example 1. Consider a general event structure comprising three events a, b and d with all subsets consistent and minimal enablings $\emptyset \vdash a, b$ and $\{a\} \vdash d$ and $\{b\} \vdash d$. Imagine concurrent treatments a and b of two doctors which sadly lead to the death d of the patient.



As its unfolding it is hard to avoid a prime event structure with events and causal dependency $a < d_a$ and $b < d_b$—the event d_a representing "death by a" and the event d_b "death by b"—with the counit of the adjunction collapsing d_a and d_b to the common event d. (If we are to apportion blame to the doctors we shall need the probabilities of d_a and d_b given a and b [8].) In order for the

[4] The latter condition is equivalent to: (i) if $e \in x \in \mathcal{F}$ there is a finite $x_0 \in \mathcal{F}$ s.t. $e \in x_0 \in \mathcal{F}$ and (ii) for distinct $e, e' \in x$, there is $y \in \mathcal{F}$ with $y \subseteq x$ s.t. $e \in y \iff e' \neq y$.

[5] A coreflection is an adjunction where the left adjoint is full and faithful, or equivalently the unit is iso.

counit to be a map we are forced to make $\{d_a, d_b\}$ inconsistent. This is one issue: why should death by one doctor's treatment be in conflict with death by the other's—they could be jointly responsible. But even more damningly the tentative counit fails the universal property required of it! Consider another prime event structure with three events comprising $a < d$ and $b < d$ ("death due to both doctors' treatments"). The obvious map to the family of configurations of the general event structure—the identity on events—fails to factor *uniquely* through the putative counit: d can be sent to either d_a or d_b; the event "death by both doctors" can be sent to either "death by a" or "death by b." This raises the second issue: if we are to obtain the required universal property we have to regard these two maps as essentially the same. □

The two issues raised in the example suggest a common solution: to enrich prime event structures with equivalence relations. This will allow a broader class of maps, settling the first issue, and introduce an equivalence on maps, settling the second. The causal unfolding of the "doctors example" will be very simple and comprise the prime event structure $a < d_a$ and $b < d_b$ with d_a and d_b equivalent events; with all events consistent. The construction of the unfolding in general is surprisingly involved; causal histories can be much more intricate than in the simple example.

3 Events with an equivalence, categories $\mathcal{E}_\equiv$ and $\mathcal{F}am_\equiv$

We build causal unfoldings in a new model, based on the obvious extension to events with an equivalence relation. An *event structure with equivalence* (an ese) is a structure

$$(P, \leq, \mathrm{Con}, \equiv)$$

where $(P, \leq, \mathrm{Con})$ satisfies the axioms of a prime event structure and $\equiv$ is an equivalence relation on P. The intention is that the events of P represent *prime causes* while the $\equiv$-equivalence classes of P represent *disjunctive events*: p in P is a prime cause of the event $\{p\}_\equiv$. Notice there may be several prime causes of the same event and that these may be parallel causes in the sense that they are consistent with each other and causally independent.

The extension by an equivalence relation on events is accompanied by an extension to families of configurations. An *equivalence-family* (ef) is a family of configurations $\mathcal{A}$ with an equivalence relation $\equiv_A$ on its underlying set $A =_{\mathrm{def}} \bigcup \mathcal{A}$ (with no further axioms). Equivalence-families are the most general model we shall consider; they support parallel causes and, later, a causal unfolding.

Let $(\mathcal{A}, \equiv_A)$ and $(\mathcal{B}, \equiv_B)$ be ef's, with respective underlying sets A and B. A map $f : (\mathcal{A}, \equiv_A) \to (\mathcal{B}, \equiv_B)$ is a partial function $f : A \rightharpoonup B$ which preserves $\equiv$, if $a_1 \equiv_A a_2$ then either both $f(a_1)$ and $f(a_2)$ are undefined or both defined with $f(a_1) \equiv_B f(a_2)$, such that

$$x \in \mathcal{A} \implies fx \in \mathcal{B} \ \& \ \forall a_1, a_2 \in x.\ f(a_1) \equiv_B f(a_2) \implies a_1 \equiv_A a_2\,.$$

Composition is composition of partial functions. We regard two maps

$$f_1, f_2 : (\mathcal{A}, \equiv_A) \to (\mathcal{B}, \equiv_B)$$

as equivalent, and write $f_1 \equiv f_2$, iff they are equidefined and yield equivalent results, *i.e.* if $f_1(p)$ is defined then so is $f_2(p)$ and $f_1(p) \equiv_Q f_2(p)$, and if $f_2(p)$ is defined then so is $f_1(p)$ and $f_1(p) \equiv_Q f_2(p)$. Composition respects $\equiv$. This yields a category of equivalence families $\mathcal{F}am_\equiv$; it is enriched in the category of sets with equivalence relations (also called setoids).[6]

Clearly from an ese $(P, \equiv_P)$ we obtain an ef $(\mathcal{C}^\infty(P), \equiv_P)$ and we take a map of ese's to be a map between their associated ef's. Write $\mathcal{E}_\equiv$ for the category of ese's; it too is enriched in the category of sets with equivalence relations. When the equivalence relation $\equiv$ of an ese is the identity we essentially have prime event structures and their maps. One virtue of ese's is that they support a hiding operation, associated with a factorisation system [3].

We sometimes use an alternative description of their maps:

Proposition 1 *A map of ese's from P to Q is a partial function $f : P \rightharpoonup Q$ which preserves $\equiv$ such that*

(i) for all $X \in Con_P$ the direct image $fX \in Con_Q$ and $\forall p_1, p_2 \in X.\ f(p_1) \equiv_Q f(p_2) \implies p_1 \equiv_P p_2$, and
(ii) whenever $q \leq_Q f(p)$ there is $p' \leq_P p$ such that $f(p') = q$.

While an ese determines an ef, the converse, how to construct the causal unfolding of an ef to an ese, is much less clear. To do so we follow up on the idea of Section 2.2 of basing minimal complete enablings on partial orders. A minimal complete enabling will correspond to a *prime extremal realisations*. Realisations and extremal realisations are our next topic.

4 Causal histories as extremal realisations

Extremal causal realisations formalise the notion of causal history in models with parallel causes, *viz.* general event structures and the most general model of equivalence-families. They will be the central tool in constructing the causal unfoldings of such models.

4.1 Causal realisations

Let $\mathcal{A}$ be a family of configurations with underlying set A. A *(causal) realisation* of $\mathcal{A}$ comprises a partial order $(E, \leq)$, its *carrier*, such that the set $\{e' \in E \mid e' \leq e\}$ is finite for all events $e \in E$, together with a function $\rho : E \to A$ for which the image $\rho x \in \mathcal{A}$ when x is a down-closed subset of E. We say a realisation is *injective* when it is injective as a function.

[6] The Appendix provides background in categories enriched in equivalence relations.

A map between realisations $(E, \leq), \rho$ and $(E', \leq'), \rho'$ is a partial surjective function $f : E \rightharpoonup E'$ which preserves down-closed subsets and satisfies $\rho(e) = \rho'(f(e))$ for all $e \in E$ where $f(e)$ is defined. It is convenient to write such a map as $\rho \succeq^f \rho'$. Occasionally we shall write $\rho \succeq \rho'$, or the converse $\rho' \preceq \rho$, to mean there is a map of realisations from ρ to ρ'. Such a map factors into a "projection" followed by a total map

$$\rho \succeq_1^{f_1} \rho_0 \succeq_2^{f_2} \rho' ,$$

where ρ_0 stands for the realisation $(E_0, \leq_0), \rho_0$ where $E_0 = \{e \in E \mid f(e) \text{ is defined}\}$ is the domain of definition of f; $\leq_0$ is the restriction of $\leq$; f_1 is the inverse relation to the inclusion $E_0 \subseteq E$; and $f_2 : E_0 \to E'$ is the total part of function f. We are using $\succeq_1$ and $\succeq_2$ to signify the two kinds of maps. Notice that $\succeq_1$-maps are reverse inclusions. Notice too that $\succeq_2$-maps are exactly the total maps of realisations. Total maps $\rho \succeq_2^f \rho'$ are precisely those functions f from the carrier of ρ to the carrier of ρ' which preserve down-closed subsets and satisfy $\rho = \rho' f$.

4.2 Extremal realisations

Let $\mathcal{A}$ be a configuration family with underlying set A. We shall say a realisation ρ is *extremal* when $\rho \succeq_2^f \rho'$ implies f is an isomorphism, for any realisation ρ'; it is called *prime extremal* when it in addition has a top element, *i.e.* its carrier contains an element which dominates all other elements in the carrier. Intuitively, an extremal realisation is a most economic causal history associated with its image, a configuration of $\mathcal{A}$; it is extremal in being a realisation with minimal causal dependencies.

Any realisation in $\mathcal{A}$ can be coarsened to an extremal realisation.

Lemma 1. *For any realisation ρ there is an extremal realisation ρ' with $\rho \succeq_2^f \rho'$.*

Proof. The category of realisations with total maps has colimits of total-order diagrams. A diagram d from a total order $(I, \leq)$ to realisations, comprises a collection of total maps of realisations $d_{i,j} : d(i) \to d(j)$ when $i \leq j$ s.t. $d_{i,i}$ is always the identity map and if $i \leq j$ and $j \leq k$ then $d_{i,k} = d_{j,k} \circ d_{i,j}$. We suppose each realisation $d(i)$ has carrier $(E_i, \leq_i)$ with $d(i) : E_i \to A$. We construct the colimit realisation of the diagram as follows.

The elements of the colimit realisation consist of equivalence classes of elements of the disjoint union $E =_{\text{def}} \biguplus_{i \in I} E_i$ under the equivalence

$$(i, e_i) \sim (j, e_j) \iff \exists k \in I.\ i \leq k \ \&\ j \leq k \ \&\ d_{i,k}(e_i) = d_{j,k}(e_j)\,.$$

Consequently we may define a function $\rho_E : E \to A$ by taking $\rho_E(\{e_i\}_\sim) = \rho_i(e_i)$. Because every $d_{i,j}$ is a surjective function, every equivalence class in E has a representative in E_i for every $i \in I$. Moreover, for any $e \in E$ there is $k \in I$ s.t.

$$\{e' \in E \mid e' \leq_E e\} = \{\{e'_k\}_\sim \mid e'_k \leq_k e_k\}\,,$$

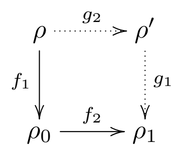
where $e = \{e_k\}_\sim$, so is finite. It follows that ρ_E is a realisation. The maps $f_i : \rho_i \succeq_2 \rho_E$, where $i \in I$, given by $f_i(e_i) = \{e_i\}_\sim$ form a colimiting cone.

Suppose ρ is a realisation. Consider all total-order diagrams d from a total order $(I, \leq)$ to realisations starting from ρ with $d_{i,j}$ not an isomorphism if $i < j$. Amongst them, by Zorn's lemma, there is a maximal diagram w.r.t. extension. From the maximality of the diagram its colimit is necessarily extremal. $\square$

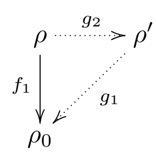
For example, as a corollary, a countable configuration of a family of configurations always has an injective extremal realisation. By serialising the countable configuration, $a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots$, where $\{a_1, \cdots, a_n\} \in \mathcal{A}$ for all n, we obtain an injective realisation ρ. By Lemma 1 we can coarsen ρ to an extremal realisation ρ' with $\rho \succeq_2^f \rho'$. As $\rho = \rho' f$ the surjective function f is also injective, so a bijection, ensuring that the extremal realisation ρ' is injective.

The following rather technical lemma and corollary are crucial.

Lemma 2. *Assume* $(R, \leq), \rho$, $(R_0, \leq_0), \rho_0$ *and* $(R_1, \leq_1), \rho_1$ *are realisations.*
(i) Suppose $f : \rho \succeq_1^{f_1} \rho_0 \succeq_2^{f_2} \rho_1$*. Then there are maps so that* $f : \rho \succeq_2^{g_2} \rho' \succeq_1^{g_1} \rho_1$*:*



(ii) Suppose $\rho \succeq_1^{f_1} \rho_0$ *where* R_0 *is not a down-closed subset of* R*. Then there are maps so* $f_1 = \rho \succeq_2^{g_2} \rho' \succeq_1^{g_1} \rho_0$ *with* g_2 *not an isomorphism:*



Proof. (i) Construct the realisation $(R', \leq'), \rho'$ as follows. Define

$$R' = (R \setminus R_0) \cup R_1$$

where w.l.o.g. we assume the sets $R \setminus R_0$ and R_1 are disjoint. Define $g_2 : R \to R'$ to act as the identity on elements of $R \setminus R_0$ and as f_2 on elements of R_0.

When $b \in R \setminus R_0$, define

$$a \leq' b \ \text{ iff } \ \exists a_0 \in R.\ a_0 \leq b \ \&\ g_2(a_0) = a\,.$$

When $b \in R_1$, define

$$a \leq' b \ \text{ iff } \ a \in R_1 \ \&\ a \leq_1 b\,.$$

To see $\leq'$ is a partial order observe that reflexivity and antisymmetry follow directly from the corresponding properties of $\leq$ and $\leq_1$. Transitivity requires an argument by cases. For example, in the most involved case, where

$$c \leq' a \text{ with } a \in R_1 \text{ and } a \leq' b \text{ with } b \in R \setminus R_0$$

we obtain

$$c \leq_1 a \text{ and } a_0 \leq b$$

for some $a_0 \in R_0$ with $f_2(a_0) = a$. As f_2 is surjective and preserves down-closed subsets,

$$c_0 \leq_0 a_0 \text{ and } a_0 \leq b$$

for some $c_0, \in R_0$ with $f_2(c_0) = c$. Consequently, $c_0 \leq b$ with $g_2(c_0) = c$, making $c \leq' b$, as required for transitivity.

Define ρ' to act as ρ on elements of $R \setminus R_0$ and as ρ_1 on elements of R_1. Then $\rho = \rho' g_2$ directly. We check ρ' preserves down-closed subsets, so is a realisation. Let $b \in R'$. If $b \in R_1$ then $\rho'[b]' = \rho_1[b]_1 \in \mathcal{C}(A)$. If $b \in R \setminus R_0$ then $\rho'[b]' = \rho g_2[b]$ the image under ρ of the down-closed subset $g_2[b]$, so in $\mathcal{C}(A)$. Because f_2 preserves down-closed subsets so does g_2. We already have $\rho = \rho' g_2$, making g_2 a map of realisations $\rho \succeq_2^{g_2} \rho'$. Define $g_1 : R' \rightharpoonup R_1$ to be the reverse of the inclusion $R_1 \subseteq R'$. Because ρ_1 is the restriction of ρ' to R_1, g_1 is a map of realisations $\rho' \succeq_1^{g_1} \rho_1$. By construction $f = g_1 g_2$.
(ii) This follows from the construction of $(R' \leq'), \rho'$ used in (i) but in the special case where f_2 is the identity map (with $R_0 = R_1$). Then $R' = R$ but $\leq' \neq \leq$ as there is $e \in R_0$ with $[e]_0 \subsetneq [e]$ ensuring that $[e]' = [e]_0 \neq [e]$. □

Corollary 1. *If ρ is extremal and $\rho \succeq^f \rho'$, then ρ' is extremal and there is ρ_0 s.t. $f : \rho \succeq_1 \rho_0 \cong \rho'$. Moreover, the carrier R_0 of ρ_0 is a down-closed subset of the carrier R of ρ, with order the restriction of that on R.*

Proof. Directly from Lemma 2. Assume ρ is extremal and $\rho \succeq^f \rho'$. We can factor f into $\rho \succeq_1^{f_1} \rho_0 \succeq_2^{f_2} \rho'$. From (i), if ρ_0 were not extremal nor would ρ be—a contradiction; hence f_2 is an isomorphism. From (ii), the carrier R_0 of ρ_0 has to be a down-closed subset of the carrier R of ρ, as otherwise we would contradict the extremality of ρ. □

It follows that if ρ is extremal and $\rho \succeq^f \rho'$ then ρ' is extremal and the inverse relation $g =_{\text{def}} f^{-1}$ is an injective function preserving and reflecting down-closed subsets, *i.e.* $g[r'] = [g(r')]$ for all $r' \in R'$. In other words:

Corollary 2. *If ρ is extremal and $\rho \succeq^f \rho'$, then ρ' is extremal and the inverse $g =_{\text{def}} f^{-1}$ is a rigid embedding from the carrier of ρ' to the carrier of ρ such that $\rho' = \rho f$.*

Lemma 3. *Let $(R, \leq), \rho$ be an extremal realisation. Then*

(i) if $r' \leq r$ and $\rho(r) = \rho(r')$ then $r = r'$;
(ii) if $[r) = [r')$ and $\rho(r) = \rho(r')$ then $r = r'$. Here $[r) =_{\text{def}} [r] \setminus \{r\}$.

Proof. (i) Suppose $r' \leq r$ and $\rho(r) = \rho(r')$. By Corollary 2, we may project to $[r]$ to obtain an extremal realisation $\rho_0 : [r] \to A$. Suppose r and r' were unequal. We can define a realisation as the restriction of ρ_0 to $[r)$. The function from $[r]$ to $[r)$ taking r to r' and otherwise acting as the identity function is a map of

realisations from the realisation ρ_0 and clearly not an isomorphism, showing ρ_0 to be non-extremal—a contradiction. Hence $r = r'$, as required.
(ii) Suppose $[r) = [r')$ and $\rho(r) = \rho(r')$. Projecting to $[\{r, r'\}]$ we obtain an extremal realisation. If r and r' were unequal there would be a non-isomorphism map to the realisation obtained by projecting to $[r]$, *viz.* the map from $[\{r, r'\}]$ to $[r]$ sending r' to r and fixing all other elements. □

In fact, by modifying condition (i) in the lemma above a little we can obtain a characterisation of extremal realisations—see the Appendix for the proof:

Lemma 4. *Let $(R, \leq), \rho$ be a realisation. Then ρ is extremal iff*

(i) if $X \subseteq [r)$, with X down-closed and $r \in R$, and $\rho(X \cup \{r\}) \in \mathcal{A}$ then $X = [r)$; and

(ii) if $[r) = [r')$ and $\rho(r) = \rho(r')$ then $r = r'$.

Lemma 5. *There is at most one map between extremal realisations.*

Proof. Let $(R, \leq), \rho$ and $(R', \leq'), \rho'$ be extremal realisations. Let $f, f' : \rho \to \rho'$ be maps with converse relations g and g' respectively. We show the two functions g and g' are equal, and hence so are their converses f and f'. Suppose otherwise that $g \neq g'$. Then there is an $\leq$-minimal $r' \in R'$ for which $g(r') \neq g'(r')$ and $g[r) = g'[r')$. Hence $[g(r')) = [g'(r'))$ and $\rho(g(r')) = \rho'(r') = \rho(g'(r'))$. As ρ is extremal, by Lemma 3(ii) we obtain $g(r') = g'(r')$—a contradiction. □

Hence extremal realisations of A under $\preceq$ form a preorder. The *order of extremal realisations* has as elements isomorphism classes of extremal realisations ordered according to the existence of a map between representatives of isomorphism classes. Alternatively, we could take a choice of representative from each isomorphism class and order these according to whether there is a map from one to the other. Recall an prime extremal realisation is an extremal realisation with a top element, *i.e.* when its carrier contains an element which dominates all other elements in the carrier. The following is a direct corollary of Proposition 4 in the next section.

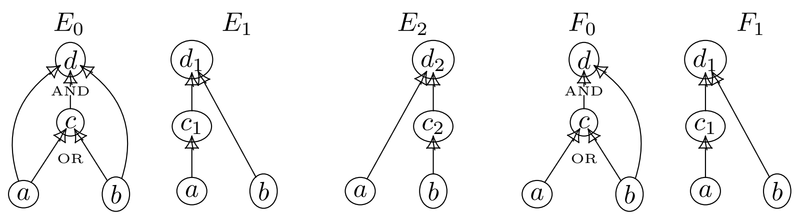
Proposition 2 *The order of extremal realisations of a family of configurations $\mathcal{A}$ forms a prime-algebraic domain [7] with complete primes the prime extremal realisations.*

The proofs of the following observations are straightforward consequences of the definitions. They emphasise that prime extremal realisations are a generalisation of (complete) primes.

Proposition 3 *Let $(A, \leq_A, Con_A)$ be a prime event structure. For an extremal realisation $(R, \leq_R), \rho$ of $\mathcal{C}^\infty(A)$, the function $\rho : R \to \rho R$ is an order isomorphism between $(R, \leq_R)$ and the configuration $\rho R \in \mathcal{C}^\infty(A)$ ordered by the restriction of $\leq_A$. The function taking an extremal realisation $(R, \leq_R), \rho$ to the configuration ρR is an order isomorphism from the order of extremal realisations of $\mathcal{C}^\infty(A)$ to the configurations of A; prime extremal realisations correspond to complete primes of $\mathcal{C}^\infty(A)$.*

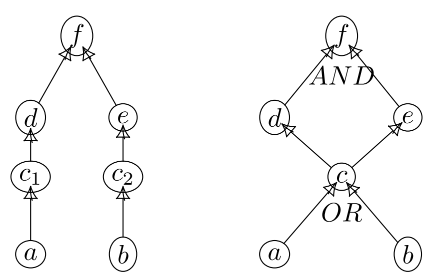
A configuration $x \in \mathcal{F}$, of a family of configurations $\mathcal{F}$, is *irreducible* iff there is a necessarily unique $e \in x$ such that $\forall y \in \mathcal{F},\ e \in y \subseteq x$ implies $y = x$. Irreducibles coincide with complete join irreducibles w.r.t. the order of inclusion. It is tempting to think of irreducibles as representing minimal complete enablings. But, as sets, irreducibles both (1) lack sufficient structure: in the formulation we are led to, of minimal complete enabling as prime extremal realisations, several prime realisations can have the same irreducible as their underlying set; and (2) are not general enough: there are prime realisations whose underlying set is not an irreducible. We conclude with examples illustrating the nature of extremal realisations; it is convenient to describe families of configurations by general event structures.

Example 2. This example shows that prime extremal realisations do not correspond to irreducible configurations. First, we show a general event structure E_0 (all subsets consistent) with irreducible configuration $\{a, b, c, d\}$ and two (injective) prime extremals E_1 and E_2 with tops d_1 and d_2 which both have the same irreducible configuration $\{a, b, c, d\}$ as their image. The lettering indicates the functions associated with the realisations, *e.g.* events d_1 and d_2 in the partial orders map to d in the general event structure.



On the other hand there are prime extremal realisations of which the image is not an irreducible configuration. Consider the general event structure F_0. The prime extremal F_1 describes a situation where d is enabled by b and c is enabled by a. It has image the configuration $\{a, b, c, d\}$ which is not irreducible, being the union of the two incomparable configurations $\{a\}$ and $\{b, c, d\}$. □

Example 3. It is possible to have extremal realisations in which an event depends on an event of the family having been enabled in two distinct ways, as in the following prime extremal realisation, on the left; it is clearly not injective.



The extremal describes the event f being enabled by d and e where they are in turn enabled by different ways of enabling c. We assume all subsets consistent.□

5 The causal unfolding: an adjunction from $\mathcal{E}_\equiv$ to $\mathcal{F}am_\equiv$

Furnished with the concept of extremal realisation, we can now exhibit an adjunction (precisely, a very simple case of biadjunction) from $\mathcal{E}_\equiv$, the category of ese's, to $\mathcal{F}am_\equiv$, the category of equivalence families. The left adjoint $I : \mathcal{E}_\equiv \to \mathcal{F}am_\equiv$ is the full and faithful functor which takes an ese to its family of configurations with the original equivalence.

The right adjoint, the *causal unfolding*, $er : \mathcal{F}am_\equiv \to \mathcal{E}_\equiv$ is defined on objects as follows. Let $\mathcal{A}$ be an equivalence family with underlying set A. Define $er(\mathcal{A}) = (P, \mathrm{Con}_P, \leq_P, \equiv_P)$ where

– P consists of a choice from within each isomorphism class of the prime extremals p of $\mathcal{A}$ —we write $top(p)$ for the image of the top element in A;
– Causal dependency $\leq_P$ is $\preceq$ on P;
– $X \in \mathrm{Con}_P$ iff $X \subseteq_{\mathrm{fin}} P$ and $top[X]_P \in \mathcal{A}$ —the set $[X]_P$ is the $\leq_P$-downwards closure of X, so equal to $\{p' \in P \mid \exists p \in X.\ p' \preceq p\}$;
– $p_1 \equiv_P p_2$ iff $p_1, p_2 \in P$ and $top(p_1) \equiv_A top(p_2)$.

Proposition 4 *The configurations of P, ordered by inclusion, are order-isomorphic to the order of extremal realisations: an extremal realisation ρ corresponds, up to isomorphism, to the configuration $\{p \in P \mid p \preceq \rho\}$ of P; conversely, a configuration x of P corresponds to an extremal realisation $top : x \to A$ with carrier $(x, \preceq)$, the restriction of the order of P to x.*

Proof. It will be helpful to recall, from Corollary 2, that if $\rho \succeq^f \rho'$ between extremal realisations, then the inverse relation f^{-1} is a rigid embedding of (the carrier of) ρ' in (the carrier of) ρ; so $\rho' \preceq \rho$ stands for a rigid embedding. Suppose $x \in \mathcal{C}^\infty(P)$. Then x determines an extremal realisation

$$\theta(x) =_{\mathrm{def}} top : (x, \preceq) \to A\,.$$

The function $\theta(x)$ is a realisation because each p in x is, and extremal because, if not, one of the p in x would fail to be extremal, a contradiction. Clearly $\rho' \preceq \rho$ implies $\theta(\rho') \subseteq \theta(\rho)$. Conversely, it is easily checked that any extremal realisation $\rho : (R, \leq) \to A$ defines a configuration $\{p \in P \mid p \preceq \rho\}$. If $x \subseteq y$ in $\mathcal{C}^\infty(P)$ then $\phi(x) \preceq \phi(y)$. It can be checked that θ and ϕ are mutual inverses, *i.e.* $\phi\theta(x) = x$ and $\theta\phi(\rho) \cong \rho$ for all configurations x of P and extremal realisations ρ. □

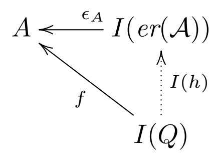
From the above proposition we see that the events of $er(\mathcal{A})$ correspond to the order-theoretic completely-prime extremal realisations [7]. This justifies our use of the term 'prime extremal' for extremal with top element.

The component of the counit of the adjunction $\epsilon_A : I(er(\mathcal{A})) \to \mathcal{A}$ is given by the function

$$\epsilon_A(p) = top(p)\,.$$

It is a routine check to see that ϵ_A preserves $\equiv$ and that any configuration x of P images under top to a configuration in $\mathcal{A}$, moreover in a way that reflects $\equiv$.

Theorem 5. *Let $\mathcal{A} \in \mathcal{F}am_{\equiv}$. For all $f : I(Q) \to \mathcal{A}$ in $\mathcal{F}am_{\equiv}$, there is a map $h : Q \to er(\mathcal{A})$ in $\mathcal{E}_{\equiv}$ such that $f = \epsilon_A \circ I(h)$, i.e. so the diagram*



commutes. Moreover, if $h' : Q \to er(\mathcal{A})$ is a map in $\mathcal{E}_{\equiv}$ s.t. $f \equiv \epsilon_A \circ I(h')$, i.e. the diagram above commutes up to $\equiv$, then $h' \equiv h$.

Proof. Let $Q = (Q, \mathrm{Con}_Q, \leq_Q, \equiv_Q)$ be an ese and $f : I(Q) \to \mathcal{A}$ a map in $\mathcal{F}am_{\equiv}$. We shall define a map $h : Q \to er(\mathcal{A})$ s.t. $f = \epsilon_A h$. (As here, in the proof we shall elide the composition symbol $\circ$, and I on maps which it leaves unchanged.)

We define the map $h : Q \to er(\mathcal{A})$ by induction on the depth of Q. The depth of an event in an event structure is the length of a longest $\leq$-chain up to it—so an initial event has depth 1. We take the depth of an event structure to be the maximum depth of its events. (Because of our reliance on Lemma 1, we use the axiom of choice implicitly.)

Assume inductively that $h^{(n)}$ defines a map from $Q^{(n)}$ to $er(\mathcal{A})$ where $Q^{(n)}$ is the restriction of Q to depth below or equal to n such that $f^{(n)}$ the restriction of f to $Q^{(n)}$ satisfies $f^{(n)} = \epsilon_A h^{(n)}$. (In particular, $Q^{(0)}$ is the empty ese and $h^{(0)}$ the empty function.) Then, by Proposition 4, any configuration x of $Q^{(n)}$ determines an extremal realisation $\rho_x : h^{(n)}x \to A$ with carrier $(h^{(n)}x, \preceq)$.

Suppose $q \in Q$ has depth $n+1$. If $f(q)$ is undefined take $h^{(n+1)}(q)$ to be undefined. Otherwise, note there is an extremal realisation $\rho_{[q]}$ with carrier $(h[q], \preceq)$. Extend $\rho_{[q]}$ to a realisation $\rho^{\top}_{[q]}$ with carrier that of $\rho_{[q]}$ with a new top element $\top$ adjoined, and make $\rho^{\top}_{[q]}$ extend the function $\rho_{[q]}$ by taking $\top$ to $f(q)$. By Lemma 1, there is an extremal realisation ρ such that $\rho^{\top}_{[q]} \succeq_2 \rho$. Because $\rho_{[q]}$ is extremal, $\rho \succeq_1 \rho_{[q]}$, so ρ only extends the order of $\rho_{[q]}$ with extra dependencies of $\top$. (For notational simplicity we identify the carrier of ρ with the set $h[q] \cup \{\top\}$.) Project ρ to the extremal with top $\top$. Define this to be the value of $h^{(n+1)}(q)$. In this way, we extend $h^{(n)}$ to a partial function $h^{(n+1)} : Q^{(n+1)} \to er(\mathcal{A})$ such that $f^{(n+1)} = \epsilon_A h^{(n+1)}$. To see that $h^{(n+1)}$ is a map we can use Proposition 1. By construction $h^{(n+1)}$ satisfies property (ii) of Proposition 1 and the other properties are inherited fairly directly from f via the definition of $er(\mathcal{A})$.

Defining $h = \bigcup_{n \in \omega} h^{(n)}$ we obtain a map $h : Q \to er(\mathcal{A})$ such that $f = \epsilon_A h$.

Suppose $h' : Q \to er(\mathcal{A})$ is a map s.t. $f \equiv \epsilon_A h'$. Then, for any $q \in Q$,

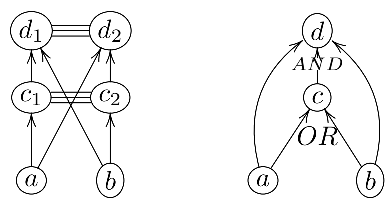
$$top(h'(q)) = \epsilon_A h'(q) \equiv_A f(q) = \epsilon_A h(q) = top(h(q)),$$

so $h'(q) \equiv_P h(q)$ in $er(\mathcal{A})$. Thus $h' \equiv h$. □

The theorem does not quite exhibit a traditional adjunction, because the usual cofreeness condition specifying an adjunction is weakened to only having uniqueness up to $\equiv$. However the condition it describes does specify an exceedingly simple case of a biadjunction (or pseudo adjunction) between 2-categories—a set together with an equivalence relation (a *setoid*) is a very simple example of a category. As a consequence, whereas with the usual cofreeness condition allows us to extend the right adjoint to arrows, so obtaining a functor, in this case following that same line will only yield a pseudo functor er as right adjoint: thus extended, er will only preserve composition and identities up to $\equiv$.

The map $(P, \equiv) \to er(\mathcal{C}^\infty(P), \equiv)$ which takes $p \in P$ to the realisation with carrier $([p], \leq)$, the restriction of the causal dependency of P, with the inclusion function $[p] \hookrightarrow P$ is an isomorphism; recall from Proposition 3 that the configurations of a prime event structure correspond to its extremal realisations. Such maps furnish the components of the unit of the adjunction.

Example 4. On the right we show a general event structure (all subsets consistent) and on its left its causal unfolding to an ese under er; the unfolding's events are the prime extremals.



□

6 Unfolding general event structures

Recall $\mathcal{G}$ is the category of general event structures. We obtain an adjunction from $\mathcal{E}_\equiv$ to $\mathcal{G}$ via an adjunction from $\mathcal{F}am_\equiv$ to $\mathcal{G}$. The right adjoint $fam : \mathcal{G} \to \mathcal{F}am_\equiv$ is most simply described. Given $(E, \mathrm{Con}, \vdash)$ in $\mathcal{G}$ it returns the equivalence family $(\mathcal{C}^\infty(E), =)$ in $\mathcal{F}am_\equiv$ comprising the configurations together with the identity equivalence between events that appear within some configuration; the partial functions between events that are maps in $\mathcal{G}$ are automatically maps in $\mathcal{F}am_\equiv$—the action of fam on maps.

For the effect of the left adjoint $col : \mathcal{F}am_\equiv \to \mathcal{G}$ on objects, define the *collapse*

$$col(\mathcal{A}) =_{\mathrm{def}} (E, \mathrm{Con}, \vdash)$$

where

- $E = A_\equiv$, the equivalence classes of events in $A =_{\mathrm{def}} \bigcup \mathcal{A}$;
- $X \in \mathrm{Con}$ iff $X \subseteq_{\mathrm{fin}} y_\equiv$, for some $y \in \mathcal{A}$; and
- $X \vdash e$ iff $e \in E$, $X \in \mathrm{Con}$ and $e \in y_\equiv \subseteq X \cup \{e\}$, for some $y \in \mathcal{A}$.

Let $(\mathcal{A}, \equiv) \in \mathcal{F}am_\equiv$. Assume that $\mathcal{A}$ has underlying set A. The unit of the adjunction is defined to have typical component $\eta_A : (\mathcal{A}, \equiv) \to fam(col(\mathcal{A}, \equiv))$ given by $\eta_A(a) = \{a\}_\equiv$. It is easy to check that η_A is a map in $\mathcal{F}am_\equiv$.

Theorem 6. *Suppose that* $B = (B, Con_B, \vdash_B) \in \mathcal{G}$ *and that* $g : (\mathcal{A}, \equiv) \to (\mathcal{C}^\infty(B), =)$ *is a map in* $\mathcal{F}am_\equiv$*. Then, there is a unique map* $k : col(\mathcal{A}, \equiv) \to B$ *in* $\mathcal{G}$ *s.t. the diagram*

$$\begin{array}{ccc} (\mathcal{A}, \equiv) & \xrightarrow{\eta_A} & fam(col(\mathcal{A}, \equiv)) \\ & \searrow^{g} & \downarrow{\scriptstyle fam(k)} \\ & & (\mathcal{C}^\infty(B), =) \end{array}$$

commutes.

Proof. The map $k : col(\mathcal{A}, \equiv) \to B$ is given as the function $k(e) = g(a)$ where $e = \{a\}_\equiv$. It is easily checked to be a map in $\mathcal{G}$ and moreover to be the unique map from $col(\mathcal{A}, \equiv)$ to B making the above diagram commute. □

Theorem 6 determines an adjunction from $\mathcal{F}am_\equiv$ to $\mathcal{G}$. The construction *col* automatically extends from objects to maps; maps in $\mathcal{F}am_\equiv$ preserve equivalence so collapse to functions preserving equivalence classes. The counit of the adjunction has components $\epsilon_E : col((\mathcal{C}^\infty(E), =)) \to E$ which send singleton equivalence classes $\{e\}$ to e. The counit is an isomorphism at precisely those general event structures E which are replete.

Composing $\mathcal{E}_\equiv \underset{I}{\overset{er}{\rightleftarrows}}_{\top} \mathcal{F}am_\equiv \underset{col}{\overset{fam}{\rightleftarrows}}_{\top} \mathcal{G}$ we obtain a pseudo adjunction $\mathcal{E}_\equiv \rightleftarrows_{\top} \mathcal{G}$. Its right adjoint constructs the *causal unfolding* of a general event structure.

The composite adjunction from $\mathcal{E}_\equiv$ to $\mathcal{G}$ cuts down to a reflection, in which the counit is a natural isomorphism, when we restrict to the subcategory of $\mathcal{G}$ where all general event structures are replete. Then the right adjoint provides a full and faithful embedding of replete general event structures (and so families of configurations) in ese's.

7 Conclusion

This concludes the construction of causal unfoldings of (very general) equivalence-families, and so, in particular, general event structures. In applications it has been useful to cut down the unfolding to subcategories. In particular, while the category of event structures with equivalence, $\mathcal{E}_\equiv$, does have bipullbacks (in which commutations and uniqueness are only up to the equivalence $\equiv$ on maps) it doesn't always have the pseudo pullbacks or pullbacks, used in defining the composition of strategies. However, an important subcategory does: define $\mathcal{EDC}$ to be the subcategory of $\mathcal{E}_\equiv$ with objects, *event structures with disjunctive causes* (edc's), satisfying: $p_1, p_2 \leq p \;\&\; p_1 \equiv p_2 \implies p_1 = p_2$. In an edc an event cannot causally depend on two distinct prime causes of a common disjunctive event, and so rules out realisations such as that mentioned in Example 3. $\mathcal{EDC}$ provides a suitable foundation for strategies with parallel causes and is handily related by adjunctions to general and prime event structures [3].

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A Equiv-enriched categories

Here we explain in more detail what we mean when we say "enriched in the category of sets with equivalence relations" and employ terms such as "enriched adjunction," "pseudo adjunction" and "pseudo pullback." The classic text on enriched categories is [5], but for this paper the articles [6] and [9] provide short, accessible introductions to the notions we use from Equiv-enriched categories and 2-categories, respectively.

Equiv is the category of *equivalence relations.* Its objects are $(A, \equiv_A)$ comprising a set A and an equivalence relation $\equiv_A$ on it. Its maps $f : (A, \equiv_A) \to (B, \equiv_B)$ are total functions $f : A \to B$ which preserve equivalence.

We shall use some basic notions from enriched category theory [5]. We shall be concerned with categories enriched in Equiv, called Equiv-enriched categories, in which the homsets possess the structure of equivalence relations, respected by composition [6]. This is the sense in which we say categories are enriched in (the category of) equivalence relations. We similarly borrow the concept of an Equiv-enriched functor between Equiv-enriched categories which preserve equivalence in acting on homsets. An Equiv-enriched adjunction is a usual adjunction in which the natural bijection preserves and reflects equivalence.

Because an object in Equiv can be regarded as a (very simple) category, we can regard Equiv-enriched categories as (very simple) 2-categories to which notions from 2-categories apply [9].

A *pseudo functor* between Equiv-enriched categories is like a functor but the usual laws only need hold up to equivalence. A *pseudo adjunction* (or biadjunction) between 2-categories permits a weakening of the usual natural isomorphism between homsets, now also categories, to a natural equivalence of categories. In the special case of a pseudo adjunction between Equiv-enriched categories the equivalence of homset categories amounts to a pair of $\equiv$-preserving functions whose compositions are $\equiv$-equivalent to the identity function. With traditional adjunctions by specifying the action of one adjoint solely on objects we determine it as a functor; with pseudo adjunctions we can only determine it as a pseudo functor—in general a pseudo adjunction relates two pseudo functors. Pseudo adjunctions compose in the expected way. An Equiv-enriched adjunction is a special case of a 2-adjunction between 2-categories and a very special case of pseudo adjunction. In this article there are many cases in which we compose an Equiv-enriched adjunction with a pseudo adjunction to obtain a new pseudo adjunction.

Similarly we can specialise the notions pseudo pullbacks and bipullbacks from 2-categories to Equiv-enriched categories. Let $f : A \to C$ and $g : B \to C$ be two maps in an Equiv-enriched category. A *pseudo pullback* of f and g is an object D and maps $p : D \to A$ and $q : D \to B$ such that $f \circ p \equiv g \circ q$ which satisfy the further property that for any D' and maps $p' : D' \to A$ and $q' : D' \to B$ such that $f \circ p' \equiv g \circ q'$, there is a unique map $h : D' \to D$ such that $p' = p \circ h$ and $q' = q \circ h$. There is an obvious weakening of pseudo pullbacks to the situation in which the uniqueness is replaced by uniqueness up to $\equiv$ and the equalities by $\equiv$—these are simple special cases of bilimits called *bipullbacks*.

Right adjoints in a 2-adjunction preserve pseudo pullbacks whereas right adjoints in a pseudo adjunction are only assured to preserve bipullbacks.

B Additional proof

The following characterisation of extremal realisations is not strictly necessary for the rest of of the paper.

▶ **Lemma 4.** Let $(R, \leq), \rho$ be a realisation. Then ρ is extremal iff

(i) if $X \subseteq [r)$, with X down-closed and $r \in R$, and $\rho(X \cup \{r\}) \in \mathcal{A}$ then $X = [r)$; and
(ii) if $[r) = [r')$ and $\rho(r) = \rho(r')$ then $r = r'$.

Proof. *"Only if"*: Assume ρ is extremal. We have already established (ii) in Lemma 3. To show (i), suppose X is down-closed and $X \subseteq [r)$ in R with $\rho(X \cup \{r\}) \in \mathcal{A}$. By Corollary 2, we may project to $[r]$ to obtain an extremal realisation $\rho_0 : [r] \to A$. Modify the restricted order $[r]$ to one in which $r' \leq r$ iff $r' \in X$, and is otherwise unchanged. The same underlying function ρ_0 remains a realisation, call it ρ_0', on the modified order. The identity function gives us a map $f : \rho_0 \succeq_2 \rho_0'$ which is an isomorphism between realisations iff $X = [r)$.
"If": Assume (i) and (ii). Suppose $f : \rho \succeq_2 \rho'$, where R', ρ' is a realisation. We show f is injective and order-preserving. As f is presumed to be surjective and to preserve down-closed subsets we can then conclude it is an isomorphism.

To see f is injective suppose $f(r_1) = f(r_2)$. W.l.o.g. we may suppose r_1 and r_2 are minimal in the sense that

$$r_1' \leq r_1 \ \& \ r_2' \leq r_2 \ \& \ f(r_1') = f(r_2') \implies r_1' = r_1 \ \& \ r_2' = r_2 \,.$$

Define $r' =_{\text{def}} f(r_1) = f(r_2)$. Then

$$[r'] \subseteq f[r_1] \ \& \ [r'] \subseteq f[r_2] \,.$$

Furthermore, by the minimality of r_1, r_2,

$$[r') \subseteq f[r_1) \ \& \ [r'] \subseteq f[r_2) \,.$$

It follows that

$$[r') \subseteq f[r_1) \cap f[r_2) = f([r_1) \cap [r_2))$$

where the equality is again a consequence of the minimality of r_1, r_2. Taking $X =_{\text{def}} [r_1) \cap [r_2)$ we have $(fX) \cup \{r'\}$ is down-closed in R'. Therefore

$$\rho(X \cup \{r_1\}) = \rho' f(X \cup \{r_1\}) = \rho'(fX \cup \{r'\}) \in \mathcal{A} \,.$$

By condition (ii), $X = [r_1)$. Similarly, $X = [r_2)$, so $[r_1) = [r_2)$. Obviously $\rho(r_1) = \rho' f(r_1) = \rho' f(r_1) = \rho(r_2)$, so we obtain $r_1 = r_2$ by (i).

We now check that f preserves the order. Let $r \in R$. Define

$$X =_{\text{def}} [\{r_1 \leq r \mid f(r_1) < f(r)\}] \,,$$

where the square brackets signify down-closure in R. Then X is down-closed in R by definition and $X \subseteq [r)$. We have $[f(r)] \subseteq f[r]$ whence

$$fX = f[r] \cap [f(r)) = [f(r)) \,.$$

Therefore $fX \cup \{f(r)\}$ is down-closed in R', so

$$\rho(X \cup \{r\}) = \rho' f(X \cup \{r\}) = \rho'(fX \cup \{f(r)\}) \in \mathcal{A} \,.$$

Hence $X = [r)$, by (ii). It follows that

$$r_1 \rightarrowtriangle r \implies r_1 \in X \implies f(r_1) < f(r) \text{ in } R' \,.$$

As the order on R is the transitive closure of immediate dependency, this in turn shows that f preserves the order. □